DOCUMENT RESUME

ED 476 613 SE 065 231

AUTHOR	Speiser, Robert, Ed.; Maher, Carolyn A., Ed.; Walter, Charles N., Ed.
TITLE	Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Snowbird, Utah, October 18-21, 2001). Volume 1 [and] Volume 2.
INSTITUTION	International Group for the Psychology of Mathematics Education. North American Chapter.
ISBN	ISBN-0-88119-001-2
PUB DATE	2001-00-00
NOTE	1008p.; For selected individual papers, see SE 067 893-909, SE 067 913-914, and SE 067 924.
AVAILABLE FROM	ERIC/CSMEE Publications, 1929 Kenny Road, Columbus, OH 43210-1080. Tel: 800-276-0462; Tel: 614-292-5680.
PUB TYPE	Collected Works - Proceedings (021) -- Reports - Descriptive (141)
EDRS PRICE	EDRS Price MF07/PC41 Plus Postage.
DESCRIPTORS	*Algebra; Educational Assessment; *Educational Research; Elementary Secondary Education; Graphs; Higher Education; *Mathematics Curriculum; *Mathematics Instruction; *Thinking Skills; Visualization
IDENTIFIERS	*Representations (Mathematics)

ABSTRACT

This book contains volume 1 of the proceedings of the 23rd annual meeting of the International Group for the Psychology of Mathematics Education (PME) held October, 2001 in Snowbird, Utah. Papers include: (1) "Opening the Dimensions of Mathematical Capability: The Development of Knowledge, Practice and Identity in Mathematics Classrooms" (Jo Boaler); (2) "Learning Mathematics as Developing a Discourse" (Anna Sfard); (3) "Gender and Mathematics: Themes Within Reflective Voices" (Diana B. Erchick and Linda Condron); (4) "Representations and Mathematics Visualization" (Fernando Hitt); (5) "Models and Modeling" (Richard Lesh, Helen Doerr, Thomas Post, Judith S. Zawojewski, and Guadalupe Carmona); (6) "Geometry and Technology" (Douglas McDougall and Jean J. McGehee); (7) "The Complexity of Learning to Reason Probabilistically" (Carolyn A. Maher and Robert Speiser); and (8) "Problem Posing Research: Answered and Unanswered Questions" (Lyn D. English). Volume 2 of the proceedings of the 23rd annual meeting of the International Group for the Psychology of Mathematics Education (PME) held October, 2001 in Snowbird, Utah is also included. Papers include: (1) "Growth in Student Mathematical Understanding Through Precalculus Student and Teacher Interactions" (Daniel R. Ilaria and Carolyn A. Maher); (2) "Investigating the Teaching and Learning of Proof: First Year Results" (Tami S. Martin and Sharon Soucy McCrone); (3) "Beliefs About Proof in Collegiate Calculus" (Manya Raman); (4) "Metaphors of the Novice: Emergent Aspects of Calculus Students' Reasoning" (Mike Oehrtman); (5) "An Examination of the Interaction Patterns of a Single- Gender Mathematics Class" (Amy Burns Ellis); (6) "The Role of Intellectual Culture of Mathematics in Doctoral Student Attrition" (Abbe H. Herzig); (7) "Group Tests as a Context for Learning How to Better Facilitate Small Group Discussion" (Judith Kysh); and (8) "Learning Paths to 5- and 10-Structured Understanding of Quantity: Addition and Subtraction Solution Strategies of Japanese Children" (Aki Murata and Karen Fuson). (MM)

Reproductions supplied by EDRS are the best that can be made
from the original document.

ED 476 613

Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Snowbird, UT, October 18-21, 2001.) Volume 1 [and] Volume 2.

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

☐ This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality

☐ Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

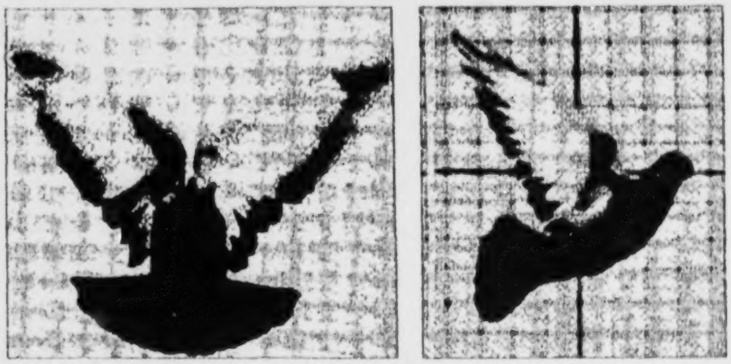
PERMISSION TO REPRODUCE AND DISSEMINATE THIS MATERIAL HAS BEEN GRANTED BY

D. Owens

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

1

SE 065 231



PME-NA XXIII

Snowbird, Utah
October, 2001

ERIC Clearinghouse for Science, Mathematics, and Environmental Education

Educational Resources Information Center

2A1

Proceedings of the
Twenty-Third Annual Meeting

North American Chapter of the International Group for the

Psychology of Mathematics Education

Volume 1: Plenaries, Working Groups, Discussion Groups, Research Reports, Short Oral Reports, Poster Presentations

PME-NA XXIII

October 18-21, 2001
Snowbird, Utah

Editors:

Robert Speiser
Carolyn A. Maher
Charles N. Walter

ERIC
Educational Resources Information Center

Clearinghouse for Science, Mathematics, and
Environmental Education, Columbus, OH

Cite as:

Speiser, R., Maher, C. A., & Walter, C. N. (Eds.). (2001). *Proceedings of the twenty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vols. 1-2). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

ERIC Clearinghouse accession number: SE 065 164

International Standard Book Number 0-88119-001-2

Clearinghouse Production Team:
Executive Editor: David L. Haury
Copyeditor: Linda A. Milbourne

Cover Design: Two Samurai Graphics

Printed in the United States of America

Order from: ERIC/CSMEE Publications, 1929 Kenny Road, Columbus, OH 43210-1080. Price per two-volume set is $45.00 plus shipping, handling, and relevant tax. All orders must be prepaid with delivery to a single address. Prices are subject to change without notice. To order by credit card, call 1-800-276-0462 or 614-292-5680.

Note. The *Proceedings* were produced with financial support provided by the U. S. Department of Education under contract no. ED-99-CO-0024. Opinions expressed in this document do not necessarily reflect the positions or policies of the U. S. Department of Education.

History and Aims of the PME Group

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education.
2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics educators.
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implementation thereof.

Reviewers

PME-NA XXIII Proceedings, 2001

Silvia Alatorre
Alice Alston
Dawn Leigh Anderson
Bridget Arvold
Leslie Aspinwall
Patricia Balderas-Canas
Jeffrey Barrett
Joanne Rossi Becker
Babette Benken
Sally Berenson
Joyce Bishop
Linda Bolte
Daniel Brahier
Terri Bucci
Christianne Maria Butto Zarzar
Marilyn Carlson
Jennifer Chauvot
Karen Koellner Clark
Jose Contreras
Ann Crawford
Christopher Danielson
Roger Day
Yolanda De La Cruz
Willi Doerfler
Christopher Drumm
Barbara Edwards
Rebekah Elliott
David Feikes
Maria L. Fernández
Jose Fonseca
John Francisco
Isobel(Bobby) Gaensler
Bradley Garner
Florence Glanfield
Karen Graham
Jose Guzman
Karen Heinz
Pao-Sheng Hsu
Elizabeth Jakubowski
Amanda Jansen
Debra Johanning
Pat Lamphere Jordan
Constance Kamii
Rochelle Goldberg Kaplan
Elhan Kazemi
Paul Kehle
Regina Kiczek
Margaret Kinzel
Anita Kitchens
Hari P. Koirala
Judith Kysh
Carol Larson
Keith Leatham
Yeping Li
LouAnn Lovin
Michael Lutz
Carolyn A. Maher
Lyndon Martin
Armando Martinez-Cruz
Kay McClain
Doug McDougall
Jean McGehee
Mercedes McGowen
David Meel
Denise Mewborn
James Middleton
Teri Jo Murphy
Janice Newton
Nancy O'Rode
Doug Owens
Stephen Pape
Erkki Pehkonen
Barbara Pence
Fred Peskoff
Catherine Pomar
Neil Portnoy
Arthur Powell
Norma Presmeg
David Pugalee
Manya Raman
Christopher Rasmussen
Kay Reinke
Ron Ritchhart
Manuel Santos
Roberta Schorr
John Selden
Annie Selden
Betty Senger
Ken Shaw
Elaine Simmt
Cynthia Smith
John P. Smith III
Robert Speiser
Elena Steencken
Kathryn Stewart
Hollylynne Drier Stohl
Walter Stone
Susan Taber
Sylvia Taube
Mourat Tchosanov
Pat Thompson
Anthony Thompson
Liz Uptegrove
Juliana Utley
Jorge Villa
Sharon Walen
Janet G. Walter
Charles Walter
John Weber
Gideon Weinstein
Dirk Wessels
Darlene Whitkenack
Dorthy Whitlow
Steve Williams
Joseph Zilliox

PME-NA Steering Committee (2000-2001)

Robert Speiser	Chair
Denise Mewborn	Chair-Elect
Barbara Pence	Treasurer
Doug McDougall	Secretary
Maria Fernandez	Past Chair
Daniel Brahier	Member-at-Large
Karen Norwood	Member-at-Large
Patrick Thompson	Member-at-Large

PME-NA XXIII Program Committee

Robert Speiser, Chair
Sarah Berenson
Maria Fernández
Carolyn A. Maher
Douglas T. Owens
Charles N. Walter
Daniel J. Brahier
Fernando Hitt
Douglas McDougall
Denise Mewborn
Karen Norwood
Barbara Pence
Patrick Thompson
Maria Trigueros

PME Proceedings in the ERIC Database

ERIC Document Reproduction Service	(800)443-ERIC
ERIC Clearinghouse for Science, Mathematics, and Environmental Education	(800)276-0462

North American Chapter

Year	Meeting	Location	ERIC Accession No.
1980	PME-NA 2/PME 4	California	ED 250 186
1981	PME-NA 3	Minnesota	ED 223 449
1982	PME-NA 4	Georgia	ED 226 957
1983	PME-NA 5	Montreal	ED 289 688
1984	PME-NA 6	Wisconsin	ED 253 432
1985	PME-NA 7	Ohio	SE 056 279
1986	PME-NA 8	Michigan	ED 301 443
1987	PME-NA 9/PME 11	Montreal	SE 055 633
1988	PME-NA 10	Illinois	SE 056 278
1989	PME-NA 11	New Jersey	SE 057 133
1990	PME-NA 12/PME 14	Mexico	ED 411 137-139
1991	PME-NA 13	Virginia	SE 352 274
1992	PME-NA 14/PME 16	New Hampshire	SE 055 811
1993	PME-NA 15	California	ED 372 917
1994	PME-NA 16	Louisiana	SE 055 636
1995	PME-NA 17	Ohio	SE 057 176
1996	PME-NA 18	Florida	SE 059 001
1997	PME-NA 19	Illinois	SE 060 729
1998	PME-NA 20	North Carolina	SE 061 830
1999	PME-NA 21	Mexico	SE 062 752
2000	PME-NA 22	Arizona	SE 064 088

International

Year	Meeting	Location	ERIC Accession No.
1978	PME 2	W. Germany	ED 226 945
1981	PME 3	England	ED 226 956
1980	PME 4	California	ED 250 186
1981	PME 5	France	ED 225 809
1982	PME 6	Belgium	ED 226 943
1983	PME 7	Israel	ED 241 295
1984	PME 8	Australia	ED 306 127
1985	PME 9	Netherlands	ED 411 130-131
1986	PME 10	London	ED 287 715
1987	PME 11	Montreal	ED 383 532
1988	PME 12	Hungary	ED 411 128-129
1989	PME 13	Paris	ED 411 140-142
1990	PME 14	Mexico	ED 411 137-139
1991	PME 15	Italy	ED 413 162-164
1992	PME 16	New Hampshire	ED 383 538
1993	PME 17	Japan	ED 383 536
1994	PME 18	Portugal	ED 383 537
1995	PME 19	Brazil	ED 411 134-136
1996	PME 20	Spain	SE 064 730-734
1997	PME 21	Finland	ED 416 082-084
1998	PME 22	South Africa	ED 427 969
1999	PME 23	Israel	ED 436 403
2000	PME 24	Japan	ED 452 031-034

Preface

If a central theme emerges, for the editors at least, from the preparation of these volumes, it might well be "Tackle major issues—don't think small." This ambitious boldness seems especially evident in the Plenary addresses, in the long-term efforts of the working groups, in the intellectual energy and ferment represented by this year's discussion groups, and in the wide variety and depth of this year's papers.

Further, as we see it, the Utah meeting demonstrates in many ways the highly representative and indeed international character of PME-NA. Our plenary speakers, for example, draw collectively on life and work experiences on at least three continents.

We are especially gratified by the productive efforts of the Working Groups, from which collective publications are beginning to emerge. In this connection, we are especially excited by the papers from the group on Modelling and Mathematics Visualization. In a similar spirit of joint innovation, the Mu-Group (Susan Pirie and her coworkers, from British Columbia) has prepared four sessions of linked research reports on the development of mathematical understanding. Further, no less than four discussion groups, also drawing from experience obtained on several continents, will address important issues from a wide range of perspectives.

For these reasons, putting together PME-NA XXIII in Utah gives us special satisfaction. We hasten to thank the sponsoring institutions: Rutgers University, through the Robert B. Davis Institute for Learning, Graduate School of Education; and Brigham Young University, through the Department of Mathematics Education (College of Physical and Mathematical Sciences) and the Center for the Improvement of Teacher Education and Schooling (McKay School of Education).

Finally, we wish especially to thank Doug McDougall for taking on the daunting challenge of establishing the new PME-NA website, and for his patience and resourcefulness over those unforgettable first months of building, learning and debugging.

The Editors

Contents

History And Aims Of The PME Group iii
Reviewers iv
PME-NA Committees v
PME Proceedings In The ERIC Database vi
Preface ix

Plenary Papers **1**

Opening the Dimensions of Mathematical Capability: The Development of Knowledge, Practice and Identity in Mathematics Classrooms 3
Jo Boaler

Learning Mathematics as Developing a Discourse 23
Anna Sfard

Working Groups **45**

Gender and Mathematics: Themes Within Reflective Voices 47
Diana B. Erchick & Linda Condron

Representations and Mathematics Visualization 53
Fernando Hitt

Models and Modeling 67
Richard Lesh, Helen Doerr, Thomas Post, Judith S. Zawojewski, & Guadalupe Carmona

Geometry and Technology 71
Douglas McDougall & Jean J. McGehee

The Complexity of Learning to Reason Probabilistically 73
Carolyn A. Maher & Robert Speiser

Discussion Groups **79**

Problem Posing Research: Answered and Unanswered Questions 81
Lyn D. English

The Quality and Role of Curriculum Materials in Mathematics Education 83
Yeping Li

Inquiry Into Videotape Data Analysis for Studying the Growth of Mathematical Understanding 84
Arthur Powell

Defining and Promoting Quality Research in Mathematics Education 87
Martin A. Simon

Algebraic Thinking 89

Research Reports

On Transformations of Functions 91
Bernadette Baker, Clare Hemenway & Maria Trigueros

Student Achievement in Algebraic Thinking: A Comparison of 3rd Graders' Performance on a State 4th Grade Assessment 99
James J. Kaput & Maria L. Blanton

Analyzing College Calculus Students' Interpretation and Use of Algebraic Notation 109
Margaret Tatem Kinzel

Students' Reactions and Adjustments to Fundamental Curricular Changes: What Are "Mathematical Transitions"? How Can We Study Them? 117
Jon Star, Dawn Berk, Carol Burdell, Amanda Jansen Hoffmann, Violeta Lazarovici, Gary Lewis & Jack Smith

Short Oral Reports

Algebraic Reasoning Demonstrated by Grade 4, 5, and 6 Students as They Solve Problems With Two and Three Unknowns 129
Galina Dobrynina

Early Access to Algebraic Ideas: The Role of Representations and Mathematics of Change 131
Teresa Rojano & Elvia Perrusquía

A Study of Students' Translations From Equations to Word Problems 133
Ana C. Stephens

Posters

School Algebra as a Language: The First Elements of Algebraic Syntax and Semantics 135
Armando Solares Rojas

Learning and Cognition 137

Research Reports

Remnant Effect of Basic School Mathematics in Mexico 139
Silvia Alatorre

12

An Investigation of Covariational Reasoning and Its Role in Learning the Concepts of Limit and Accumulation 145
Marilyn P. Carlson, Sean Larsen & Sally Jacobs

The Role of Children's Spontaneous Concepts on the Development of Representations for Mathematics Learned in School 155
Rochelle Goldberg Kaplan

The Notion of Limit and Learning Problems 169
Fernando Hitt & Rosa Páez

A Modeling Perspective on Metacognition in Everyday Problem-Solving Situations 177
Margret Hjalmarson

Learning Multi-Digit Multiplication by Modeling Rectangles 187
Andrew Izsák

Conceptual Understanding of Functions: A Tale of Two Schemas 195
Mindy Kalchman & Karen C. Fuson

Young Children's Representation of Groups of Objects: The Relationship Between Abstraction and Representation 207
Constance Kamii

"I Wouldn't Want to Get My Head Whomped Off by a Butcher's Knife!" Understanding Children's Performance on State Assessments 213
Elham Kazemi

Set Forming and Set Ordering: Useful Similes in Observing the Growth of Mathematical Understanding? 223
Thomas Kieren

A Over B: Meanings Over Algorithms 235
Susan J. Lamon

Growing Mathematical Understanding: Teaching and Learning as Listening and Sharing 245
Lyndon C. Martin

The Role of Task-Analysis Cycles in Supporting Students' Mathematical Development 255
Kay McClain

The Emergence of Quotient Understandings in a Fifth-Grade Classroom: A Classroom Teaching Experiment 263
James A. Middleton, Teruni de Silva, Zulbiye Toluk & Wendy Mitchell

Arithmetic Thinking Strategies and Low Achieving Junior High School Students in Australia 273
Bob Perry & Peter Howard

Creating the Space of Solution Functions to Differential Equations 281
Chris Rasmussen & Michelle Stephan

A Teaching Proposal About Ratio and Proportion Worked With Students of Elementary School 291
Elena Fabiola Ruiz Ledesma & Marta Valdemoros Álvarez

Toward Consensus on Computational Strategies for Single-Digit Multiplication 301
Bruce Sherin

Part-Whole Concept Understanding in a Population of 2nd and 3rd Graders 313
Dorothea A. Steinke

Mental Representations - The Interrelationship of Subject Matter and Pedagogical Content Knowledge--The Case of Exponential Functions 321
Günter Törner

Does Teaching Matter? Exploring the Relationship Between Teaching and the Growth of Students' Mathematical Understanding 331
Jo Towers

Theory, Video and Mathematical Understanding: An Examination of What Different Theoretical Perspectives Can Offer 343
"The μ Group"

1. Analysis, Lies, and Videotape 346
   Susan E. B. Pirie
2. Analysis and Structure: Using Schoenfeld's Table to Help Understand Students' Understanding 350
   Katharine Borgen
3. Exploring the Notion of Epistemological Obstacles 356
   Sitaniselao Stan Manu
4. Making Decisions: Analysis Through a Teacher's Eyes 362
   Donna Jenner
5. Rectangular Pyramids, Objects of Thinking, and Mathematical Understandings 367
   Jennifer Thom

6. Understanding as a Dynamical Process: How Do They Get to Where They Are? 372
Lyndon C. Martin

Short Oral Reports

Increasing the Comprehension of Function Notion From Virtual Experience Using Cabri-II 381
Verónica Hoyos

Non-Analogous Reasoning in an Analogous Situation of Repeating Pattern 383
Peter Liljedahl

The Development of Children's Understanding of Equivalence Relationships in Numerical Symbolic Contexts 385
Anne Morris

The Influence of Exploring Patterns and Generalizing on the Formation of the Concepts of Sequences and Series 387
Dirk Wessels, Glenda Nixon & Cornelia Naude

Posters

Equivalence: Concept Building in a Fifth Grade Classroom 389
Ann Baker & Amy McDuffie

Mapping Pedagogical Growth Across Layers of the Pirie-Kieren Model Within a Group Lesson-Planning Activity 391
Matthew R. Clark

Preservice Elementary Teachers' Solution Processes and Their Interpretations of Solutions to Problematic Word Problems Involving Division of Fractions 393
José N. Contreras

Motion or Transactions: Developing Teachers' Understanding of Rates of Change by Connecting Ideas From Different Contexts 395
José Aires de Castro-Filho & Jere Confrey

A Peircean Nested Model of Semiotic Chaining Linking Home and School Mathematics 397
Norma C. Presmeg

Undergraduates' Conceptions of Congruence of Integers 399
Jennifer Christian Smith

Probabilty and Statistics **401**

Research Reports

Probability Modeling: Clarifying the Categories 403
Carol T. Benson & Edward S. Mooney

Students' Developments in Solving Data-Handling Ends-in-View Problems 413
Lyn D. English

The Stability of Probabilistic Reasoning 425
Regina D. Kiczek & Carolyn A. Maher

Refining a Framework on Middle School Students' Statistical Thinking 437
Edward S. Mooney, Cynthia W. Langrall, Pamela S. Hofbauer & Yolanda A. Johnson

Students' Reasoning About Sampling Distributions and Statistical Inference 449
Luis A. Saldanha & Patrick W. Thompson

Short Oral Reports

The Role of Probalisitic Simulations and Tree Diagrams in Determining and Nurturing Students' Primitive Mathematical Intuitions 455
Leslie Aspinwall & Kenneth L. Shaw

Problem Solving **457**

Research Reports

Students' Use and Understanding of Different Mathematical Representations of Tasks in Problem Solving Instruction 459
Fernando Barrera & Manuel Santos

The Process Involved in Formulation and Reformulation of Problems or Questions in the Learning of Mathematics 467
Juan Estrada

Students Connecting Mathematical Ideas: Possibilities in a Liberal Arts Mathematics Class 475
Barbara H. Glass

High School Curriculum Contents From an International Perspective 483
Lourdes Guerrero & Antonio Rivera

Problem-Solving in Integral Calculus: One Role of Metacognitive Thinking 491
Stephen J. Hegedus

Does the Acquisition of Mathematical Knowledge Make Students Better Problem Solvers? An Examination of Third Graders' Solutions of Division-With-Remainder (DWR) Problems 501
Yeping Li

A Comparative Study of the Selected Textbooks From China, Japan, Taiwan and the United States on the Teaching of Ratio and Proportion Concepts 509
Jane-Jane Lo, Jinfa Cai & Tad Watanabe

Transforming Students' Fragmented Knowledge Into Mathematical Resources to Solve Problems Through the Use of Dynamic Software 521
Manuel Santos-Trigo

Analysis of Students' Strategies When Solving Systems of Differential Equations in a Graphical Context 529
María Trigueros G.

Building Mathematical Meaning Without Teacher Intervention: Analysis of the Discourse of Precalculus Students 539
Janet G. Walter

A Developmental and Social Perspective on Problem Solving Strategies 549
Judith S. Zawojewski & Guadalupe Carmona

Short Oral Reports

College Students' Preferences for Prose and Tabular Representations When Constructing Functional Equations 561
Susan S. Gray

Building Meaning for Mathematical Ideas 563
Tania A. Myrtil & Alice Alston

Fostering Students' Mathematical Thinking Through the Use Qualitative Tasks 565
Armando Sepúlveda

Verbal Problems' Classification of Senior High School's Textbooks With Technological Industrial Education Orientation in Mexico 567
Jorge Villa & José Guzman

Posters

Multiple Embodiments in Model-Eliciting Activities 569
Edith Gummer & Hollis Duncan

Problem Solving in Mathematics: Barriers to Problem-Centered Learning in an Eighth Grade Classroom 571
Loan H. Nguyen & Amy Roth-McDuffie

Middle School Students' Sense Making of Solutions to Mathematical Word Problems 573
Clara Nosegbe

Plenary Papers

OPENING THE DIMENSIONS OF MATHEMATICAL CAPABILITY: THE DEVELOPMENT OF KNOWLEDGE, PRACTICE, AND IDENTITY IN MATHEMATICS CLASSROOMS

Jo Boaler
Stanford University

Introduction

Over the last ten years I have studied the learning opportunities provided to students in different mathematics classrooms, with different teaching approaches. The goal of these studies has been to understand the ways in which the different approaches have shaped students' knowledge of mathematics, and to begin to tease apart the complex relationships between teaching and learning, between knowledge and practice, and between learning and believing. This has provided me with the opportunity to learn about learning, as I have been fortunate enough to watch thousands of mathematics lessons, and analyze students' mathematical development as it has progressed over time. I have done this at the time of what some have described as a cognitive revolution (Schoenfeld, 1999; Resnick, 1993) as views of learning have radically shifted and changed. In this paper I will set out some of the changed perspectives on learning and 'knowledge transfer' that I have developed through my studies in England and California, describing a little of three different studies. I will document a path through my own learning about learning in order to trace an expansion of the dimensions that I have come to believe constitute the learning experience. In my presentation I will also show a video of some high school mathematics teaching in order to consider the complex dimensions of student capability that emerge in practice.

For many years educational theories have been based upon the assumption that knowledge is a relatively stable, individual characteristic that people develop and carry with them, transferring from place to place. Knowledge, in such theories, 'consists of coherent islands whose boundaries and internal structures exist, putatively, independently of individuals' (Lave, 1988, p43). Behaviorists, for example, proposed that the best way for people to learn mathematics would be to gain multiple opportunities to practice methods, thus re-enforcing certain behaviors (Greeno & MMAP, 1998). This view was based on an assumption that students learned what was taught, and that knowledge that was clearly communicated and received would be available for use in different situations. Constructivists offered a very different perspective, opposing the view that learners simply receive what is taught, proposing instead that students need to make sense of different ideas and actively organize them into their own cognitive schema, selecting, adapting and reorganizing knowledge as part of their own constructions (Lerman, 1996). Both of these perspectives on learning, as different as they are, represent knowledge as a characteristic of people that may be developed and then used in different situations.

Situated perspectives on learning offer a radically different interpretation, representing knowledge, not as an individual attribute, but as something that is distributed between people and activities and systems of their environment (Lave, 1988; Greeno & MMAP, 1998; Boaler, 2000; Cobb, 2000). This perspective emerged from recognition that people use knowledge differently in different situations and that knowledge, rather than being a stable, individual entity, is co-constructed by individuals and by other people with whom they are interacting and aspects of the situation in which they are working. Hence the term situated learning. One of the implications of this shift in the representation of knowledge has been a focus upon the practices and activities of learning. Cognitive structures are still considered (Greeno, 1997), but these are not abstracted out of their learning environments, they are examined as part of the broader system in which they emerge (Greeno & MMAP, 1998). The idea that knowledge is not the sole property of individuals has been viewed suspiciously by some scholars as we have a long history of viewing knowledge differently. But in this paper I would like to propose that these recent views of knowledge have profound practical implications for students' learning of mathematics, some of which I will explore by reviewing three different studies, considering the different dimensions of learning that each study served to highlight.

The Relationships between Knowledge and Practice

In England I conducted a three-year study of students learning mathematics in two schools (Boaler, 1997). The schools catered to similar populations of students, in terms of ethnicity, gender, social class and prior mathematical attainment, but they taught mathematics in totally different ways. One of the schools – Amber Hill – used a traditional approach to the teaching of mathematics, based upon teacher demonstration and student practice. The other school – Phoenix Park – required students to work on 2-3 week long, open-ended projects that the teachers had designed. The aim of my research was to conduct a detailed investigation into the relationship between teaching approach, student beliefs and student understanding in the two schools. I therefore monitored a cohort of students (approximately 300 in all) over a three-year period, from when they were 13 to when they were 16. A variety of qualitative and quantitative methods were employed, including approximately 100 one-hour lesson observations in each school; questionnaires given to 300 students each year; in-depth interviews with 4 teachers and 40 students from each school; and a range of open, closed and authentic assessments. I also conducted analyses of the students' responses to the national school leaving examination in mathematics. There were no significant differences in the mathematical attainment of the two cohorts of students when the study began.

One of the findings of that three-year study was that students' knowledge development in the two schools was *constituted* by the pedagogical practices in which they engaged. Thus it was shown that practices such as working through textbook exercises, in one school, or discussing and using mathematical ideas, in the other, were not merely vehicles for the development of more or less knowledge, they shaped the forms of knowledge produced. One outcome, was that the students at Amber Hill who had learned mathematics working through textbook exercises, performed well in similar textbook situations, but found it difficult using mathematics in open, applied or discussion-based situations. The students at Phoenix Park who had learned mathematics through open, group-based projects developed more flexible forms of knowledge that were useful in a range of different situations, including conceptual examination questions and authentic assessments. The students at Phoenix Park significantly outperformed the students at Amber Hill on the national examination, despite the fact that their mathematical attainment had been similar three years earlier, before the students at Phoenix Park embarked upon their open-ended approach. In addition, the national examination was unlike anything to which the Phoenix Park students were accustomed.

One of the indications of the differences in the students' learning at the two schools was shown by an analysis of their performance on the national examination. I had divided all the questions on the examination into two categories – conceptual and procedural (Hiebert, 1986), and then recorded the marks each student gained for each question. At Amber Hill the students gained significantly more marks on the procedural questions (which comprised two-thirds of the examination papers) than the conceptual questions. At Phoenix Park, there were no significant differences in the students' performance on the conceptual and procedural questions, even though the conceptual questions were, by their nature, often more difficult than the procedural questions. The Phoenix Park students also solved significantly more of the conceptual questions than the Amber Hill students.

The students at the two schools also developed very different beliefs about mathematics. I interviewed forty students from each school and talked with them about their mathematical beliefs, asking them whether they used mathematics in their day-to-day lives. All the students at both schools said that they did, some of them had part-time jobs outside of school that they described. When I asked the students whether the mathematics they used outside school was similar or different to that which they used inside school, the students at the two schools gave very different responses. All of the Amber Hill students said that it was completely different, and that they would never make use of any of the methods they used in school:

JB: When you use maths outside of school, does it feel like when you do maths in school or does it feel....

K: No, it's different.

S: No way, it's *totally* different. (Keith and Simon, Amber Hill, year 10, set 7)

R: Well when I'm out of school the maths from here is nothing to do with it to tell you the truth.

JB: What do you mean?

R: Well, it's nothing to do with this place, most of the things we've learned in school we would never use anywhere. (Richard, Amber Hill, year 10, set 2)

The students at Amber Hill seemed to have constructed boundaries around their knowledge (Siskin, 1994) and they believed that school mathematics was useful in only one place – the classroom. The students at Phoenix Park responded very differently and three-quarters of the students said that there were no differences between mathematics of school and the real world, and that in their jobs and lives they thought back to their school mathematics and made use of it:

JB: When you do something with maths in it outside of school does it feel like when you are doing maths in school or does it feel different?

G: No, I think I can connect back to what I done in class so I know what I'm doing.

JB: What do you think?

J: It just comes naturally, once you've learned it you don't forget. (Gavin and John, Phoenix Park, year 10, MC)

H: In books we only understand it as in the way how, what it's been set, like this is a fraction, so alright then.

L: But like Pope's theory I'll always remember – when we had to draw something, I'll always remember the projects we had to do.

H: Yeah they were helpful for things you would use later, the projects. (Linda and Helen, PP, year 10, MC)

D: I think back to here.

JB: Why do you think that?

A: I dunno, I just remember a lot of stuff from here, it's not because it wasn't long ago, it's just because .. it's just in my mind. (Danny & Alex, Phoenix Park, year 11, JC)

The students gave descriptions of the different and flexible ways they used mathematics that were supported by their positive performance in a range of different assessments (Boaler, 1998).The students' related these capabilities to the approach of the school:

L: Yeah when we did percentages and that, we worked them out as though we were out of school using them.

V: And most of the activities we did you could use.

L: Yeah, most of the activities you'd use, not the actual same things as the activities, but things you would use them in. (Lindsey and Vicki, PP, year 11, JC)

A: Well if you find a rule or a method you try and adapt it to other things; when we found this rule that worked with the circles we started to work out the percentages and then adapted it, so we just took it further and took different steps and tried to adapt it to new situations. (Ann, PP, year 10, JC)

One conclusion that may be drawn from that study, that would fit with cognitive interpretations of learning, would be that the students in the traditional school did not learn as much as the students who learned mathematics through open-ended projects, and they did not understand in as much depth, thus they did not perform as well in different situations. That interpretation is partly correct, but it lacks important subtleties in its representation of learning. A different analytical frame, that I found useful, was to recognize that the students learned a great deal in their traditional mathematics classrooms at Amber Hill – they learned to watch and faithfully reproduce procedures and they learned to follow different textbook cues that allowed them to be successful as they worked through their books. Problems occurred because such practices were not useful in situations outside the classroom. At Amber Hill, the students worked hard and completed a great many textbook exercises, but they often achieved this by learning to interpret a range of textbook cues that helped them select the right mathematical procedures at the right time. These cues included the following:

- When working through exercises, students expected to use the method they had just been taught on the board. If a question required the use of a different method, they would often get the answer wrong, or become confused and ask for help. This occurred even when students knew how to use the required method.
- When moving between exercises students expected to use a slightly harder method in the second exercise, so they would adapt their method slightly when changing exercises, whether or not this was necessary.
- The students always expected to use all of the numbers given to them in a question, or all of the lines present on a diagram. If students didn't use them all they thought they were doing something wrong and changed their methods to ones that could include all of the numbers or lines.

- If a question required some real world knowledge, or non-mathematical knowledge, students would stop and ask for help, even if they knew how to use the mathematics involved.

These different cues were important in many ways. First, they demonstrated an important classroom practice in which students engaged that impacted their performance in different situations. If students encountered textbook situations that departed from their expectations, they became confused, not because of the extent of their mathematics 'knowledge', but because of the regularities of the mathematics classroom to which they had become attuned (Greeno & MMAP, 1998). In assessments the students tried to use classroom cues to help them, but generally found that they were absent or different:

> T: You can get a trigger, when she says like *simultaneous equations* and *graphs, graphically*. When they say like - and you know, it pushes that trigger, tells you what to do.
>
> JB: What happens in the exam when you haven't got that?
>
> T: You panic. (Trevor, Year 11, Set 3)

In straight-forward tests of their mathematical methods and procedures, the students performed relatively well, but in more applied assessments the students tried to follow classroom cues, which invariably resulted in failure. The students' performance on such assessments could be taken as an indication of their knowledge—or lack of it—and nothing more, but a more accurate portrayal of their performance would address the *practices* of their mathematics learning, and the ways that these interacted with their knowledge. When the Amber Hill students encountered mathematics questions and problems they tried to engage in the practices they learned in the classroom but found that these were of limited use. This meant that they did not, or could not, use the mathematics they had learned. These classroom practices were also interesting because they emerged *in practice*, at the intersection of teachers, students, and curriculum materials (Cohen, Raudenbusch & Ball, 2000). They did not come from the textbook, although the books played a part in their production; they did not come from the teacher, although the teacher was also important, and the students did not arrive in class expecting or wanting to use cues. A more focused analysis of teaching *or* learning, describing the teacher, the students, or the curriculum materials, rather than the interactions between them (Brown & Borko, 1992) would not have revealed an important dimension of the students' mathematical capability. As I read more about situated theory and I learned more about the students in the two schools, I came to understand that the students at Amber Hill learned a great deal in their mathematics classrooms—they became extremely proficient in using subtle cues and they were generally successful in class, as long as the affordances that their knowledge relied upon were present:

> A: It's stupid really 'cause when you're in the lesson, when you're doing work - even when it's hard - you get the odd one or two wrong, but most of them you get right and you think well when I go into the exam I'm gonna get most of them right, 'cause you get all your chapters right. But you don't. (Alan, AH, year 11, set 3)

Problems occurred for the Amber Hill students because their classroom practices were highly specific to the mathematics classroom and when they were in different situations, even the national examination, they became confused, because they tried to follow the cues they had learned in the classroom and discovered that this practice did not help them:

> G: It's different, and like the way it's there like - not the same. It doesn't like tell you it, the story, the question; it's not the same as in the books, the way the teacher works it out. (Gary, Year 11, Set 4)

In other situations, such as their out-of-school jobs and everyday lives, the students engaged in communities that were sufficiently different for them to regard the mathematics they had learned within school as an irrelevance. One of the main conclusions I drew from that three year study was that knowledge and practices are intricately related and that studies of learning need to go beyond knowledge to consider the practices in which students engage and in which they need to engage in the future. There is a pervasive public view that different teaching pedagogies only influence the amount of mathematics knowledge students develop. If this were true, then it may make sense to teach all mathematics through demonstration and practice, as the Amber Hill teachers did, as that is probably the most 'efficient' way to impart knowledge. But students do not only learn knowledge in mathematics classrooms, they learn a set of practices and these come to define their knowledge. If they only ever reproduce standard methods that they have been shown, then most students will only learn that particular practice of procedure repetition, which has limited use outside the mathematics classroom. Thus, I concluded from that study that students at Phoenix Park were able to use mathematics in different situations because they understood the mathematical methods they met, but also because the practices in which they engaged in the mathematics classroom were present in different situations (Boaler,1999). In class they adapted and applied mathematical methods, and they discussed ideas and solutions with different people. When they used mathematics in the 'real world' they needed to engage in similar practices and they readily did so, drawing upon the mathematics they learned in school. At Amber Hill the students engaged in an esoteric set of practices that were not represented elsewhere and that reduced their ability and propensity to use the mathematics they had 'learned' in different situations. I moved from thinking about mathematical capability as a function only of knowledge to viewing it as a complex relationship between knowledge and practice. Figure 1 represents that shift:

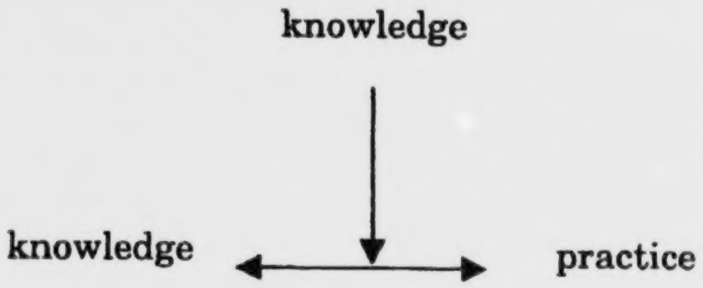


Figure 1.

The situated lens that I employed in that study opened two important avenues of exploration and understanding for me. First, it suggested a focus on classroom practices, pushing me to consider the relationship between students' knowledge production and the characteristics of their teaching and learning environments. Second, it helped me to understand that students did not learn *less* at Amber Hill, they learned *different* mathematics and that my understanding of the students' mathematical learning opportunities and capabilities at the two schools, needed to extend beyond knowledge to the practices in which students engaged in the classroom *and the relationship between the two*. The site of knowledge transfer had shifted, in my understanding, from students' minds to the mathematical practices in which they engaged. Thus the Phoenix Park students were more able to 'transfer' mathematics, not because their knowledge was secure and available for transport, but because they engaged in a set of practices in the classroom that were present elsewhere. As Greeno and MMAP (1998) have suggested:

> "*Learning*, in this situative view, is hypothesized to be becoming attuned to constraints and affordances of activity and becoming more centrally involved in the practices of a community (Lave & Wenger, 1991), and *transfer* is hypothesized to depend on attunement to constraints and affordances that are invariant or modifiable across transformations of a situation where learning occurred to another situation in which that learning can have an effect (Greeno, Smith & Moore, 1993)." (1998, p. 11)

This complex representation of learning as a relation between knowledge and practice seemed generative, but future studies of mathematics teaching and learning in which I engaged revealed a need to further expand my conceptions of learning to include a third dimension, that goes beyond knowledge and practice.

Relationships between Knowledge, Practice and Identity

In a recent study that Jim Greeno and I have written about (Boaler & Greeno, 2000), we were given further opportunities to investigate the nature of learning in different teaching environments. Megan Staples and I interviewed eight students from each of 6 Northern Californian high schools. The 48 students were all were attending advanced placement (AP) calculus classes. In that study four of the schools taught using traditional pedagogies similar to those at Amber Hill – the teachers demonstrated methods and procedures to students, who were expected to reproduce them in exercises. In the other two schools, students used the same calculus textbooks, but the teachers did not rely on demonstration and practice, they asked the students to discuss the different ideas they met, in groups. In that study we found that students in the more traditional classes were offered a particular form of participation in class that we related to Belenky, Clinchy, Goldberger, & Tarule's notion of 'received knowing' (1986, p4). Mathematics knowledge was *presented* to students and they were required to learn by attending carefully to both teachers' and textbook demonstrations. The mathematical authority in the classrooms was external to the students, resting with the teacher and the textbooks (Ball, 1993), and the students' knowledge was dependent upon these authoritative sources. In these classrooms it seemed that the students were required to receive and absorb knowledge from the teacher and textbook and they responded to this experience by positioning themselves as *received knowers* (Belencky et al., 1986). The students who were learning in these traditional classrooms were generally successful, but we found that many students experienced an important conflict between the practices in which they engaged, and their developing identities as people. Thus many of the students talked about their dislike of mathematics, and their plans to leave the subject as soon as they were able, not because of the cognitive demand, but because they did not want to be positioned as received knowers, engaging in practices that left no room for their own interpretation or agency. The students all talked about the kinds of person (Schwab, 1969) they wanted to be – people who used their own ideas, engaged in social interaction, and exercised their own freedom and thought, but they experienced a conflict between the identities that were taking form in the ebb and flow of their lives and the requirements of their AP calculus classrooms:

K: I'm just not interested in, just, you give me a formula, I'm supposed to memorize the answer, apply it and that's it.

Int: Does math have to be like that?

B: I've just kind of learned it that way. I don't know if there's any other way.

K: At the point I am right now, that's all I know. (Kristina & Betsy, Apple school)

The disaffected students we interviewed were being turned away from mathematics because of pedagogical practices that are unrelated to the nature of mathematics (Burton, 1999a, b). Most of the students who told us about their rejection of mathematics in the 4 didactic classrooms – 9 girls and 5 boys, all successful mathematics students – had decided to leave the discipline because they wanted to pursue subjects that offered opportunities for expression, interpretation and human agency. In contrast, those students who remained motivated and interested in the traditional classes were those who seemed happy to 'receive' knowledge and to be relinquished of the requirement to think deeply:

> J: I always like subjects where there is a definite right or wrong answer. That's why I'm not a very inclined or good English student. Because I don't really think about how or why something is the way it is. I just like math because it is or it isn't. (Jerry, Lemon school)

The students in didactic classes who liked mathematics did so because there were only right and wrong answers, and because they did not have to consider different ideas and methods. They did not need to think about 'how or why' mathematics worked and they seemed to appreciate the passive positions that they adopted in relation to the discipline. For the rest of the students in the traditional classes, such passive participation was not appealing and this interfered with their affiliation and their learning. In the other two calculus classes in which teachers engaged students in mathematical discussions, a completely different picture emerged. Such differences in students' perceptions were unexpected as we had gone into the study believing that all 6 classes were taught in similar ways. But we found that the students in the discussion-oriented classes had formed very different relationships with mathematics that did not conflict with the identities they were forming in the rest of their lives. The students in these classes regarded their role to be learning and understanding mathematical relationships, they did not perceive mathematics classes to be a ritual of procedure reproduction. This lack of conflict was important – it meant that the students who wanted to do more than receive knowledge, were able to form plans for themselves as continued mathematics learners. The following girl is just one of the students we interviewed in the discussion-oriented calculus classes that planned to major in mathematics:

> Sometimes you sit there and go 'it's fun!' I'm a very verbal person and I'll just ask a question and even if I sound like a total idiot and it's a stupid question I'm just not seeing it, but usually for me it clicks pretty easily and then I can go on and work on it. But at first sometimes you just sit there and ask – 'what is she teaching us?' 'what am I learning?' but then it clicks, there's this certain point when it just connects and you see the connection and you get it. (Veena, Orange school)

One of the interesting aspects of Veena's statement about mathematics class is her description of herself as a 'verbal person'. This was the reason that many of the students in the more traditional classes gave for rejecting mathematics. Indeed it seems worrying, but likely, that Veena may have rejected mathematics if she had been working in one of the four other schools in which the discussions and connections she valued were under-represented.

The type of participation that is required of students who study in discussion-oriented mathematics classrooms is very different from that required of students who learn through the reception and reproduction of standard methods. Students are asked to contribute to the judgment of validity, and to generate questions and ideas. The students we interviewed who worked in discussion based environments were not only required to contribute different aspects of their selves, they were required to contribute *more* of their selves. In this small study we found the notion of identity to be important. Students in the different schools were achieving at similar levels on tests but they were developing very different relationships with the knowledge they encountered. Those students who were only required to receive knowledge described their relationships with mathematics in passive terms and for many this made the discipline unattractive. Those who were required to contribute ideas and methods in class described their participation in active terms that were not inconsistent with the identities they were developing in the rest of their lives. Wenger's (1998) depiction of learning as a process of 'becoming' was consistent with the students' reported perceptions:

> Because learning transforms who we are and what we can do, it is an experience of identity. It is not just an accumulation of skills and information, but a process of becoming – to become a certain person or, conversely, to avoid becoming a certain person. Even the learning that we do entirely by ourselves contributes to making us into a specific kind of person. We accumulate skills and information, not in the abstract as ends in themselves, but in the service of an identity. (Wenger, 1998, p. 215)

This was a small study but it served to illuminate the importance of students' relationship with the discipline of mathematics that emerged through the pedagogical practices in which they engaged. This helped my understanding of learning to expand further to include the identities students were developing as learners and as people, as they engaged in different practices (see figure 2).

But these were early ideas and this representation seemed incomplete, as whilst identity seemed important, and clearly connected with pedagogical practices, I was unclear about the way this notion related to knowledge – the missing side of the tri-

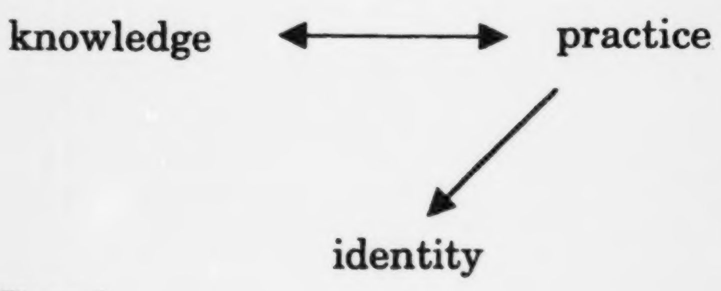


Figure 2.

angle in figure 2. The most recent study in which we are currently engaged, as well as the writing of Andrew Pickering, a sociologist of knowledge, has provided an important site for the continued exploration of these ideas, in particular for the investigation of relationships between knowledge and identity.

Developing Relationships with the Discipline of Mathematics

The final study that I will describe is a follow-up study to that which I conducted in England. We are in the first year or a four year study that will monitor the learning of over 1000 students as they go through three different high schools. Two of the schools offer a choice of mathematics curriculum, which they describe as 'traditional' and 'reform' oriented. This study is providing important opportunities to observe students' engagement in very different classroom practices and to examine the relationship between such practices and the knowledge and identities that students develop. In order to illustrate some of the relationships that we are finding interesting I will show a video of some fourth year high school teaching in a 'reform' curriculum called the Interactive Mathematics Program (IMP). In the lesson the students have been posed the problem: 'What is the maximum sized triangle that can be made with 2 sticks, of lengths 2 and 3 meters?' – the length of the 3rd side and all of the angles were to be determined by the students.

As we work to understand the capabilities that are being encouraged by these examples of classroom interaction we are again finding the notion of agency to be important. The students in this classroom, as in the Phoenix Park classrooms I studied, are given the opportunity to use and apply mathematics, a process which confers upon them considerable amounts of human agency. Students are required to propose 'theories', critique each other's ideas, suggest the direction of mathematical problem solving, ask questions, and 'author' some of the mathematical methods and directions in the classroom. One conclusion we could draw from these interactions would be that the students have *more* agency than those in more traditional classrooms, but whilst this may be true, such an observation feeds into debates about 'traditional' and 'reform' teaching methods in unfortunate ways. There is a common perception that students in 'reform' curriculum programs simply have more agency and more authority (Rosen, 2000), which often leads to fears that students are not learning enough, that they are left to wander in different, unproductive directions, and that they learn only "fuzzy" mathematics (Becker & Jacob, 2000). But we are finding that the *nature* of the agency in which students engage in these classrooms is related to the discipline of mathematics and the practices of mathematicians in important ways. Such insights have emanated from an analytic frame proposed by Andrew Pickering (1995). He has studied the work of professional mathematicians and concluded that their work requires them to engage in a 'dance of agency' (1995, p. 116). Pickering proposes that there are different types of agency and that conceptual advances require the interchange of human agency and the 'agency of the discipline' (1995, p. 116). Pickering considers some of the World's important mathematical advances and identifies the times at which mathematicians use their own agency – in creating initial thoughts and ideas, or by taking established ideas and extending them. He also describes the times when they need to surrender to the 'agency of the discipline', when they need to follow standard procedures of mathematical proof, for example, subjecting their ideas to widely agreed methods of verification. Pickering draws attention to an important interplay that takes place between human and disciplinary agency and refers to this as 'the dance of agency.'

In Pickering's terms, an advance in mathematics involves three processes, called *bridging, transcription,* and *filling.* Bridging involves a proposal for making some extension of a base model—that is, a set of accepted concepts and methods (establishing a "bridgehead"). Transcription involves transferring components of the base model analogically to the bridgehead—that is, attempting to treat the contents of the new topic with methods that are previously accepted. Filling involves providing additional definitions of terms in the new domain or modifying (preferably by generalization) methods from the base model. In the process of transcription, the mathematician performs procedures that he or she is not free to vary. It is in bridging and filling that the agency of mathematical work resides with the human mathematical thinkers. In Pickering's words:

> As I conceive them, bridging and filling are activities in which scientists display choice and discretion, the classic attributes of human agency. ... Bridging and filling are free moves, as I shall say. In contrast, transcription is where discipline asserts itself, where the disciplinary agency just discussed carries scientists along, where scientists become passive in the fact of their training and established procedures. Transcriptions, in this sense, are disciplined forced moves. Conceptual practice therefore has, in fact, the familiar form of a dance of agency, in which the partners are alternately the classic human agent and disciplinary agency. (Pickering, 1995, p. 116) (From Boaler & Greeno, 2000, p. 194).

Pickering's framework seems important for our analyses of the different practices of teaching and learning we observe. 'Traditional' classrooms are commonly associated with disciplinary agency, as students follow standard procedures of the discipline. 'Reform' classrooms, by contrast, are associated with student agency, with the idea that students use their own ideas and methods. We see something different in our observations of 'reform' classrooms. Rather than a group of students wandering unproductively, inventing methods as they go, we see a collective engaged in the 'dance of agency'. The students spend part of their time using standard methods and procedures and part of the time 'bridging' between different methods, and modifying standard ideas to fit new situations ('filling'). In the video we have seen there are examples both of students working at this interplay and of the teacher encouraging that work. In many of the traditional classrooms I have observed, in this and previous years, students have received few opportunities to engage in the 'dance of agency', and when they need to engage in that 'dance', in new and 'real world' situations, they are ill prepared to do so. Melissa Sommerfeld and I interviewed the entire class of IMP 4 students and found that the students all described an interesting relationship with mathematics that contrasted with the students working at similar levels of mathematics in traditional AP calculus classes. As part of the interviews we asked students what they do when they encounter new mathematical situations that require more than the reproduction of a standard formula. In the extracts below the students give their responses – some of them refer to the 'triangle' problem (as seen on the video clip), that they had recently met:

> K: I'd generally just stare at the problem. If I get stuck I just think about it really hard and then just start writing. Usually for everything I just start writing some sort of formula. And if that doesn't work I just adjust it, and keep on adjusting it until it works. And then I figure it out. (Keith)
>
> D: With this triangle thing there's different ways you can do it and there's different ways you can look at it. So it's sort of like, you have to..there's certain things you have to do. But you can do it in different ways or whatever way makes sense.
>
> J: Yeh you can really, like i was saying, you can look at it from different angles. Like with this triangle problem, you know, it's totally, it's kind of like a perspective thing. There's all sorts of ways you can use the math to get where you're going. (David and Jack)
>
> A: Well, first of all my group, we were remembering all the trigonometry that we've learned, but then we were like "Oh well we don't have right triangles to work with here" so we got stuck for a minute there, and then we just kind of came to our own conclusion that we would make it into right triangles and work with that. (Alissa)
>
> B: A lot of times we have to use what we've learned, like previous, and apply it to what we're doing right now, just to figure out what's going on It's never just, like, *given*. Like "use this formula to find this answer" You always have to like, change it around somehow a lot of the time. (Benny)

These students seem to be describing a 'dance of agency' as they move between the standard methods and procedures they know and the new situations to which they would apply them. They do not just talk about their own ideas, they talk about adapting and extending methods and the interchange between their own ideas and standard mathematical methods. The student below talks in similar terms as he reflects upon his decision to enter the IMP program:

> E: As far as the thought processes that you use in IMP are different from the standard parroting back of facts about algebra, I just really think that it's changed the way I think about a lot of things besides math that I really appreciate.
>
> Int: Like what?
>
> E: Like..if nothing else, it's breaking out of the pattern of just taking something that's given to you and accepting it and just going with it. It's just looking at it and you try and point yourself in a different angle and look at it and reinterpret it. It's like if you have this set of data that you need to look at and find an answer to, you know, if people just go at it one way straightforward you might hit a wall. But there might be a crack somewhere else that you can fit through and get into the meaty part.

Many of the students at Amber Hill frequently 'hit a wall' when they were given mathematics problems to solve. They would try and remember a standard procedure, often using the cues they had learned. If they could remember a method they would try it, but if it did not work, or if they could not see an obvious method to use, they would give up. The students we interviewed in the IMP4 classes described an important practice of their mathematics classroom – that of working at the interplay of their own and disciplinary agencies – that they used in different mathematical situations. Additionally the students seemed to have developed identities as mathematics learners who were willing to engage in the interplay of the two types of agency. The students had developed what we are regarding as a particular *relationship with the discipline of mathematics* that served to complete the triangle in figure 3.

A number of researchers have written about the importance of productive beliefs and dispositions (Schoenfeld, 1992; McLeod, 1992) but the idea of a 'disciplinary

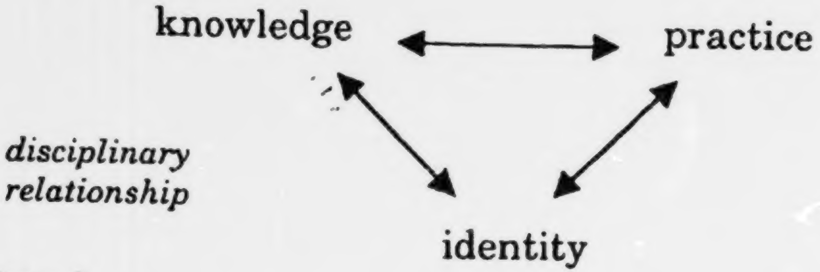


Figure 3.

relationship' serves to *connect* knowledge and belief in important ways. Herrenkohl and Wertsch (1999) have suggested a notion that addresses this connection, that they call the 'appropriation' of knowledge. They distinguish between mastery and appropriation, saying that too many analyses have focused only upon students' mastery of knowledge, overlooking the question of whether students 'appropriate' knowledge. They claim that students do not only need to develop the skills they need for critical thinking, they also need to develop a disposition to use these skills. In claiming that students need to 'appropriate' knowledge, they suggest a *connection* between the content students are learning and the ways they relate to that knowledge. The fact that the Phoenix Park students were able to use mathematics in different situations may reflect the similarity in the practices they met in different places, but it also reflects the fact that they had developed a positive, active relationship with mathematics. They expected to be able to make use of their knowledge in different situations and the identities they had developed as learners included an active relationship with mathematics. This relationship reflected their engagement in the dance of agency. Thus they were able to 'transfer' mathematics, partly because of their knowledge, partly because of the practices in which they engaged, and partly because they had developed an active and productive relationship with mathematics. This idea seems to pertain to theories of learning transfer and expertise in important ways, expanding notions of capability beyond knowledge and practice to the dispositions they produce and the relations between them.

Discussion and Conclusion

Debates about the effectiveness of different curriculum approaches generally focus upon knowledge – as researchers, parents, mathematicians and others consider which approaches enable students to understand the most. Few people have considered the practices of classrooms or the relationships that such practices afford with the disciplines students are learning. Different pedagogical practices are considered to be more or less effective or interesting, but they are rarely examined for the different opportunities they provide for affiliation and identification. In tracing a path through three of my recent studies I hope to have brought some useful analytic lenses to the question of knowledge transfer and mathematical capability. I have outlined a shift from a focus only upon knowledge, to one that attends to the inter-relationships of knowledge, practice, and identity. This seems to offer new perspectives on knowledge use and capability that fit with the behavior of experts. If we consider a mathematician at work, for example, she may be given a new problem to solve, but lack the knowledge needed to solve it. In such a situation it seems likely that she will still make progress, as she has learned a set of mathematical practices that she may use in trying to solve the problem (Ball & Bass, 2001) – practices such as representing the situation graphically, generalizing to different sets of numbers, or 'bridging' from a method she knows. She has also developed a productive relationship with the discipline of mathematics that means she will try different methods, garner helpful resources and make use of the knowledge and practices she has learned. She will work at the base of the triangle in figure 4 in the production of knowledge, yet many analyses of knowledge transfer fail to recognize these dimensions of capability.

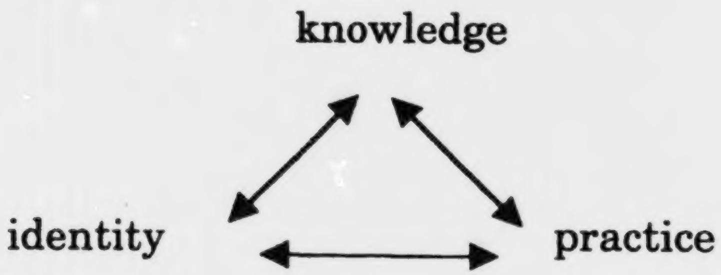


Figure 4.

The analysis I have offered in this paper leaves many unanswered questions – about the specificity of the relationships that pertain between knowledge and identity for example. Is it enough to develop a productive relationship with the discipline of mathematics or do learners need to appropriate *particular* knowledge, with relationships pertaining to specific domains of mathematics? Are all learners advantaged by opportunities to engage in a dance of agency, or do some learners advance through a more passive relationship with the discipline? How do identities of race, class and gender intersect with those of mathematics? These and other questions I will continue to consider, but this particular description of my learning trajectory ends with an idea about students' use of mathematics that goes beyond knowledge and practices to the inter-relations of knowledge, practices, and identities that emerge in different environments.

References

Ball, Deborah L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary mathematics. *The Elementary School Journal, 93* (4), 373-397.

Ball, Deborah L., & Bass, Hyman. (2001). Personal Communication.

Becker, J., & Jacob, B. (2000). California school mathematics politics: The anti-reform of 1997-1999. *Phi Delta Kappan, March*, 529-537.

Belenky, M. F., Clinchy, B. M., Goldberger, N. R., & Tarule, J. M. (1986). *Women's ways of knowing: The development of self, voice and mind*. New York: Basic Books Inc.

Boaler, J. (1997). *Experiencing school mathematics: Teaching styles, sex and setting.* Buckingham: Open University Press.

Boaler, J. (1998). Alternative approaches to teaching, learning and assessing mathematics. *Evaluation and Program Planning, 21*, 129-141.

Boaler, J. (1999). Participation, knowledge and beliefs: A community perspective on mathematics learning. *Educational Studies in Mathematics, 40*, 259-281.

Boaler, J. (2000). Exploring situated insights into research and learning. *Journal for Research in Mathematics Education.*

Boaler, J., & Greeno, J. (2000). Identity, agency and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 171-200). Westport, CT: Ablex Publishing.

Brown, C., & Borko, H. (1992). Becoming a mathematics teacher. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-371). New York: Macmillan.

Burton, L. (1999). Exploring and reporting upon the content and diversity of mathematicians' views and practices. *For the Learning of Mathematics, 19* (2), 36-38.

Burton, L. (1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics, 37*, 121-143.

Cobb, P. (2000). The importance of a situated view of learning to the design of research and instruction. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 45-82). Westport, CT: Ablex Publishing.

Cohen, D., Raudenbush, S., & Ball, D. L. (2000). *Instruction, resources, and research.* Seattle, WA: Center for Teaching Policy, University of Washington.

Greeno, J. (1997). On claims that answer the wrong questions. *Educational Researcher, 25* (1), 5-17.

Greeno, J., Smith, D., & Moore, J. (1993). Transfer of situated learning. In D. Detterman & R. Sternberg (Eds.), *Transfer on trial* (pp. 99-167). Norwood, NJ: Ablex Publishing Corporation.

Greeno, J. G., & MMAP. (1998). The situativity of knowing, learning and research. *American Psychologist, 53* (1), 5-26.

Herrenkohl, L. R., & Wertsch, J. (1999). The use of cultural tools: Mastery and appropriation. In I. Sigel (Ed.), *Development of mental representation* . Mahwah, NJ: Lawrence Erlbaum.

Hiebert, J. (1986). *Conceptual and procedural knowledge: The case of mathematics.* Mahwah, NJ: Erlbaum.

Lave, J. (1988). *Cognition in practice*. Cambridge: Cambridge University Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York: Cambridge University Press.

Lerman, S. (1996). Intersubjectivity in mathematics learning: A challenge to the radical constructivist paradigm? *Journal for Research in Mathematics Education, 27* (2), 133-150.

McLeod, D. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-371). New York: Macmillan.

Pickering, A. (1995). *The mangle of practice: Time, agency, and science.* Chicago: University of Chicago Press.

Resnick, L. B. (1993). Shared cognition: Thinking as social practice. In L. B. Resnick, J. M. Levine, & S. D. Teasley (Eds.), *Perspectives on socially shared cognition* (pp. 1-22). Washington, DC: American Psychological Association.

Rosen, L. (2000). *Calculating concerns. The politics of representation in California's "Math Wars".* , University of California San Diego.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-371). New York: Macmillan.

Schoenfeld, A. (1999). Looking toward the 21st century: Challenges of educational theory and practice. *Educational Researcher, 28* (7), 4-14.

Schwab, J. J. (1969). *College curriculum and student protest.* Chicago: University of Chicago Press.

Siskin, L., S. (1994). *Realms of knowledge: Academic departments in secondary schools*. London: Falmer Press.

Wenger, E. (1998). *Communities of practice: Learning, meaning and identity.* Cambridge: Cambridge University Press.

LEARNING MATHEMATICS AS DEVELOPING A DISCOURSE

Anna Sfard
The University of Haifa

1. Posing the question: What is it that changes when one learns mathematics?

In the field of mathematics education, the terms *discourse* and *communication* seem to be in everybody's mouth these days. They feature prominently in research papers, they can be heard in teacher preparation courses, and they appear time and again in variety of programmatic documents that purport to establish instructional policies (see e.g., *Principles and standards for school mathematics*, NCTM 2000). All this could be interpreted as showing merely that we became as aware as ever of the importance of mathematical conversation for the success of mathematical learning. In this talk, I will try to show that there is more to discourse than meets the ears, and that putting communication in the heart of mathematics education is likely to change not only the way we teach but also the way we think about learning and about what is being learned. Above all, I will be arguing that communication should be viewed not as a mere aid to thinking, but as almost tantamount to the thinking itself. The *communicational approach to cognition*, which is under scrutiny in this talk, is built around this basic theoretical principle.

To begin with, let us pay a brief visit to two classrooms where learning of a new mathematical topic has just started. The first class is just being introduced to the concept of negative number. The teacher takes her place in the front of the group of twelve-year old seventh graders and initiates the conversation[1].

Episode 1: The first lesson on negative numbers

[N1] Teacher: Have you ever heard about negative numbers? Like in temperatures, for example?

[N2] Omri: Minus!

[N3] Teacher: What is minus?

[N4] Roy: Below zero.

[N5] Teacher: *Temperature* below zero?

[N6] Sophie: Below zero… it can be minus five, minus seven… Any number.

[N7] Teacher: Where else have you seen positive and negative numbers?

[N8] Omri: In the bank.

[N9] Teacher: And do you remember the subject “Altitude”? What is *sea level*?

[N10] Yaron: Zero

[N11] Teacher: And above sea level? More than zero?

[N12] Yaron: From one meter up.

Since we are interested in learning, and learning means change, we may analyze this episode by trying to describe the modifications that have yet to occur in the children's ways of dealing with negative numbers. At first sight, this future learning is not just a matter of a change; rather, it requires creating something completely new. The children, although not entirely ignorant of negative numbers, can do little more at the moment than associate the topic with certain characteristic terms, such as *minus* or *below zero*. It seems, therefore, that they will have to work on the subject almost from scratch. To put it in the traditional language, we may say that the children have yet to *acquire the concept of negative number* or *to construct this concept* for themselves.

Rather than trying to fathom the operative meaning of these last words, let me now turn to another episode, in which two first graders, Shira and Eynat, begin learning some basic geometry. The girls are first shown a number of geometric figures and are asked by the teacher to mark those that can be called triangles. Once the task is completed, the following conversation between the girls and the teacher takes place[2]:

Episode 2: The first meeting about triangles

[T1] Eynat: [*Pointing to shape A*] This is a triangle but is also has other lines.

[T2] Teacher Well, Eynat, how do you know that triangle is indeed a triangle?

[T3] Eynat: Because it has three..aah…three… well.. lines.

……. …….

[T22] Teacher: [*Pointing to shape B*] This one also: one, two three..

[T23] The girls: Yes

[T24] Teacher: So, it *is* triangle? Why didn't you mark it in the beginning?

[T25] Eynat: 'Cause then… I did not exactly see it.. I wasn't sure [*While saying this, Eynat starts putting a circle also around shape C*]

……. ……….

[T28] Shira: [*Looking at shape C that Eynat is marking*] Hey, this is not a triangle. Triangle is wide and this one is thin.

[T29] Eynat: So what? [*but while saying this, she stops drawing the circle*]

[T30] Teacher: Why? Why this is not a triangle [points to shape B]? Shira said it is too thin. But haven't we said…

[T31] Eynat: There is no such thing as too thin. [*but while saying this, she erases the circle around shape C*]

[T32] Teacher: Triangle -- must it be of a certain size?

[T33] Shira: Hmmmm…Yes, a little bit… It must be wide. What's that? This is not like a triangle – this is a stick!

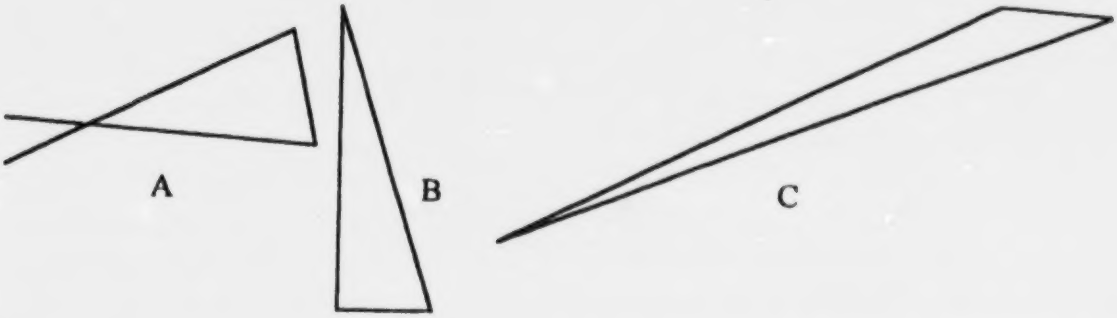


Figure1: Triangle or not triangle?

Here, unlike in the case of negative numbers, the students are already well acquainted with the mathematical objects in question, the triangles. And yet, neither the way they speak about these shapes nor the manner in which they act with them is fully satisfactory from the point of view of the teacher. In her search for triangles Shira disqualifies any shape that seems to her too thin. Eynat, even though aware of the formal definition of triangles (T3) and apparently convinced that "there is no such thing as too thin" (T31), still cannot decide whether the stick-like shape in the picture is a triangle or not: Originally, she has not marked shape C as triangle; now, following the conversation on shape B, she encircles this shape (T25), only to change her mind again (T29), and to erase the circle, eventually (T31). The teacher will be eager to induce some changes in the ways children think, speak and act with triangles. The development she is anxious to see is of a different kind than the one required in the case of negative numbers. Still, we can describe this new change in terms of concept acquisition and conceptual change, just as we did before: We can say that the children face the formidable task of *overcoming their misconceptions about triangles*.[3]

In this talk, I will reformulate this last statement after introducing a somewhat different way of talking about learning. My preference for the framework that will be called *communicational* stems mainly from the conviction that theories which conceptualize learning as personal acquisitions can tell us only so much about the complex phenomenon of learning. The acquisitionist approach relies heavily on the idea of cognitive invariants that cross cultural and situational borders. Consequently, the theories that come from this tradition are geared toward finding and investigating what remains constant when the situation changes. And yet, as has been convincingly argued by many scholars (e.g. Lave, 1988; Cole, 1996), human learning is too dynamic and too sensitive to ongoing social interactions to be fully captured in the terms of decontextualized mental schemes, built according to universal rules. In fact, my point of departure in this talk is that most of our *learning is nothing else than a special kind of social interaction aimed at modification of other social interactions*. Thus, rather than looking for those properties of the individual that can be held responsible for the constancy of this person's behavior, I am opting for a framework that allows me to stay tuned to the interactions from which the change arises. Let me add however, that my choice of the framework should not be interpreted as a rejection of the long-standing acquisition metaphor. The communicational approach should be regarded as a framework that has a potential to subsume this more traditional outlook, while modifying its hidden epistemological infrastructure.

2. Communicational approach to learning

Let me go back to the two episodes we have just seen and try to describe the required change, focusing on patterns in discourse rather than on schemes in students' minds[4]. While listening to the two brief conversations between the children and their teachers we had good reasons to wonder about the quality of the communication that was taking place. In the first scene, although it was obvious that the children were already familiar with the key term *negative number*, it was also clear that they could not say much about the topic of the exchange. It is significant that they answered teacher's questions with single- or double-word exclamations, such as "Minus!" or "Below zero", rather than in full sentences. We can say that at this point, the students could identify the discourse on negative numbers when they heard it, but they were not yet able to take an active part in it. In the second episode the situation, although different, still asked for a change. True, the children eagerly participated in the discourse on triangles; and yet, the way they did this was unlike that of their teacher.

It is important to note that while introducing children to new ways of communicating seems to be the teacher's principal goal, the work never starts from zero. Whether the discourse to be taught is on negative numbers or on triangles, it will be developed out of the discourses in which the children are already fluent. Thus, whatever the topic of learning, the teacher's task is to modify and exchange the existing discourse rather than to create a new one form scratch. If so, we can define learning as the *process of changing one's discursive ways in a certain well-defined manner*. More specifically, a person who learns about triangles or negative numbers alters and extends her discursive skills so as to become able to communicate on these topics with members of mathematical community. The new discourse may also be expected to make it possible to solve problems that could not be solved in the past.

At this point somebody may object and say that there is more to learning than modifying communication. Learning, the critic would say, is first and foremost about changing the ways we *think*, and the issue of how we communicate this thinking, although important, is still of only secondary significance. Let me then argue that thinking has not been excluded from my communicational account of learning. This point becomes immediately clear when we realize that the traditional split between thinking and communicating is untenable, and that *thinking is a special case of the activity of communicating.*[5] Indeed, a person who thinks can be seen as communicating with herself. This is true whether the thinking is in words, in images, or in any other symbols. Our thinking is clearly a dialogical endeavor, where we inform ourselves, we argue, we ask questions, and we wait for our own response. If so, becoming a participant in mathematical discourse is tantamount to learning to *think* in a mathematical way.

Let us now go back to our two classroom episodes and reformulate the initial query in communicational terms. Asking what the children have yet to learn is now equivalent to inquiring how students' way of communicating should change if they are to become skilful participants of mathematical discourse on triangles and negative numbers. Let me begin with the younger students. Clearly, Eynat and Shira have to modify their use of the keyword *triangle*. This seemingly superficial and rather marginal change is, in fact, quite profound and by no means easy to implement. Indeed, this change will not occur unless the students adopt new criteria for judging the appropriateness of the word use. So far, the children's decisions to call different shapes the same name *triangle* was based mainly (at least by Shira) or also (as in the case of Eynat) on the their perception of the overall visual similarity of these shapes. This is true whether the shapes were actually seen or just remembered. If Shira had difficulty identifying the thin shape in Figure 1 as a triangle, it is because it struck her with its similarity to a stick. According to the rules that govern the child's discourse at the moment, different names mean different shapes, and thus what is called stick cannot be called triangle as well. The mapping between things and names must be a single-valued function. This is the only possibility in the world where names are part and parcel of the shapes and like the shapes themselves, are externally given rather than being products of human decisions. All this will have to change in the process of learning. From now on, the students will have to seek the advice of verbal definitions and treat the latter as the exclusive basis for their decisions about what can count as "the same" or as different. These decisions will be mediated by language and will involve certain well-defined procedures. Among others, before the children decide about the name that should be given to a shape, they will have to scan this shape in a linear way, splitting it in separate parts and counting the elements thus obtained (note that counting is a verbal action without which the new kind of decision procedure, and thus the new type of sameness, would not be possible). This is a far- reaching change, one that affects the *meta-discursive rules* that regulate the ways discursive decisions are made.[6]

In the case of negative numbers, even more extensive changes are required. When it comes to words, it is not just a matter of modifying their use. The students will have to extend their vocabulary and to learn to operate with such new terms as "negative two" or "negative three and a half". Unlike in the case of triangles, where one can identify the object of talk with the help of pictures (e.g., triangles drawn on the paper), the students will now need new, specially designed visual means to mediate the communication. Some special symbols, such as -2 or -3.5, and geometric models, such as extended number line, will soon be introduced. Like in the case of triangles, a change will also be required at the meta-discursive level. I will elaborate on this last theme later.

For now, let me generalize these last observations. The analysis of the two episodes has shown that the children's present discourse differs from typical school discourse along at least three dimensions:

- Its *vocabulary*.
- The visual means with which the communication is mediated.
- The *meta-discursive rules* that navigate the flow of communication and tacitly tell the participants what kind of discursive moves would count as suitable for this particular discourse, and which would be deemed inappropriate.

Thus, if learning mathematics is conceptualized as a development of a mathematical discourse, to investigate learning means getting to know the ways in which children modify their discursive actions in these three respects. In the rest of this talk I will be analyzing the ways in which the required change can take place. While doing this I hope to show that adoption of the communicational approach to cognition is not an idle intellectual game, and that it influences both our understanding of what happens when children learn mathematics and our ideas about what should be done to help students in this endeavor.

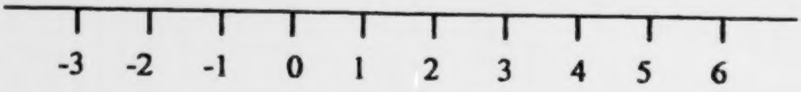
3. How do we create new uses of words and mediators?

According to a popular, commonsensical vision of the sequence of events that take place in the course of learning, the student must first have an idea of a new mathematical object, then give this idea a name and, eventually, he or she must also practice its use. This picture of learning may well be one that underlies the principle of "learning with understanding" that stresses the importance and primacy of conceptual understanding over formalization and skill (see e.g., Hiebert & Carpenter, 1992). The child is supposed to understand a mathematical idea, at least to some extent, before she starts using special mathematical names and symbols that "represent" it, and before she becomes proficient in these uses.

Conceptualization of learning as an introduction to a discourse leads me to doubt this popular model and makes the case for a different course of learning. Let us take the learning of negative numbers as an example. I will be arguing now that introduction of new names and new signifiers is the beginning rather than the end of the story. First, let me show the virtual impossibility of teaching a new discourse without actually speaking about its objects from the very first moment. Let us have a look at the way in which negative numbers are introduced in a school textbook (see Figure 2).

The crux of this definition is in the interesting conceptual twist: points on the number axis are marked with decimal numerals preceded by dash and, subsequently,

Let's choose a point on a straight line and name it "zero." Let's choose a segment and call it "the unit of length." Let's place the unit head-to-tail repeatedly on the line to the right of the point "zero." ***The points made this way will be denoted by 1, 2, 3 and so on ...***



To the left of the point "zero," we put the unit segment head-to-tail again and ***denote the points obtained in this way with numbers -1, -2, -3,... The set of numbers created in this way is called the set of negative numbers.***

Figure 2: From a school textbook (Mashler, 1976, *Algebra for 7th grade*), translated from Hebrew.

they are called *negative numbers.* One may wonder how this verbal acrobatics – giving *new names* to *points* and saying these are *numbers* – can enable the child an access to a discourse on the negatives. At the first sight, the learning sequence that begins with giving a new name to an old thing seems somehow implausible. And yet, such an order of things may be inevitable, and it may also be more effective than we tend to think.

It is *inevitable* because in order to initiate children to a discourse on new objects, one already has to use this discourse. The objects of the discourse must thus be identified in one way or another, in words or symbols. This is probably why the teacher in *Episode 1* cannot refrain from using words like "negative numbers", "minus two", etc. while introducing the topic for the first time. Clearly, she feels compelled to do it in spite of the fact that the children have little idea abut the uses to which these words can be put. The proposed order of things in the process of learning is also more *effective* than we tend to think because of the simple fact that the new objects – the negatives – have been associated, and introduced with, the word *number.* The familiar notion evokes in the student expectations with respect to the possible uses of the new signifiers, such as -1, or -2.5. The children know that some numerical operations are involved. They know that many rules that hold for numbers will now hold for the negatives. For better for worse, the children seem to know quite a lot about this something to which they might have been exposed through a single sentence. In the episodes that we are going to see next we will have a chance to see how they make use of their former discursive experience with numbers.

Let us return to our seven graders then, to see how well they are doing as the newcomers to the discourse on negative numbers. In the new classroom scene that follows, we can see how the expectations evoked by he word *number* help the students find their ways into the new discourse. Some of these ways are like those of the expert participants, and some have to be deemed mistaken. At the present stage, three weeks and sixteen one-hour meetings later, the children already know how to add signed numbers and are trying to figure out for themselves how to multiply a positive by negative. First, they do it in small groups. In one of these groups, the following exchange takes place after the teacher asked what 2 (-5) could be equal to:

Episode 3: The teacher asked what 2 (-5) could be equal to.

[N13] Sophie: Positive two times negative five…

[N14] Ada: Two times negative five..

[N15] Sophie: Aha, hold on… hold on… It's as if you said negative five multiplied two times…. So, negative five multiplied two times it's negative ten…

So far, so good. By projecting in metaphorical manner from their former discursive experience, the children discovered for themselves the rule which is, indeed, generally accepted. I will now show that this is not always the case. During the classroom discussion that took place after the work in pairs was completed, the following exchange took place in response to the same question as before:

Episode 4: In response to the question, "What 2 (-5) could be equal to?"

[N16] Roy: Negative ten.

[N17] Teacher: Why?

[N18] Roy: We simply did… two times negative five equals negative ten because five is the bigger number, and thus… uhmm… It's like two times five is ten, but [it's] negative ten because it is negative five.

……

[N42] Noah: And if it was the positive seven instead of positive two?

[N43] Yoash: Then there will be positive thirty five

[N44] Sophie: Why?

[N45] Yoash: Because the plus [the positive] is bigger.

On the first sight, Roy's idea may sound somehow surprising. On the a look, it is as justified as the one proposed by Sophie: like the girl before him, Roy draws on previously developed discursive habits, except that this time the choice does not fit with the one made along history by the mathematical community. Indeed, in the first case, the children substitute new numbers for old numbers: The negatives slide into the slot of the second multiplier, occupied so far exclusively by unsigned numbers. In the second case, the students substituted operation for operation: The *multiplication* of signed numbers was obtained from the multiplication of unsigned numbers more or less in the way in which the *addition* of signed numbers has been previously obtained from the addition of the unsigned (see a symbolic presentation of the templates in Fig. 3). As already noted, while the choice of the first group may be deemed successful because it happens to adhere to what counts as proper in the mathematical discourse, the choice of the other group fails to meet the standard. What is most important, however, from the researcher's point of view, is the similarity between the two cases rather than the difference: in both episodes we have seen students trying to incorporate the newly encountered negatives into the discourse on numbers, and in both episodes they did it by using old discursive templates for the new signifiers.

Successful try: **substitution into the discursive template**

$$a \cdot b = b + b$$

Unsuccessful try: **substitution into the discursive template**

$$(+a) + (-b) = \begin{cases} |a - b| & \text{if } a > b \\ -\,|a - b| & \text{if } a \leq b \end{cases}$$

in which *a* and *b* are "unsigned" and both + and – are substituted with ·

Figure 3. Recycling old discursive templates in the new context.

Let me now go beyond the present examples and speculate on two distinct stages of learning likely to follow the introduction of a new signifier. As we have just seen, the use of a new signifier is at first *template-driven*, that is, based on substituting new signifiers into old discursive templates. The mechanism of recycling old discursive habits, bearing a family resemblance to Lakoff's conceptual metaphor (Lakoff 1993), is at work here. This stage is characterized by a rather inflexible use of the new signifier and by treating symbols, such as -5, as things in themselves, not standing for anything else. If all goes well, and if there is no visible to reason to renounce the templates that have been chosen, both these features will eventually disappear thanks to what will be called here *objectification* of the discourse. The signifiers will eventually begin to be referred to as *representations* of another entities believed to have independent existence of sorts. This is when, for the student, -5 becomes but a representation of a *negative number*, with the latter conceived as intangible object that can also be represented in many other ways (e.g., as 15–20 or as a point on the number axis) and which exists independently of the human mind. At this stage, the use of the new signifiers becomes much more flexible. New number words and symbols get life of their own and they can now be incorporated into linguistic structures that were unheard of within the old, more restricted discourse. The new discourse makes it also possible to say much more in a much smaller number of words.

This two-phase process of discourse development, summarized in Fig. 4, is kept in motion by what I once described metaphorically as a mechanism of a pump (Sfard, 2000a): Introduction of a symbol is like lifting the piston in that it creates a new semantic space – a need for a new meaning, a new discursive habit. The gradual objectification of the discourse is analogous to the procedure of filling in the space thus created. It is through an intermittent creation of a space "hungry" for new objects and through its subsequent replenishment with new discursive forms and relations that the participants of mathematical discourse steadily expand its limits.

Stage one: Template-driven use of signifiers

Templates *come in clusters*
The new signifier -- *only within certain phrases.*
Only a *weak sense* of signified (object)

Stage two: Objectified use of symbols

A symbol becomes a representation
Other signifiers are used as equivalent
Flexibility and generality
Economy of expression
Experience (sense of) a signified (object)

Figure 4. "Pump mechanism": Two phases in extending a mathematical discourse (adding new objects).

Turning to old discursive habits may be the only way to deal with the somewhat paradoxical nature of mathematical learning. At a closer look, the process of objectification turns to be inherently circular: If mathematical objects, such as negative numbers, are discursive constructions, we have to talk about them in order to bring them into being. On the other hand, how can we talk about something that does not yet exist for us? Or, to put it differently, signifieds can only be built through discursive use of the signifiers, but at the same time, the existence of these signifieds is a prerequisite for the meaningful use of the signifiers. This circularity, although an infallible source of difficulty and a serious trap for the newcomers to a discourse, is in fact the driving force behind this discourse's incessant growth. This is what fuels the process of co-emergence in which the new discursive practices and the new signifieds spur each other's development. In this process, the discursive forms and meanings, as practiced and experienced by interlocutors, are like two legs which make moving forward possible due to the fact that at any given time one of them is ahead of the other.

To sum up, in the Episodes 3 and 4 we have seen students who, in their attempt to develop the discourse on negative numbers, have just entered the template-driven phase. The dilemma which of the two possible templates for multiplying positive number by negative has not yet been resolved. The teacher temporarily refrains from giving any advice and the children, left to themselves, have yet a long way to go. The situation with the girls who are learning the about triangles is quite different, and it is different in two crucial respects. First, the children's discourse on geometrical shapes, although quite unlike that of their teacher, is already objetified. Indeed, for the girls triangles are externally given objects and not an arbitrary construct the identity of which can be discursively decided. Second, quite unlike in the former example, it is now the teacher who will try to establish new uses of the word triangle and will attempt to change the rules that govern these uses. Thus, in this case, the teacher is an active initiator of new discursive habits.

4. How do we create new meta-discursive rules and turn them our own?

So far, we have been focusing on discursive changes that take place following an extension of vocabulary (e.g., introduction of number names, such as "negative one" or "negative ten"), an addition of new mediating means (such as new numerical symbols or extended number line) or an alteration of word use (Shira's teacher proposes to apply the word 'triangle' to a shape the girls are reluctant to call this name). We have been talking about two developmental phases in the use of new discursive means: the phase of template-driven use and the phase of objectified use. It is now important to recall that together with the alterations in use, another change, this time on the meta-discursive level, must take place: More often than not, the rules that govern interlocutors' discursive decisions will evolve as well. Such change must certainly happen in children's discourse on numbers if they are to be able to decide which of the two ways of multiplying positive by negative – the one offered by Sophie or the one designed by Roy should be accepted as the proper one.

Indeed, let us pause for a moment and ask ourselves what the children need in order to decide between the two possibilities. That the task is demanding is evidenced, among others, by the following testimony of the French writer Stendhal, who recalls the difficulties he experienced as a student when trying to find out the reasons for the related rule, "negative times negative is positive":

> I thought that mathematics ruled out all hypocrisy… Imagine how I felt when I realized that no one could explain to me why *minus times minus yields plus*.... That this difficulty was not explained to me was bad enough …. What was worse was that *it was explained to me by means of reasons that were obviously unclear to those who employed them.* (quoted in Hefendehl-Hebeker, 1991, p. 27).

It is noteworthy that Stendhal's complaint is about the nature of the justification he has heard rather than about the absence thereof. Let us try to figure out what this justification could be. Here is one possibility. Taking as a point of departure the request that the basic laws of numbers, as have been known so far, should not be violated, and assuming that the law "*plus times minus is minus*" and the rule $-(-x) = x$ have already been derived from these laws (Stendhal seemed to have had no problem with these!), the explainer may now argue that for any two positive numbers, a and b, the following must hold.

On the one hand,

$$(1)\quad 0 = 0 \cdot (-b) = [a + (-a)]\,(-b)$$

and on the other hand, because of the distributive law which is supposed to hold,

$$(2)\quad [a + (-a)]\,(-b) = a(-b) + (-a)(-b)$$

Since it was already agreed that $a(-b) = -ab$, we get from (1) and (2):

$$ab + (-a)(-b) = 0$$

From here, and from the law $-(-x) = x$, one now gets:

$$(-a)(-b) = -(-ab) = ab$$

One may say that it is the degree of formality of this argument that makes it unconvincing in the eyes of the student. Thus, let us try to imagine an alternative. A different, more effective kind of explanation could only come from everyday discourse. Indeed, secondary-school students' classroom conversations, not yet a case of a fully-fledged mathematical discourse, are typically a result of cross-breeding between everyday discourse and modern mathematical discourse. In the everyday discourse, claims about objects count as acceptable (*true*) if they seem necessary and inevitable, and if they are conceived as stating a property of a mind-independent 'external world'. This applies not only to material objects, but also to numbers, geometrical forms and all other mathematical entities to be implicated in colloquial uses. It is this "external reality" which is for us a touchstone of inevitability and certainty. In mathematics, like in everyday discourse, the student expects to be guided by something which can count as being beyond the discourse itself and existing independently of human decisions. This is what transpires from the words of the student by the name Dan who tried to explain to me his difficulty with the negatives:

Episode 5: Dan explains his difficulty with negative numbers

[1] Dan : *Minus is something that people invented.* I mean… we don't have anything in the environment to show it. I can't think about anything like that.

[2] Anna: Is everything that regards numbers invented by people?

[3] Dan: No, not everything…

[4] Anna: For instance?

[5] Dan: For example, the basic operation of addition, one plus one [is two] and *according to the logic of the world* this cannot be otherwise.

[6] Anna: And half plus one-third equals five sixths. Does it depend on us, humans or…

[7] Dan: Not on us. You can show it in the world.

[8] Anna: I see… and 5 minus 8 equals –3. It's us or not us?

[9] Dan: It's us.

[10] Anna: Why?

[11] Dan: Because in our world there is no example for such a thing.

Thus, the safest way for the student toward understanding and accepting the negative numbers and the operations on these numbers would be to make them a part of his or her everyday discourse. Alas, in the present case this does not seem possible. Although people usually can incorporate negative numbers into sentences that concern everyday matters, these discursive appearances are incomplete in that they rarely include operations on numbers and thus, in fact, refer to such entities as -2, or -10.5 as labels rather than fully-fledged numbers. This is evidenced by the results of my experiment in which eighteen students who have already learned about negative numbers were asked to construct sentences with the number -3, as well as questions the answers to which could be -2. In both cases, they were encouraged to look for utterances with "everyday content". As can be seen from the results presented in Fig. 5, not all the students were up to the task. The few "everyday uses" of negative numbers were made solely in the context of temperature, latitude and bank overdraft. In all these cases, the negative numbers were applied as labels rather than as measures of quantity.[7]

Sentence with –3:
- **The temperature went down to -3. (42%)**

Questions the answer to which may be -2:
- **Temperature went down 12 degrees from 10 degrees.**
  What is the temperature now? (42%)
- **How much money do you owe [*sic*!] to John? (25%)**

Figure 5. Examples of "everyday" utterances involving negative numbers given by students.

We are then left with the justification grounded in meta-discursive rule that underlied the formal argument presented above. According to this rule the only criterion for the extension of the existing mathematical discourse is the inner coherence of the resulting extended discourse. More specifically, all that is needed to accept a proposed use of a new type of number is its consistency with certain properties of the sets of numbers this definition is going to broaden. There is little chance that the young students for whom mathematical discourse is a description of the existing reality rather than the means for creating a new one will accept this new meta-rule, let alone reinvent it by themselves. (This, by the way, may well be the reason why teaching the negative numbers has been grounded for ages in the didactic principle epitomized in this unforgettable rhyme "Minus times minus is plus, the reason for this we need not discuss"; W.H. Auden, quoted in Kline, 1980).

To see how true it is that the students may have no grounds for preferring one old template over another, let us go back to the seven graders whom we left puzzling over the question what should be the result of multiplying a positive by negative. The debate went on for two full periods and the class got eventually convinced by Roi, who claimed that the sign should be like that of the multiplier with the bigger absolute value. The teacher seemed quite desperate.

Episode 6: Why choose one template rather than another?

[N46] Teacher: You keep repeating what Roi said last Monday, and I want to know *why* you think it must be so.

[N47] Yoaz: Because this is what Roi said.

[N48] Teacher: But Roi himself didn't explain why it is the magnitude that counts.

[N49] Roi: Because there must be a law, one rule or another

[N50] Teacher: There must be some rule, so it is better that we do it according to magnitude?

[N51] Leegal: The bigger is the one that counts

....

[N83] Teacher: Why [did you agree that six times is twelve]? Six times negative two is negative twelve – is this too complicated?

[N84] Roi: But I am more charismatic… I managed to influence them all.

The picture we get from here is as follows: The children know that if they deal with numbers, there must be rules; and yet, they have no idea where these rules should come from. Their helplessness finds it expression in Roi's humorous declaration that the personality of the inventor of a definition is as good a reason for the acceptance of this definition as any other (after all, don't we repeat, time and again, that mathematics is a human creation?!). If you think about it, whatever change in meta-rules is to occur, it can only be initiated by the teacher. Indeed, unlike the object-level rules of mathematical discourse, the meta-level rules cannot be shown to be necessary or inevitable. As eloquently argued by Wittgenstein (1953; see also Sfard 2000b), in a certain deep sense they are but conventions. These conventions have their historical reasons, but the historical reasons are not today's students' reasons. The possibility that the students would re-construct these rules for themselves is thus highly implausible. Children can only arrive at these rules by interacting with an expert participant, at least part of the time.

This last didactic suggestion sounds quite straightforward. And yet, if we now go back to the first graders who are learning about triangles, we will see that children do not easily accept changes in meta-discursive rules even if the initiative comes from a very determined teacher.

Episode 7: Trying to convince Shira that shape C is a triangle

[T35] Teacher: But you told me… Hold on, you told me that triangle… well…. Eynat, you told me, and Shira agreed, that that in triangle there must be three lines, right?

[T36] Eynat: Right.

[T37] Teacher: So, come on, tell me how many lines do we have here? [*in the shape presented in Fig. 2*]

[T38] Shira: One, two, three….

[T39] Teacher: So, maybe this *is* a triangle? So, Eynat erased [the circle] for no reason? You are not sure. About this one you said it is triangle [*shows another, more "canonic" triangle*].

[T40] Shira: Because it is wide and it fits to be a triangle. It is not thin like a stick [*illustrates with hand movements and laughs*]

[T41] Teacher: How do we know that a triangle… whether a shape is triangle? What did we say? What did we say? To say that shape fits to be triangle, what do we need?

[T42] Shira: Three points… three vertices… and…

[T43] Teacher: Three vertices and…?

[T44] Shira: Three sides.

[T45] Teacher: And three sides. Good. If so, this triangle [*!*] fits. Look, one side… and here I have one long side, and here I have another long side. So, here we have a triangle.

[T46] Shira: And one vertex, and a second vertex, and a… point?!

[T47] Teacher: Look here: one vertex, second vertex, third vertex

[T48] Shira: So it *is* a triangle?

In this episode, the long debate on the status of the stick-like shape reaches its climax. To show the difficulty of the required discursive change let me analyze the brief conversation while trying to answer a number of questions, as specified below.

1. How do the meta-rules of the children's discourse still have to change? A partial answer to this question has already been given in brief when I was analyzing *Episode 2*. Let me repeat and elaborate. The meta-rule that has to change is the one that regulates children's activity of giving names to geometrical shapes. At present, Eynat and Shira perform the naming task unreflectively, on the basis of their previous visual experiences. They recognize triangles and squares the way they recognize people's faces, that is in an intuitive way and without giving the reasons for their choices. From now on, they will be requested to communicate to others not only their decisions but also the way these decisions were made. This can only be done in words. The introduction of language imposes a linearity: Rather than satisfying themselves with the holistic visual impression which cannot be communicated to another person, the girls will have to tell their interlocutors how a shape should be scanned before the decision regarding its name is made. The scanning procedures (which I have once called an *attended focus*; see Sfard 2000c) are mediated by, and documented in, language. In fact, they are only possible as a part of verbal communication. When we check whether a shape is a triangle, we have to count its sides. The counting is a linguistic act and the result of counting is a word (*three*, in the case of triangles). The new way of making decisions about the names of geometric figures will thus be done *by analyzing words associated with the shapes* and without any reference to the overall visual impressions that were the basis of the naming procedures so far.

This new meta-discursive rule entails a change of yet another meta-discursive principle. So far, giving names has been an act of splitting the world into *disjoint* sets of objects. I highlighted the word *disjoint* in this last sentence to stress that within the initial geometric discourse, different names mean different objects. This fact is of crucial importance, as it entails the meta-rule according to which one cannot call a shape both *triangle* and *stick*, or both *square* and *rectangle*. This will have to change once the naming decisions are based on the results of detailed scanning procedures of varying complexity rather than on holistic visual appreciation. Indeed, scanning procedures can be ordered according to the relation of inclusion (for example, the procedure for identifying rectangles is, quite literally, a part of the procedure for identifying squares), and thus may be hierarchically organized. The hierarchical organization of the scanning procedures becomes, in turn, a basis for hierarchical categorization of geometrical shapes.

How does the teacher try to induce this change? The transition from the old to new meta-discursive rules must clearly take place before Eynat and Shira become fully convinced that stick-like shape is a triangle. Impatient to see the transition happening, the teacher repeatedly reminds the criterion which should be used in deciding: First she says that "in triangle there must be three lines" (T35), then she asks "To say that shape fits to be triangle, what do we need?" (T41), and finally eagerly confirms Shira's answer "three vertices and three sides" (T42). Although none of the teacher's formulations explicitly indicates the fact that having three sides is a sufficient condition for a shape to be a triangle, the sufficiency is there, signaled by the teacher with non-direct means. Over and over again the teacher initiates scanning the shapes and counting their elements. Invariably, the words "one, two, three" are followed with the telling "So.." (see T24, T39, T45), and, eventually, with the statement asserting that the shape is triangle. The word "so" is very effective in suggesting that whatever comes next is an inevitable entailment of the "one, two three" sequence.

How successful is the teachers' effort? On the face of it, the teacher's method of discursive alignment works: Shira soon learns to complete the procedure of counting to three with the words "So, this is a triangle" (children are incredibly dexterous at detecting and picking up discursive patterns!). In T48 she hurries to state this conclusion on her own accord, clearly aware of the rules of the game set by the teacher. And yet, the fact that she utters this conclusion as a question rather than as a firm assertion signals that she may be declaring a surrender rather than a true conviction. The girl knows *what* she is expected to say, but she does not know *why*. The claim that shape C is a triangle contradicts the meta-discursive rule according to which she has been making her naming decisions so far. What the teacher considers to be a necessary and sufficient condition for "triangleness", for the girl is a necessary condition, at best. The lack of certainty can be felt also in Eynat's contributions, in spite of her evident awareness of the meta-discursive rules that elude her partner (see T3 and T31). The ultimate evidence for the fact that old meta-discursive habits die hard will come some time later, when the children are asked to distinguish between rectangles and other polygons. Both girls will then adamantly reject the teacher's suggestion that a square can also be called rectangle and they will stick to their version for a long time in spite of the teacher's insistence.

One could conjecture that in the case we have been analyzing, the slowness of learning resulted not so much from the stubbornness of the old discursive habits as from the ineffectiveness of the teaching method. Moreover, since this method was based on demonstrating the application of the new meta-rules rather than on arguing for them explicitly, some people may criticize the teacher for violating the principle of learning with understanding. This is thus the proper place to remind ourselves that unlike the object-level rules of mathematics, each of which is logically connected to all the others, the meta-rules are not dictated by the logical necessity. In consequence, one cannot justify them in a truly convincing, rational way. The children, if they wish to communicate with others, will have to accept these rules just because they regulate the game played by more experienced players.[8] They will have to become participants of the new discourse before they can fully appreciate its advantages.

5. Final remarks: How does all this affect practice?

In this talk I proposed to think about learning mathematics as developing a special type of discourse. It is now time to demonstrate that suggesting this conceptual shift was not a mere intellectual exercise. In this last section I wish to argue that the change of perspective is bound to affect some of our beliefs on teaching mathematics.

Let me begin with the nowadays popular principle according to which, whatever learning we are trying to induce, we should keep it *meaningful* all along the way. The slogan *learning with understanding* stresses the importance and primacy of conceptual understanding over formalization and skill. As I mentioned several times, the conceptualization of learning as an introduction to a discourse leads me to question this insistence on sustained understanding. From what has been said so far, I would rather conclude that one can make sense of mathematical discourse only through a persistent participation, and not prior to it. In fact, too great a stress on understanding may eventually become counterproductive, as it is likely to undermine students' willingness to engage in mathematical discourse at times of insufficient understanding. Let me thus reiterate the advice given by Cardan more than half a millenium ago to those who criticized him for his use of the "imaginary numbers": Whoever wishes to become fully fluent in mathematical communication has to persist in practicing mathematical discourse, and must do it "putting aside the mental tortures involved", if necessary.

Since, indeed, objectified mathematical discourse can only arise in the delicate dialectic between engaging in the mathematical communication and trying to understand, another popular pedagogical view must be questioned. Because of the existence of calculators and computers, some writers have been insisting that the student may be exempted from trying to attain procedural efficiency (see e.g., Devlin 1997). Some educators would go so far as to say that formalism and skills may be completely removed from school curricula. And yet, as I was arguing along these pages, it is a mistake to think of symbolization as a matter of finishing touches – of giving new "expression" to the "old thought". Rather, special symbolic mediators are necessary to generate the mathematical communication in the first place.

According to yet another popular claim, the best way to assure effective learning is to keep mathematics embedded in real-life context. In discursive terms, this means that school mathematical discourse should always remain a part of everyday discourse. While discussing the learning of negative numbers I have just shown how unrealistic this goal is. Besides, if learning mathematics means an initiation to a special type of discourse, staying within the confines of everyday discourse would contradict our aim!

Perhaps the most widely accepted assumption about learning is that the student is the builder of his or her own knowledge. This Piagetian claim is often misinterpreted as saying that children should construct their knowledge more or less on their own, in the course of collaborative problem solving. In discursive terms, this would mean that the students are expected to develop mathematical discourse while interacting with each other. Our data, followed by the discussion on the notion of meta-discursive rule and on the ways in which these rules evolve have shown the untenability of this belief as well.

To sum up, the communicational vision of learning implies that while teaching mathematics, we should keep in mind the inherent difficulty of the endeavor, stemming from the endemic circularity of the learning process and from the fact that at least some of the meta-rules of mathematical discourse are not logically inevitable. Neither of the resulting dilemmas can be overcome by a purely rational effort. Coming to terms with negative numbers or with meta-discursive rules that change what seems to be a law of nature requires time and patience. This may be why the mathematician von Neumann has been quoted saying to a journalist "One does not understand mathematics, young man, one just gets used to it." Naturally, this statement should not be taken as a serious, or sufficient, basis for a pedagogical advice. And yet, the communicational approach proposed in this talk does show the importance of discursive habits and the impossibility of developing them on a purely rational basis. Consequently, this approach brings with it a number of practical suggestions, some of which appear quite different from what is being practiced in schools these days.

References

Cole, M. (1996). *Cultural psychology.* Cambridge, MA: The Belknap Press of Harvard University Press.

Devlin, K. (1997). Editorial: Reduce skills teaching in math class. *Focus, 17(6),* pp. 2-3.

Edwards, D. (1997). *Discourse and cognition.* London: Sage.

Harre, R., & Gillet, G. (1995). *The discursive mind.* Thousand Oaks, CA: SAGE.

Hefendehl-Hebeker, L. (1991). Negative numbers: Obstacles in their evolution from intuitive to intellectual constructs. *For the Learning of Mathematics, 11*(1), 26-32.

Hiebert, J. & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *The handbook of research on mathematics teaching and learning* (pp. 65-100). New York: Macmillan.

Kline, M. (1980). *Mathematics: The loss of certainty.* Oxford: Oxford University Press.

Lakoff, G. (1993). The contemporary theory of metaphor. In Ortony, A. (Ed.), *Metaphor and thought, 2nd edition* . Cambridge: Cambridge University Press. pp. 202-250.

Lave, J. (1988). *Cognition in practice*. Cambridge: Cambridge University Press.

NCTM (National Council of Teachers of Mathematics) (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Sfard, A. (2000a). Symbolizing mathematical reality into being: How mathematical discourse and mathematical objects create each other. In P. Cobb, K. E. Yackel, & K. McClain (Eds), *Symbolizing and communicating: perspectives on Mathematical discourse, tools, and instructional design* (pp. 37-98). Mahwah, NJ: Erlbaum.

Sfard, A. (2000b). On reform movement and the limits of mathematical discourse. *Mathematical Thinking and Learning, 2* (3), 157-189.

Sfard, A. (2000c). Steering (dis)course between metaphor and rigor: Using focal analysis to investigate the emergence of mathematical objects. *Journal for Research in Mathematics Education, 31(3)*, 296-327.

Van Hiele, P. M. (1985). The child's thought and geometry. In D. Fuys, D. Geddes, & R. Tischler (Eds.). *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. van Hiele* (pp. 243-252). Brooklyn, NY: Brooklyn College, School of Education.

Vygotsky, L. S. (1987). Thinking and speech. In R. W. Rieber, & A. C. Carton (Eds.), *The collected works of L. S. Vygotsky* (Vol. 1, pp. 39-285). New York: Plenum Press.

Wittgenstein, L. (1953). *Philosophical investigations* (G. E. M. Anscombe, Trans.). Oxford, UK: Blackwell.

Notes

[1] These data are taken from the study conducted with Sharon Avgil. This and all subsequent segments of transcripts have been translated from Hebrew by the author.

[2] These data are taken from a study conducted with Orit Shalit-Admoni and Pnina Shavit.

[3] Alternatively, in this latter case we may say, inspired by Vygotsky (1987), that the teacher tries to help children in making the transition from *spontaneous* to *scientific* concept of triangle.

[4] Let me stress once again: a statement like this should not be read as a denial of the existence of mental structures. Rather, it is a methodological claim; it is a declaration that, as a researcher, I feel on a firmer ground when I can base my arguments on observable aspects of the phenomena under inquiry.

[5] The communicational approach presented in this talk is similar to, although not identical with, the *discursive psychology* promoted, among others, by Harre & Gillett (1994) and by Edwards (1997).

[6] Note that my present description of the required change is quite similar – one can

say isomorphic—to one that could be given based on the van Hiele theory of the development of geometrical thinking (van Hiele 1985). Still, the two descriptions are put apart by their different epistemological/ontological underpinnings: While van Hiele's analysis, firmly rooted in the Piagetean framework, would produce a story of mental schemes, the present description is the description of students' ways of communicating. What makes the latter version qualitatively different from the former is that it presents the development of child's geometrical thinking as part and parcel of the development of her communicational skills, and thus makes salient the principal role of language, of contextual factors and of social interaction.

[7] It is also noteworthy that many of the "everyday" questions to which the answer was supposed to be -2 suffered from out-of-focus syndrome; that is, although the negative quantity was somehow involved in the situation presented in the question, the actual answer to the question should be 2 rather than -2 (see the last example in Figure 5).

[8] This means that new meta-rules can only be dictated by the teacher. Let me immediately add that the kind of dictation we are talking about is not necessarily an imposition. As long as all the parties involved are willing to play the game, the introduction of its rules by an experienced player cannot be considered as violating anybody's freedom. More often than not, this is the case in the classroom situation: The teacher is willing to teach, whereas the children, even if they do not seem too eager, are willing to learn. Indeed, both sides are keen on having an effective communication. It is also tacitly agreed by all parties involved that children should adjust their discursive ways to those of the teacher, and not the other way round. After all, if the children were not ready to follow the discursive lead of the grownups, they would never become able to communicate with other people. So, it is only understandable that the children change their discursive ways by reading meta-discursive hints, by guessing what is proper and by imitating patterns appearing in the discursive actions of other interlocutors. All this is done for no other reason than the wish to improve communication by aligning themselves with their more experienced partners.

Working Groups

GENDER AND MATHEMATICS: THEMES WITHIN REFLECTIVE VOICES

Diana B. Erchick
The Ohio State University at Newark
Erchick.1@osu.edu

Linda Condron
The Ohio State University at Marion
Linda.Condron @osu.edu

At each PME-NA since the 1998 PME-NA conference in Raleigh, NC, the Gender and Mathematics Working Group has defined, explored and evaluated the needs of the field comprised by our work. We have explored absences, represented the complexity, determined needs, and identified the direction of the research agendas of the scholars participating in our group as well as those in the gender and mathematics education community at large. In reflecting back across the Gender and Mathematics Working Group's foci over each the past 3 years, we find a natural progression in the emphasis of each year's work, a progression that has brought us to our current efforts and what

We began in Raleigh at PME-NA XX in 1998 with "Gender and Mathematics: Integrating Research Strands." Here we explored the absences in the research and defined both components and the complexity of their connections at the intersections of "the doing of mathematics" and what Suzanne Damarin identified as a "sex-gender system" (Damarin, 1998 PME).

In Cuernavaca in 1999, The Gender and Mathematics Working Group of PME-NA XXI took a closer look at the work being done by the participants. From this closer look we set goals to keep our scholarship visible in the mathematics education community and work toward integrating our research findings into mathematics education and mathematics teacher education. Toward that end, we generated the following suggestions for how we might accomplish our goals: 1) continue our original working group goal of producing a monograph on gender and mathematics for the *Journal for Research in Mathematics Education*; and 2) support, conduct and supervise research that contributes to an understanding of the ways in which mathematics educators and mathematics teacher educators can integrate gender research findings into the classroom.

Finally, at PME-NA XXII in Tucson in 2000, we reviewed and organized topics that emerged from our work to date into categories for the call for papers for the proposed monograph for *JRME*. Those topics included but were not limited to the following: a) The development of epistemological voice; b) The integration of gender research into the mathematics classroom, K-16; c) The integration of gender research into the mathematics education classroom; d) The role of the content in addressing gender issues in mathematics education research; e) Mathematical success in fast-track and other programs for girls in mathematics; and f) Mathematical success for women in mathematics and math-using fields. We then developed a working thematic structure for the monograph we are committed to produce. The thematic structure is one where "Reflective Voices" is the concept guiding the structure. We intend to organize work around multiple perspectives that include researcher, historical, teacher, student and critical perspectives. These perspectives include feminist, methodological, self-reflective and empirical standpoints; applications include the mathematics classroom, the mathematics education classroom and research and publication environments. Below is the working outline for our monograph.

Monograph Working Outline/Organization

A. Working Title – Research, Reflections, and Revelations on Gender and Mathematics: Multiple Perspectives (Standpoints)
B. Introduction: Vision of past/ discussion of present presence?
C. Content Theme: Reflective Voices
    1. Author/researcher perspectives: scholarly voices
        a. Feminist theoretical perspectives
        b. Research reports (empirical perspectives)
        c. Self-reflective perspectives
        d. Methodological perspectives
    2. Historical standpoints: Voices from the past
        a. Voices of feminist influence
    3. Teacher standpoints
    4. Children and students' standpoints
        a. successful women and girls
        b. unsuccessful women and girls
        c. integrating gender research into the mathematics classroom
        d. integrating gender research into the mathematics education classroom.
    5. Critical perspectives on gender
    6. (Non)marginalization of the Gender and Mathematics research agenda
        a. gate-keeping within the professional
        b. editorial restrictions
        c. celebration of positionality
    7. The gendered nature of mathematics (voice of mathematics)
D. Closing: Future Directions

Participants in this the 2001 PME-NA XXIII Gender and Mathematics Working Group in Snowbird bring to these sessions work in preparation for the monograph. Their work now makes real for us the full potential of the monograph. For instance, consider the contribution of Laurie E. Hart and Martha Allexsaht-Snider "Strategies for Achieving Equity in Mathematics Education." These scholars have chosen to build upon prior work on equity in mathematics education in K-12 education that focussed on race, ethnicity and socio-economic status (SES). Hart and Allexsaht-Snider now integrate a discussion of gender into their work. As they describe it:

> We analyze and critique the recommendations made in mathematics education standards documents and synthesize research about strategies for accomplishing equity in mathematics education. Structural aspects of school districts, beliefs about students and the learning of mathematics and classroom processes, including teaching practices, appear to be important.

These scholars bring to the discussion both the integration of gender discussions into work focused on other social constructs as well as critical perspectives on mathematics education standards documents.

Jae-Hoon Lim, in her paper, "Sociocultural Context of Young Adolescent Girls' Motivation for Learning School Mathematics," focuses on the sociocultural context of young adolescent girls' motivation for learning school mathematics. Hers is an ethnographic case study with two ability-based 6th grade mathematics classes in a rural middle school. Lim explores the sociocultural context of young adolescent girls' motivation for learning school mathematics and focuses on the impact of various sociocultural factors upon the girls' experiences with and thoughts about school mathematics. This researcher critically examines the impact of instructional and organizational practices in the schooling system. She also examines the impact of the system's implicit as well as explicit ideology upon the girls' experiences with school mathematics. The students' standpoints place this work within the study of both successful and unsuccessful girls in terms of their relationships with mathematics.

Another researcher, Dawn Leigh Anderson, contributes work that explores the experiences of accomplished women mathematicians. Her paper, "Narratives of Women who have been Successful in Mathematics" reports on a study with six women mathematicians and describes factors and experiences that led each to become successful in mathematics. As we find is true for much of the recent work emerging from the field, Anderson's work emphasizes voice and listening to and interpreting participants' voices. Thus, the experiences these women had and have with mathematics are described in terms of their understanding of their own experiences. Their narratives include talk of personal background, self-identity, relationships, professional development and career, and the role that gender plays in their lives. In this work are some consistencies: these women were encouraged and supported by parents and teachers; they identified that support as one of the reasons for their success; and they had lives that did not revolve solely around their careers. However, evidence also suggests some inconsistencies across the participants: some held strong views about themselves while others felt insecure and inadequate; and the role gender played in their success was perceived differently by all. In all, we find in such work as this perspectives from those women who are *successful*, those who can tell us, in the reflective manner that the gender and mathematics working group finds so valuable, what, from their perspective, has supported the growth of a strong relationship to mathematics.

Linda Condron has also studied women who are successful with mathematics, Her work is entitled, "What We Learn From Women In Mathematics-Using Fields."

She notes that half of the women computer professionals working in engineering environments she interviewed had not started out in technical careers at all. Rather, they changed careers in their 20s or 30s after majoring/working in other areas such as psychology, education, clerical work, or even the post office. Many had stories of having disliked mathematics or having been discouraged from studying mathematics early on, only to return to it later and find great success. These women defy the common notion that it is important to future success to get into the "mathematics pipeline" early on and stay in it through high school and college to the life long technical career. The problem with the pipeline metaphor is that there is only one way *into* the pipeline while there are multiple ways *out of* the pipeline. Her research makes it clear that women are able to find multiple paths into mathematics-using careers, given some personal "E/elemental encounter" (Erchick, in review). Thus, she argues for an expanded discourse about mathematical possibilities for women, and mathematics educational realities for women.

Teachers talk in the work that Diana B. Erchick contributes to the working group. Her work, "Women and Mathematics: Developing Voice Within and Around the Content" focuses on teacher perspectives on the role of the content itself in the development of their relationship with mathematics. Erchick works with "successful" women, but not those traditionally seen as successful. Rather, she solicits research participation from teachers who teach children mathematics in K-6 classrooms for this work, teachers who from a number of perspectives are found to be developing successful relationships with mathematics. These are teachers who seek out professional development experiences in mathematics education, who work toward a stronger and more comfortable for them relationship with the content.

In "Calculated Control: Feminist Pedagogies as Tools of Self-Regulation," Robert Klein explores the ways that the categorical notions of "woman" and the "self-regulated student" are constructed and deconstructed within a computer calculus course that incorporates "female friendly" (Rosser, 1997) pedagogies. He traces how these pedagogies are re-deployed as social practices that allow for centralized control through self-regulation. The very pedagogies thought to be "female friendly" may be deployed in ways that run counter to critical feminist desires for transformative educational practices. Issues of agency and authorization of "official knowledge" are important concepts used in this paper to begin (re)writing the "histories of our subjectivities" (Butler, 1992, p. 25) that open up spaces for action.

Terri Teal Bucci's "Paradigm Pedagogy with Respect to Women and Girls in the Mathematics Classroom," addresses teacher standpoints and lays the ground work for conversations about teacher pedagogical alliances and their influences on women and girls. This work opens the door for us to consider the paradigms from which secondary mathematics teachers teach. In terms of our efforts in the Gender and Mathematics Working Group, Bucci's work raises questions for us to consider. For instance: What do paradigmatic foundations mean for pedagogy? How do particular paradigmatic perspectives correlate with research in gender and mathematics? Finally, How does it all impact our efforts to support women and girls in the mathematics classroom?

With this and other work contributed to our sessions, we will conduct the two Gender and Mathematics Working Groups sessions for 2001. In Session 1 we will share current research projects, findings and reports of participants in attendance. These papers will form the foundation for the discussion in the session. We will integrate this ongoing work and the monograph themes. We will redefine, as needed, the monograph themes based on our own work and that of the scholarly community at large addressing the topic of gender and mathematics; and we will identify absences in either the monograph or our contributions to it. In Session 2 we will identify a working plan. In this session the group will develop a call for additional papers for the monograph; finalize a dissemination plan for the call for papers; and develop a working plan for producing the monograph. In this session we also will solidify participation plans initiated at PME-NA XXII, participation that includes tasks, timelines, writing and administration in the creation of the monograph proposal and the subsequent monograph itself.

References

Erchick, Diana B. (in review). E/elemental encounters within the space/barrier: (Re) conceptualizing women's experience with mathematics. *Journal for Research in Mathematics Education*.

Butler, J. (1992). Contingent foundations: Feminism and the question of "postmodernism." In J. Butler & J. Scott (Eds.), *Feminists theorize the political* (pp. 3-21). New York: Routledge.

Rosser, S.V. (1997). *Re-engineering female friendly science.* New York: Teacher's College Press.

REPRESENTATIONS AND MATHEMATICS VISUALIZATION

Fernando Hitt
Matemática Educativa del Cinvestav-IPN, México
http://www.matedu.cinvestav.mx

Over the last four years, the working group on Representations and Mathematics Visualization focused on the students' construction of concepts under theories on representations. The wide scope of the works presented and discussed in the different reunions of the group show the richness of the research on this field. In the last meeting held in Tucson, Arizona we had the opportunity to discuss special issues presented by Manuel Santos, Kay McClain, Norma Presmeg and Patricia Balderas, Alfinio Flores, Anita H. Bowman, Ana-Isabel Sacristán, and Fernando Hitt. Here we present a summary of their work. A large version of the articles discussed in the meetings of the working group and the aims of our group are available online at http://www.matedu.cinvestav.mx

Summary Report 1

STUDENTS' APPROACHES TO THE USE OF TECHNOLOGY IN MATHEMATICAL PROBLEM SOLVING

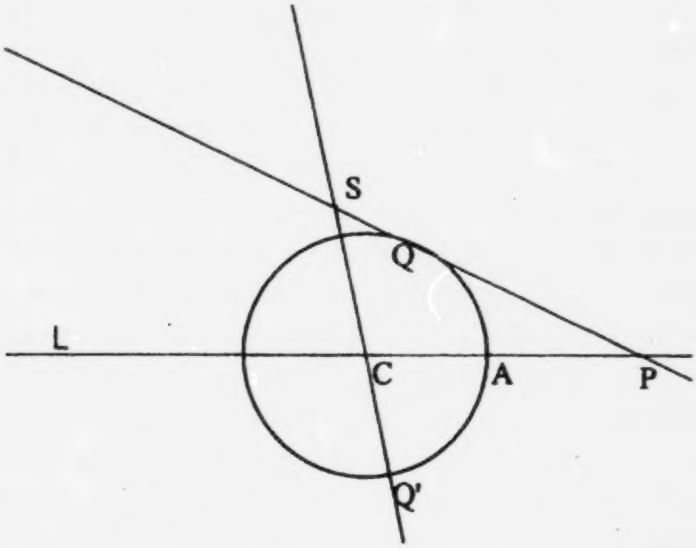
Manuel Santos
msantos@mail.cinvestav.mx

Recent mathematics curriculum reforms have pointed out the relevance of using technology in the learning of mathematics. Indeed, the use of graphic calculator or particular software such as dynamic geometry has produced changes not only in the type of tasks and questions that students examine during their solution processes; but also in the role played by both teachers and students throughout the development of the class. The National Council of Teachers of Mathematics (2000) identifies the use of technology as one of the key organizer principles of Pre-K-12 curriculum. A remarkable feature that distinguishes the use of technology is that it allows students to experiment and examine mathematical relationships from diverse angles or perspectives. Balacheff & Kaput (1998) identify two dimensions in which the use of computers makes an impact in students' mathematical experiences:

1. Symbolic, by changing the representational medium in which mathematics is expressed; and
2. Interactivity, by changing the relationships between learners and the subject matter and between learners and teachers—by introducing a new partner (p. 495).

To what extent does the use of representations (achieved via technology) help students in the process of understanding and solving tasks or problems? What is the role of teachers/instructors in an enhanced technology class? Is it possible to identify what aspects or features of students' mathematical learning can be enhanced by the use of technology? To what extent do mathematical arguments or ways to approach problems vary from traditional approaches (paper and pencil)? These questions are used as frame to discuss different approaches in using technology as a tool in students' learning of mathematics. In particular, we document the use of Cabri Geometry software and the TI-92 Plus calculator as a means to:

1. **Searching for mathematical constructions and meaning**. Here, a simple construction that involves a circle, lines and symmetric properties is used to generate all the conics that normally are studied in an entire course of analytic geometric. Students have opportunities to verify properties and understand definitions of each conic.



2. **Accessing basic mathematical resources to work and solve non-routine problems**.Several non-routine problems that require powerful mathematical resources for their solution with traditional approaches can become accessible to a variety of students when are approached via technology.
3. **Finding and exploring different conjectures.** The use of technology offers great potential for students to search for invariants and propose corresponding conjectures.
4. **Generalizing mathematical patterns using defined and recursive functions**. An important goal during learning mathematics is that students develop different strategies to find and analyze the behavior of mathematical relationships that emerge from the consideration of particular cases.

Different aspects of the mathematical practice are displayed while using technology to approach problems. In particular, representing data seems to be an important step for accessing basic mathematical resources that become crucial to approach the tasks. Working with dynamic environments helps students identify and examine particular conjectures. It is also shown that the process of working with the problems via technology introduces a natural environment for posing and pursuing related questions.

Summary Report 2

COMPUTER-BASED TOOLS FOR DATA ANALYSIS: SUPPORT FOR STATISTICAL UNDERSTANDING

Kay McClain
Vanderbilt University

The context for my discussion is a twelve-week classroom teaching experiment conducted with seventh-grade students that focused on statistical data analysis (Cobb, 1999; McClain, Cobb, & Gravemeijer, 2000). One of the goals of the teaching experiment was to investigate ways to proactively support middle-school students' ability to reason about data while developing statistical understandings related to exploratory data analysis. The research team's[1] interest was motivated by current debates about the role of statistics in school curricula (cf. National Council of Teachers of Mathematics, 2000; Shaughnessy, 1992). The image that emerged for the members as they read this literature was that of students engaging in instructional activities in which they both developed and critiqued databased arguments.

As the research team began to design the instructional sequence to be used in the seventh-grade classroom, we attempted to identify the "big ideas" in statistics. Our plan was to develop a single, coherent sequence and thus tie together the separate, loosely related topics that typically characterize middle-school statistics curricula. In doing so, we came to focus on the notion of distribution. This enabled us to treat notions such as mean, mode, median, and frequency as well as others such as "skewness" and "spread-outness" as characteristics of distributions. It also allowed us to view various conventional graphs such as histograms and box-and-whiskers plots as different ways of structuring distributions.

In our development work, we were guided by the premise that the integration of computer tools was critical in supporting our mathematical goals. Students would need efficient ways to organize, structure, describe, and compare large data sets. This could best be facilitated by the use of computer tools for data analysis. However, we tried to avoid creating tools for analysis that would offer either too much or too little support. This quandary is captured in the current debate about the role of technologies in supporting students' understandings of data and data analysis. This debate is often cast in terms of what has been defined as expressive and exploratory computer models (cf. Doerr, 1995). In one of these approaches, the expressive, students are expected to recreate conventional graphs with only an occasional nudging from the teacher. In the other approach, the exploratory, students work with computer software that presents a range of conventional graphs with the expectation that the students will develop mature mathematical understandings of their meanings as they use them. The approach that we took when designing computer-based tools for data analysis offers a middle ground between the two approaches.

As we began to develop tasks, we reasoned that students would need to encounter situations in which they had to develop arguments based on the reasons for which the data were generated. In this way, they would need to develop ways to analyze and describe the data in order to substantiate their recommendations. We anticipated that this would best be achieved by developing a sequence of instructional tasks that involved either describing a data set or analyzing two or more data sets in order to make a decision or a judgment. The students typically engaged in these types of tasks in order to make a recommendation to someone about a practical course of action that should be followed. The students' investigations were grounded in the use of the tools. As the sequence progressed, students began to develop inscriptions to support their arguments. The development of these inscriptions can be traced to the emergence of practices that evolved from the students' use of the computer-based tools. As such, the tools offered means for supporting students' developing ways of reasoning statistically.

Note. Other members of the research team were Paul Cobb, Koeno Gravemeijer, Maggie McGatha, Cliff Konold, Jose Cortina, and Lynn Hodge.

Summary Report 3

GRADUATE STUDENTS' VISUALIZATION IN TWO RATE OF CHANGE PROBLEMS

Norma C. Presmeg
Illinois State University

Patricia E. Balderas-Cañas
Universidad Nacional Autónoma de México

Abstract: The research reported in this paper is part of a larger study to investigate the presence, role, extent, and constraints of visual thinking in the problem solving processes of graduate students as they solve nonroutine problems. This report gives details of the solving of two rate of change problems by three students. As evidenced by three descriptors, namely, drawing, verbal report, and gesture, visual imagery was used by each of the three students for both problems. Visual imagery was reported even in instances where no diagram was drawn and the solution appeared to be purely algebraic. The roles of visualization were investigated in four main moments of the solution processes, which we have called preparation, solution, conclusion, and hindsight. The types of imagery and their roles in these moments cause us to differentiate between use of imagery to make sense and to solve, as two distinct aims of visualization. Affect, base knowledge, spatial reasoning, and metaphors that may enable or constrain, all played a role in the graduate students' use of visualization.

Introduction

Construction and use of imagery of any kind in mathematical problem solving are processes that challenge research approaches because of the difficulty of apprehending these processes without changing them. Visual imagery used in mathematics is frequently of a personal nature, not only related with conceptual knowledge and belief systems, but often laden with affect (Presmeg, 1997, Goldin, 2000). But it is these very personal aspects that may enable or constrain the mathematical solution processes of an individual (Aspinwall et al., 1997; Wheatley, 1997; Presmeg, 1997), and thus it is important to investigate these issues.

Theoretical Considerations

The theoretical framework that underlies this study is rooted in Krutetskii's (1976) position that all mathematical reasoning relies on logic, whether or not diagrams and mental imagery are employed.

C-3. A boy walks from home to school in 30 minutes, and his brother takes 40 minutes. His brother left 5 minutes before he did. In how many minutes will he overtake his brother?

C-6. A train passes a telegraph pole in 1/4 minute and in 3/4 minute it passes completely through a tunnel 540 meters long. What is the train's speed in meters per minute and its length in meters?

Two Problems

As part of the larger study, the doctoral students were tape recorded as they solved problems "aloud" in task-based interviews. In this paper we report some of the data for Mr. Silver, Mr. Gold, and Mr. Green (pseudonyms). The following two problems were selected from the six problems in section C (intended for high school mathematics teachers) in Presmeg's (1985) preference for visuality instrument.

When and how did these students visualize?

All three of these graduate students drew unsolicited diagrams (indicating the presence of visual imagery) for both of these problems. To facilitate a deeper understanding of the issues of when, how, and with what effect solvers used imagery in this study, the following analysis is reported.

In conclusion, these problems were rich sources in the sense that our analysis of the data reported here could continue: for instance, we have not yet reported the enabling and constraining roles that metaphors played in the students' representations and problem solving. However, the data reported in this paper show convincingly that visualization may play a useful role in the problem solving strategies even of teachers of college level mathematics.

Summary Report 4

GEOMETRIC REPRESENTATIONS IN THE TRANSITION FROM ARITHMETIC TO ALGEBRA

Alfinio Flores
Arizona State University

Introduction

The continuities and discontinuities between arithmetic and algebra are complex and need to be addressed from multiple perspectives. There are several aspects that teachers need to take into account in order to help students make the transition from arithmetic to algebra, and to develop algebraic skills with understanding (see for example Kieran & Chalouh, 1993; Wagner & Parker, 1993, Lodholz, 1990). Using geometric representations can help students to go from statements about particular numbers to the corresponding generalized statements using variables.

- Learn to extract pertinent relation from problem situations and express those relations using algebraic symbols;
- Make explicit the procedures they use in solving arithmetic problems;
- Consider strings of numbers and operations as mathematical objects, rather than processes to arrive at an answer;
- Gain explicit awareness of the mathematical method that is being symbolized by the use of both numbers and letters;
- Focus on method or process instead of on the answer;
- Pose and compose problems;
- Write conjectures, predictions, and conclusions.

Example for the classroom: Sums of odd numbers and sums of cubes.

Students can look at the triangle of odd numbers in Table 1. They can compute the sum on each line, and verify that the sum is equal to a cubic number. We can represent the sums in each row in table 2 with the rectangles in Figure 1.

Students can express verbally the suggested relationship. Each rectangle represents, on one hand, the sum of consecutive odd numbers (each odd number is a thin straight or L shaped strip within each rectangle). On the other hand, each rectangle is the cube of a number, because it has n squares of area n^2. Each rectangle has a growing number of strips, 1 the first, 2 the second, 3 the third, and so on. Thus, the total number of strips up to a given rectangle is the corresponding triangular number $(1 + 2 + ... + n)$. For example, the first two rectangles have $1 + 2$ strips, therefore the third rectangle will start with the fourth odd number $7 = 2 \times 4 - 1$, which can also be written as $2 \times 3 + 1$, and it includes three consecutive odd numbers. In general, the n-th rectangle is the sum of n odd numbers, starting with $2(1+ 2 + ...+ (n - 1)) + 1$ and finishing with $2(1+ 2 + ...+ n) - 1$.

Conclusion

All too often are asked to make the transition from arithmetic to algebra too quickly. Students need time, opportunity, and support to make the transition. Concrete

Table 1. Sums of odd numbers

1	1	1^3
3 + 5	8	2^3
7 + 9 + 11	27	3^3
13 + 15 + 17 + 19	64	4^3

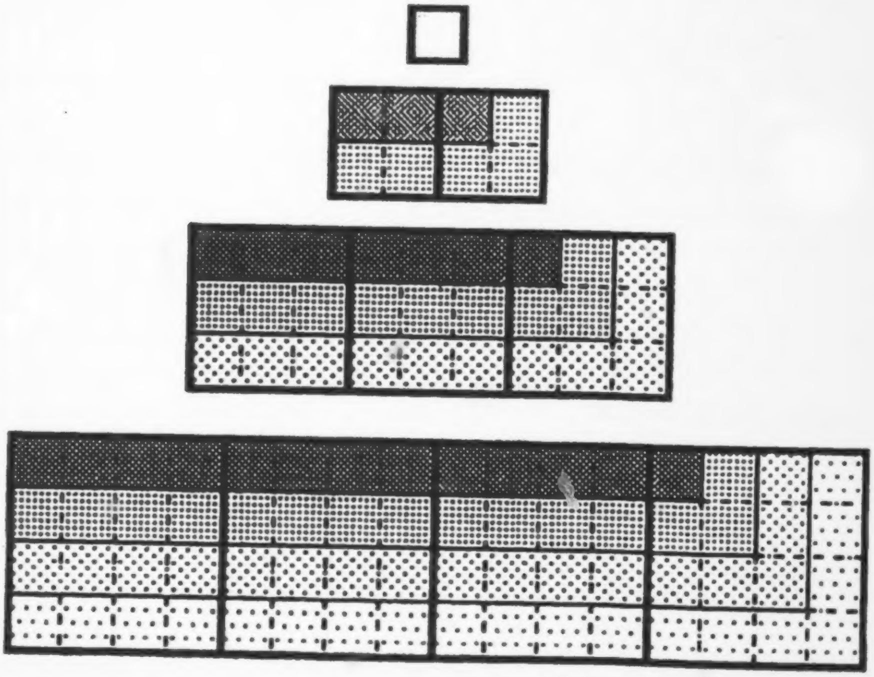


Figure 1. Sums of odd numbers and sums of cubes.

representations of numbers and their relations in the form of manipulative materials, puzzles, and visual displays can help students give meaning to the symbolic expressions.

Algebraic notation has the power to carry much of the weight of thinking when deriving or proving mathematical results. Its abstractness allows us to forget the meaning of the terms as we manipulate the symbols. However, in the beginning, while students develop their skills with the symbols, it is important that they connect meaning to what the manipulation of symbols represent, so that their manipulation does not become senseless. Students need to avoid what Sowder and Harel (1998) have described as treating *"the symbols as though they have a life independent of any meaning or any relationship to the quantities in the situation in which they arose"* (p. 5). Geometrical representations can provide guidance and understanding as to why each step or term in a chain of algebraic manipulations is correct. Later, when students have developed skill in manipulating symbols and their algebraic (or symbol) sense, geometric representations can still be helpful. They can be additional sources of discoveries and inspiration.

Summary Report 5

SHORT PROPOSAL FOR WORKING GROUP REFLECTION: REPRESENTATIONS AND MATHEMATICS VISUALIZATION

Anita H. Bowman
The University of North Carolina – Greensboro

Now that Representation has been introduced as a new Process Standard in *Principles and Standards for School Mathematics* (2000), it seems pertinent to begin reflections concerning what roles the concept of representation might play in the communication of important ideas within the broad realm of mathematics education. Beginning with a statement of the representation standard, we might find it helpful to consider how a focus on this standard might lead to frameworks that may both (a) help important stakeholders (e.g., students, teachers, curriculum designers, and educational researchers) conduct their work within the mathematics education community and (b) provide bridges for communication among the various stakeholder groups. That is, for a specific mathematical concept, it seems reasonable that representations of the concept might provide structure for students in developing important concept understanding, structure for teachers in instructional planning and decision making, structure for curriculum designers in developing educational materials, and structure for researchers in designing experiments and analyzing results. In turn the use of a common structure, across stakeholders, might then provide the "common ground" necessary for dialogue – communication about important aspects of mathematical concept development.

To clarify my general vision of the role of representations, I propose to discuss how the concept of mathematical function along with five external representations (i.e., table, graph, algebraic formula, verbal description, and situation) form a model that might be used by each stakeholder group, individually, and then to discuss how, through the use of such a framework, important information might be shared among the various stakeholder groups. The representation framework I propose to present is what I refer to as the "pentagonal model," a model derived from Janvier's (1987) "star model for understanding mathematical function." In the pentagonal model the vertices of the pentagon correspond to the five representations listed above and the line segments forming the sides and diagonals of the pentagon correspond to the 20 possible one-way translations between pairs of representations.

This discussion of the how the pentagonal model might be used by various stakeholders is proposed only as an example of the potential of representations in helping the mathematics community connect elements of mathematical concept development in such a way as to further the cause of helping learners better learn mathematics.

Summary Report 6

COORDINATING REPRESENTATIONS THROUGH PROGRAMMING ACTIVITIES: AN EXAMPLE USING LOGO

Ana Isabel Sacristán Rock
Cinvestav, México
asacrist@mail.cinvestav.mx

The research literature has pointed to the difficulties encountered by students in interpreting and establishing links between different types of representational registers. For instance, there is evidence that reading and interpreting visual objects is not straightforward (e.g., Dreyfus & Eisenberg, 1990; Artigue, 1990; among others). Here we present results from a study (see Sacristán, 1999) that investigated, among other things, how students articulated and coordinated the different representations of a Logo-based exploratory environment (microworld) in order to construct meanings for the infinite, in particular for infinite processes.

The microworld was a programming[1] environment where infinite processes —infinite sequences and series—were explored through the construction (using Turtle Geometry) of different visual models for representing them, such as "unfolding[2]" spirals and fractal figures, with a complementary numerical analysis (e.g., students constructed tables of values of, for instance, the distances forming the figures at different levels of the construction process). Thus, the infinite processes were explored through three types of representations:

- The symbolic code.
- Different types of visual models (such as unfolding spirals; and fractals).
- Numeric representations, to complement and validate the visual observations (taking advantage of the computing capabilities of the computer for reaching high terms in numerical (or other types of) sequences).

Because the production of the graphical and numerical representations was carried out through the construction of the Logo symbolic programming code, the different types of representations were explicitly linked one with the other through the first: the procedural code. In fact the code can act as an isomorphism between the different (visual) models, as well as acting as the link between all the representations, and the subject. For instance, programming the computer to draw on a computer screen can be thought of as a process of interaction between contexts, from the symbolic code to the visual and conversely. The symbolic code of the computer language can serve, within the computer context, to "explain", model, or represent the process and it encapsulates the structure behind the process. Thus, the programming activities emphasised the interaction between different types of (re-constructed) representations of the infinite processes.

Here we give a specific example from the activities with the microworld to illustrate some of the ways in which students used and coordinated the elements of the exploratory medium *to construct meanings for the infinite*. We illustrate how the microworld gave the students means to *make sense of what they saw on the screen via the programming code*: the interactions between the code and its outputs (see Figure 1). We show the role of the structure of the procedures, particularly the *iterative or recursive* structure (which was present in all the procedures for constructing the visual models of infinite sequences), and its relationship to the visual structure. In relation to the role of the recursive structure of the procedures, we describe two facets to the phenomenon: (a)The link of the endlessness of the process represented on the screen, with the iterative structure of the code; and (b) The use of the symbolic recursive structure of the code to visualise the self-similar visual behaviour.

Endnotes

[1] In the programming environment students wrote, created, modified and explored procedures that represented, in different ways, the infinite processes under study.

[2] The visual models under study could be seen in a dynamic way as they were constructed (as they "unfolded") on the screen: this allowed for the observation of the evolution in time and behaviour of the underlying processes, eliminating the limitation of only observing the final state (the result of the process).

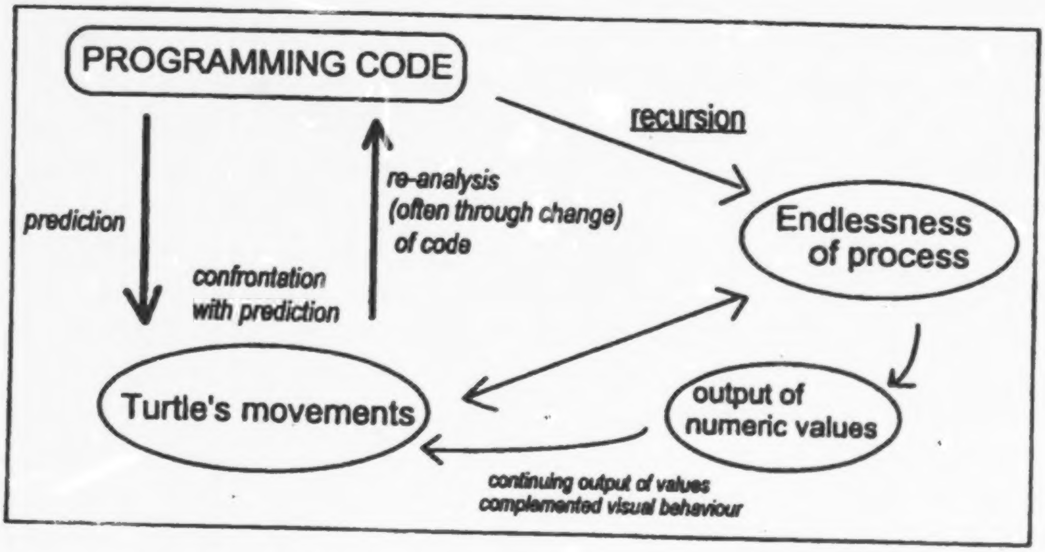

PROGRAMMING CODE
recursion
prediction
re-analysis
(often through change)
of code
Endlessness of process
confrontation with prediction
Turtle's movements
output of numeric values
continuing output of values
complemented visual behaviour

Figure 1. The interplay between the code and its output to make sense of the endless movement: the graphical image gains meaning from the symbolic representation.

Summary Report 7

CONSTRUCTION OF MATHEMATICAL CONCEPTS AND COGNITIVE FRAMES

Fernando Hitt
Departamento de Matemática Educativa del Cinvestav-IPN, México
fernando.hitt@matedu.cinvestav.mx

Abstract: The development of students' abilities to carry out operations and understanding in mathematics has been featured as separate entities in the past. Today, they are studied from an overall perspective as to the construction of concepts. The scheme concept and the internal and external representations of concepts are not recent in the literature. Nevertheless, possibly due to the technological development of graphs and figures representations, such concepts came again under analysis. Current theoretical aspects such as representational semiotic systems and their implications in the articulation of internal representations may profit from previous research (Duval 1993, 1995). As far as the construction of concepts and their implications in certain tasks are concerned, the notions of "conceptual knowledge" and "procedural knowledge" (Hiebert & Lefevre, 1986) have gained relevance when studying learning phenomena. Furthermore, we see the need of constructing structures strongly associated with

understanding –which are more general than specific to the construction of a mathematical concept. According to Perkins and Simmons (1988), and Perkins and Salomon (1989), these cognitive structures they labeled as content, problem solving, epistemic and inquiry structure, they are important to the learning of mathematics.

Introduction

Research in Mathematics Education has –on its strive to unfold the way knowledge is built- motivated the study of the role of semiotic representations and processes in the construction of knowledge. This has entailed revisiting papers concerning the role of sign, and the consideration of new theories involving the construction of internal networks in individuals who participate in the construction of such knowledge.

What we present here has to do with a certain evolution of the notions of sign, semiotic representation, understanding, construction of internal networks, the idea of transference, the notion of cognitive obstacle, and the construction of cognitive structures which are key component to achieve better understanding. We attempted to collect research work on the above mentioned notions, and tried to focus them under the same light. We shall start our analysis with the scheme notion.

The scheme notion and the construction of concepts

It is clear that mathematical objects cannot be directly accessed by the senses, but only through semiotic representations. Skemp (1971, p. 35 & 45) adds to the scheme idea:

> *...each of these by its very nature is embedded in a structure of other concepts. Each (except primary concepts) is derived from other concepts and contributes to the formation of yet others, so it is part of a hierarchy.* (p. 35)
>
> *... To understand something means to assimilate it into an appropriate schema.* (p. 43)

Hiebert & Carpenter (1992) explain those ideas in the frame of networks formed by internal representations which, in turn, were generated by the manipulation of external representations.

Duval (1993, 1995, 2000a) features a semiotic system of representation as a register it allows for three cognitive activities associated to semiotics: 1) The presence of a identifiable representation... 2) The treatment of a representation which is the transformation of a representation within the same register where it was formed...3) the conversion of a representation which is the transformation of the representation into one of different register which preserves the totality or part of the meaning of the initial representation...

The authors focus on the discussion of the semiotic representations' importance, their interaction and power to construct networks that connect knowledge. In what follows we will discuss some difficulties students might face in the construction of those networks of mathematical knowledge.

Discussion

We have seen that the research work of Hiebert and Lefevre (1986), Hiebert and Carpenter (1992) and those of Duval (1993, 1995, 2000a), provide fundamental components to think of a new theoretical orientation that enhance the importance of the semiotic systems of representations in the construction of concepts.

Contrasting the work of Hiebert and Lefevre (1986) with that of Duval (1993, 1995) we can appreciate that from the problem-solving point of view, we must take into account the construction of a network of knowledge that can permit the students interaction between the conceptual and procedimental knowledge.

The approaches we made to the problem of the construction of mathematical concepts and the construction of systems of schemata (cognitive frames) show that one aspect that is under study and difficult to grasp is that of transfer.

Also we have emphasized that under the theoretical approach, here developed, the analysis of errors must be from an holistic point of view. Indeed, the valuation of students' knowledge must be analyzed via activities that deals with the possible connections (articulations) made by the students when they constructed the concept in question. Then, valuation of students' knowledge is more holistic than punctual.

MODELS AND MODELLING

Richard Lesh
Purdue University

Helen Doerr
Syracuse University

Thomas Post
University of Minnesota

Judith S. Zawojewski
Purdue University

Guadalupe Carmona
Purdue University

In the Fall of 2001, Lawrence Erlbaum Associates, Inc will be publishing the book: *Beyond Constructivism: A Models & Modeling Perspective on Mathematics Problem Solving, Learning & Teaching* (Doerr & Lesh, Eds.). As the following table of contents shows, most of the authors in this book have been participants in past PME/NA Working Group Sessions on Models & Modeling (M&MWG)—and many of the topics that are addressed were discussed at past working group sessions. In particular, the final chapter of the book includes a large table (that can be accessed by going to the following url: http://tcct.soe.purdue.edu/M&MWG/M&Mtable.html), which summarizes many of the most significant ways that models & modeling perspectives move beyond constructivism. Because such extensive collections of materials are available, at PME/NA 2001, the M&MWG will not rely heavily on formal presentations. Instead, participants should bring 1-page "posters" (handouts) to share with other M&MWG participants. M&MWG discussion groups will be organized around models & modeling perspectives on: (i) teacher development, (ii) problem solving beyond school, (iii) influences of technology on models & modeling, and (iv) extensions of M&M perspectives beyond the middle school grades.

Table of Contents

SEC 1 INTRODUCTION TO A MODELS & MODELING PERSPECTIVE

1 Foundations of a Models & Modeling Perspective on Mathematics Teaching and Learning
Richard Lesh and Helen Doerr
2 Model Development Sequences
Richard Lesh, Kathleen Cramer, Helen Doerr, Thomas Post, and Judith Zawojewski
3 Origins and Evolutions of Model-based Reasoning in Mathematics and Science
Richard Lehrer and Leona Schauble
4 Piagetian Conceptual Systems & Models for Mathematizing Everyday Experiences
Richard Lesh and Guadalupe Carmona

5 A Semiotic Look at Modeling Behavior
Paul Kehle and Frank Lester

SEC II A MODELS & MODELING PERSPECTIVE ON TEACHER DEVELOPMENT

6 A Modeling Perspective on Teacher Development
Helen Doerr and Richard Lesh

7 A Modeling Approach for Providing Teacher Development
Roberta Schorr and Richard Lesh

8 A Modeling Approach to Describe Teacher Knowledge
Karen Koeller Clarke and Richard Lesh

9 Task-analysis Cycles as Tools for Supporting Students' Mathematical Development
Kay McClain

10 Explanations why? The role of explanations in answers of (assessment) problems
Martin van Reeuwijk and Monica Wijers

SEC III MODELS AND MODELING AS VIEWED BY HEAVY USERS OF MATHEMATICS

11 Twenty-first Century Conceptual Tools (TCCT)
Richard Lesh and Guadalupe Carmona

12 The EPICS Model in Engineering Education: Perspective on Problem Solving Abilities Needed for Success Beyond Schools
William Oakes and A.G. Rud

13 The Case for Cases
Geza Kardos

14 Introduction to an Economic Problem: A Models and Modeling Perspective
Charalambos D. Aliprantis and Guadalupe Carmona

15 Models and Modeling Perspective on Technology-based Representational Media
Tristan Johnson and Richard Lesh

16 A Models and Modeling Perspective on Skills for the High Performance Workplace
Melissa Dark

SEC IV MODELS AND MODELING IN PROBLEM SOLVING AND LEARNING

17 End-in-View Problems
Lynn English and Richard Lesh

18 A Models and Modeling Perspective on Problem Solving Strategies
Judith Zawojewski and Richard Lesh

19 A Models and Modeling Perspective on the Role of Small Group Learning Activities
Judith Zawojewski, Richard Lesh, and Lynn English

20 Local Conceptual Development of Proof Schemes in a Cooperative Learning Setting
Guershon Harel and Richard Lesh

21 Models & Modeling Perspective on Metacognitive Functioning in Everyday Situations Where Mathematical Constructs Need to be Developed
Richard Lesh, Frank Lester, and Margret Hjalmarson

22 Interest, Identity, and Social Functioning: Central Features of Modeling Activity
James Middleton, Richard Lesh, and Michelle Heger

SEC V **MODELS AND MODELING BEFORE AND AFTER MIDDLE SCHOOL**

23 Beyond Constructivism: An Improved Fitness Metaphor for the Acquisition of Mathematical Knowledge
Susan J. Lamon

24 Using a Translation for Curriculum Development and Classroom Instruction
Kathleen Cramer

25 Integrating a Models and Modeling Perspective with Existing Research and Practice
Marilyn Carlson, Sean Larsen, and Richard Lesh

26 Models of Functions and Models of Situations: On the Design of Model Based Learning Environments
Beba Shternberg and Michal Yerushalmy

SEC VI **NEXT STEPS**

27 From Problem Solving to Modeling: The Evolution of Thinking About Research on Complex Mathematical Activity
Frank Lester and Paul Kehle

28 Beyond Constructivism
Helen Doerr and Richard Lesh

Chapters in the book are accompanied by 15-minute slide show overviews, digital appendices and, other web-based resources such as downloadable problem solving activities for students or teachers, videotaped and/or transcribed problem solving sessions, and accompanying tools for teachers. These resources can be downloaded from the web site for Purdue's *Center for Twenty-first Century Conceptual Tools* (http://tcct.soe.purdue.edu).

The project that led to the preceding book was supported in part by the University of Wisconsin National Center for National Center for Improving Student Learning & Achievement in Mathematics & Science Education. Pilot versions of the book were used in a series of doctoral seminars within Purdue & Indiana University's jointly sponsored *Distributed* Doctoral Program in Mathematics Education. The DDP uses internet-based multi-media communication technologies to facilitate interactions and continuing collaborations among graduate students and experienced researchers in a variety of leading research institutions throughout the United States, Canada, Mexico, England, and Australia. In particular, during the Fall of 2001, the DDP will be hosting two multi-campus courses that will feature collaborators in the PME's Working Group on Models & Modeling. The course on Software Development in Mathematics Education will include sessions led by nearly a dozen of the world's leading software developers in mathematics education; and, it will focus on tools and simulations that are especially useful when modeling perspectives are emphasized in learning and problem solving activities. The course on Research Design in Mathematics Education will emphasize research methodologies that are especially useful when modeling perspectives are emphasized research & development activities. Also, in the Spring of 2002, the DDP will host another multi-campus course focused explicitly on Models & Modeling in Mathematics & Science Education.

GEOMETRY AND TECHNOLOGY

Douglas McDougall
OISE/University of Toronto
dmcdougall@oise.utoronto.ca

Jean J. McGehee
University of Central Arkansas
jeanm@mail.uca.edu

The working group on Geometry and Technology has met since the PME-NA XX conference at Raleigh, North Carolina. The objectives were set to explore: the technology environment, student perspective, and teacher perspective. In the first year we:

- Investigated research questions,
- Coordinated future research in this area,
- Identified commonalities in the research findings, and
- Identified critical questions that are not being addressed.

The primary discussion centered on the student in an environment created and caused by microworlds. Some participants planned projects to study what features of the dynamic geometric environment encourage the development of student understanding of geometry.

In Cuernavaca at PME-NA XXI, participants shared their past year's experience in the investigation of geometry and technology. Using the van Hiele framework to categorize college students' geometrical understanding, one participant investigated dynamic software's role in bridging the gap between empirical work with sketches and deductive thought to related proofs. Another participant began an investigation of geometry software in the teacher education program. These two investigations led to a discussion focus on the development of proof skills and student understanding of the role of proof in mathematics. At the second meeting of the working group, we applied the discussion on proof to teacher education. The following questions were posed for investigation:

- What skills and abilities to preservice teachers demonstrate with proof?
- What do preservice teachers understand about the role of proof in mathematics?
- How can dynamic geometry software impact preservice teachers' understanding of proof and skills?

We chose to use the Tucson conference to coordinate future research and to investigate various instruments and methods of research in geometry and technology. Jean McGehee shared a multiple-choice instrument based on Usiskin's van Hiele Geometry Test (1982). Tami Martin and Sharon McCrone showed the instrument from their study of students' understanding of proof. Two dominant ideas have emerged in the three years for this working group: the interest in proof and the investigation of the role of software in moving students from conjectures about drawings to theoretical work with sketches. While there has been variety in the groups studied (college students, secondary students, preservice teachers), all investigations seem to study some aspect of teaching proof.

Nicholas Jackiw, developer of *The Geometer's Sketchpad*, and Steve Rasmussen from Key Curriculum Press participated in a discussion about the intentions of programming and design and their observations of the uses of the software were insightful. For example, Jackiw has noticed that students are reluctant to drag objects in a sketch. In other words, students initially see sketches as static drawings as opposed to dynamic constructions that logically obey the axiomatic structure of Euclidean geometry.

Correspondence with the network established in the working group indicates that participants would like to continue sharing instruments and methods from ongoing projects. Some interest has been expressed in looking at *Rethinking Proof* (de Villiers, 1999) as a model for developing performance tasks in research. Participants will also investigate intuitive geometry, justification and proof for prospective teachers, teachers, and students. This will continue some of the work of the first working group and a paper presented at the Cuernavaca conference (Pence, 1999).

References

de Villiers, M. D. (1999). *Rethinking proof.* Emeryville, CA: Key Curriculum Press.

Pence, B. J. (1999). Proof schemes developed by prospective elementary school teachers enrolled in intuitive geometry. In F. Hitt & M. Santos (Eds.), *Proceedings of the Twenty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 429-435). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Usiskin, Z. (1982). *van Hiele levels and achievement in secondary school geometry.* (ERIC Document Reproduction Service No. ED 220228).

THE COMPLEXITY OF LEARNING TO REASON PROBABILISTICALLY

Carolyn A. Maher
Rutgers University
cmaher3@home.com

Robert Speiser
Brigham Young University
speiser@mathed.byu.edu

Over the last several years, we have given serious attention to how students learn to reason probabilistically; that is, how learners construct mathematical models, and how these models interrelate with each other and with data. A special focus has been on how students build and work with information. Some of work along these lines has been reported and discussed at Singapore (ICOTS-5, June 21-26, 1998), at PME-NA 20 (North Carolina State University, Raleigh, North Carolina, October 31–November 3, 1998), at the third Robert B. Davis (RBD) Working Conference (Snowbird, Utah, May 22-26, 1999) and at PME-NA 21 (Cuernavaca, Mexico, October 23-26, 1999). Continuing discussion, investigation and collaboration draw on work at sites around the world.

Issues

At PME–NA 20 (Raleigh, 1998), the Working Group began to formulate a joint agenda for research, discussion and investigation. At Cuernavaca, the Working Group at PME-NA 21 developed this informal research direction further, together with presentations of data and research results. Discussions at PME-NA 22 (Tucson) continued and extended the past discussion, with particular emphasis on identifying important mathematical and psychological issues that our work suggests should be addressed by learners, researchers and teachers. Central issues which the group discussed include:

- Gaining more detailed understanding of how learners work with data, through analyses of learners' images, representations, models, arguments and generalizations. Attention to learners' successes as well as learners' difficulties, in the unifying context of research on the development of understanding.
- Attending to how models, reasoning and thinking function in communities of learners, teachers and researchers. Examination of the roles of given tasks, of classroom environments, of student-teacher interactions, and of how learners, in a range of settings, share ideas, reasoning, and information.
- Placing more fundamental emphasis upon the development of mathematical ideas through time, with learners of different cultures, ages, and social backgrounds; and with different prior mathematical and scientific experience. Recognition of the importance of detailed analysis of learners' and researchers' changing views of underlying psychological, mathematical and scientific issues.

To help focus and develop this agenda, the Cuernavaca discussion took as starting points the interplay of combinatorial and probabilistic reasoning for constructing images and models in the course of task investigations. At Tucson, the complex role of such key ideas as sample spaces and data distributions came into the foreground, as well as very stimulating questions and discussion centered on the role of simulations and experiments in the building, by learners, of important models and ideas.

Theoretical Framework

Recent research emphasizes the complexity and subtlety of probabilistic reasoning, even in very basic situations. Models and representational strategies can easily extend distortions, even while they help support the growth of understanding. Indeed, the variety of representations which learners find useful, and the complex relationships among the models learners build and the data which they seek to explicate, provide rich opportunities for research investigations focused how learners, in social settings, construct, present, revisit and reconsider key ideas and ways of working. Indeed, given its complexity, the development of probabilistic thinking seems to *demand* reflective building over time. The tools available, the ways the tools are used, the ways in which ideas and information move among the learners, the teacher's questions, ideas and interventions, all contribute (or perhaps fail to contribute) in important ways. In our group's emerging view, both research and teaching need to take the need for long-term building, as well as the complexity even of very basic tasks, into account.

Background

Related cross-cultural research on particular dice games, by researchers in several countries, using different methods of analysis across a range of settings and learner populations, was reported in joint sessions at the International Conference on the Teaching of Statistics (ICOTS-5, Singapore, June 21-26, 1998). The Singapore reports (Amit, 1998; Fainguelernt & Frant, 1998; Maher, 1998; Speiser & Walter, 1998; Vidakovic, Berenson & Brandsma, 1998) helped motivate the work at Raleigh. Further discussions at the third RBD Working Conference (Snowbird, Utah, June 1999) addressed important aspects of the Working Group's agenda in the context of the growth of understanding.

The present Working Group, first at Raleigh, then at Cuernavaca, and again at Tucson, built upon this shared research, enlisted new collaborators, and helped continue an evolving and quite lively conversation. An incomplete but perhaps representative list of active members of the Working Group would include Sylvia Alatorre, Fernando Hitt and Araceli Limon Segovia from Mexico; and Alice Alston, Sally Berenson, George Bright, Hollylynne Stohl Drier, Susan Friel, Regina Kiczek, Clifford Konold, Carolyn A. Maher, Bob Speiser, Pat Thompson, Draga Vidakovic, and Chuck Walter from the United States. Further colleagues, in several countries, are known to be engaged in work related to the Group's agenda and concerns.

Plan for Involvement of Participants

At Raleigh, the Working Group considered data drawn from sixth-graders' work on two dice games (Maher, Speiser, Friel & Konold, 1998) which led to an extremely rich discussion. Based on this experience, a list evolved which at Tucson had come to include four tasks, which we invited participants at different sites to explore with diverse learner populations. Here are current versions of these tasks.

> *A game for two players*. Roll one die. If the die lands on 1, 2, 3 or 4, Player A gets one point (and Player B gets 0). If the die lands on 5 or 6, Player B gets one point (and Player A gets 0). Continue rolling the die. The first player to get 10 points is the winner. Is this game fair? Why or why not?
>
> *Another game for two players*. Roll two dice. If the sum of the two is 2, 3, 4, 11 or 12, Player A gets one point (and Player B gets 0). If the sum is 5, 6, 7, 8 or 9, Player B gets one point (and Player A gets 0). Continue rolling the dice. The first player to get 10 points is the winner. Is this game fair? Why or why not?
>
> *The World Series Problem*. In a "world series" two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the "world series." Assuming that both teams are equally matched, what is the probability that a "world series" will be won: (a) in four games? (b) in five games? (c) in six games? (d) in seven games?
>
> *The problem of points*. Pascal and Fermat, in correspondence, discuss a simple game. They toss a coin. If the coin comes up heads, Fermat receives a point. If tails, Pascal receives a point. The first player to receive four points wins the game. Each player stakes fifty francs, so that the winner stands to gain one hundred francs, and then they play. Suppose, however, that the players need to terminate the game before a winner is determined. Further, suppose this happens at a moment when Fermat is ahead, two points to one. In correspondence, Pascal and Fermat discuss the question: How should the 100 francs be divided?

The first two tasks, and extensions, for example with tetrahedral dice, were developed for sixth-graders in the Rutgers-Kenilworth longitudinal study by Carolyn A. Maher and her collaborators.. Related work includes (Kiczek & Maher, 1998; Maher & Martino, 1997; Maher & Martino, 1996; Maher, Davis, & Alston, 1991; Maher & Speiser, 1997; Martino, 1992; Martino & Maher, 1999; Muter, 1999; Muter & Maher, 1998). The last two tasks were developed initially for eleventh-graders in the Rutgers-Kenilworth study.

Research on several of these tasks has already taken place at several sites around the world. Work in Brazil (Fainguelernt & Frant, 1998), in Israel (Amit, 1998), and in at least four places in the United States (Berenson, 1999), (Kiczek & Maher, 1998), (Maher, 1998), (Speiser & Walter, 1998), (Vidakovic, Berenson & Brandsma, 1998) has already been reported. Closely related findings, including (Alatorre, 1999) and (Berenson, 1999), were discussed in detail by the Working Group at Cuernavaca.

At the Tucson sessions of the Working Group, still more recent research was presented. In particular, the group discussed an extended videotaped student discussion of the World Series Problem, from the Kenilworth long-term study, directed by Carolyn A. Maher. At the coming Utah meeting, we anticipate that the group's continuing discussions will be further broadened and extended. Points which we anticipate might receive special emphasis include (1) the roles played by experiments and simulations in the building of ideas by learners, (2) the multiple ways in which learners can present and reason about data, and (3) the complex uses, in actual practice, of software and related tools in the building, by learners, of interpretations and ideas, through their work on rich, extended explorations.

Anticipated Follow-Up Activities

Collaborative work, based on case studies drawing on a focused set of tasks, and upon related research from a variety of points of view, at different sites in several countries, has already helped to focus and extend discussion and collaboration. Based on our experience last year at Tucson, further work with learners, in a range of settings, as well as further sharing and collaboration, are to be anticipated. In this spirit, we cordially invite further participants to join a growing and productive enterprise.

Connections to the Goals of PME

This Working Group has emphasized research into the nature and development of probabilistic and statistical understanding, based on collaboration between researchers in several countries, focused by a shared, continually developing collective research emphasis. Recent work by members of the group, which draws on a rich background of psychological, pedagogical and mathematical ideas, has opened opportunities for further study and collaboration.

References

Alatorre, S. (1999). Adults' intuitive answers to probability problems: Methodology. *Proceedings of the Twenty-First Annual Conference of the North American Group for the Psychology of Mathematics Education* (Vol. 2, pp. 451-458). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Berenson, S. (1999). Students' representations and trajectories of probabilistic thinking. *Proceedings of the Twenty-First Annual Conference of the North American Group for the Psychology of Mathematics Education* (Vol. 2, pp. 459-465). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Amit, M. (1998). Learning probability concepts through games. In Pereira-Mendoza, Kea, Kee & Wong (Eds.), *Proceedings of the International Conference on the Teaching of Statistics* (ICOTS-5, Vol. 1, pp. 45-48). Singapore.

Fainguelernt, E., & Frant, J. (1998). The emergence of statistical reasoning in Brazilian school children. In Pereira-Mendoza, Kea, Kee & Wong (Eds.) *Proceedings of the International Conference on the Teaching of Statistics* (ICOTS-5, Vol. 1, pp. 49-52). Singapore.

Kiczek, R., & Maher, C. A. (1998). Tracing the origins and extensions of mathematical ideas. *Proceedings of the Twentieth Annual Conference of the North American Group for the Psychology of Mathematics Education*. (pp. 377-382). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Maher, C.A. (1998). In Pereira-Mendoza, Kea, Kee & Wong (Eds.), Is this game fair? The emergence of statistical reasoning in young children. *Proceedings of the International Conference on the Teaching of Statistics* (ICOTS-5, vol. 1, pp. 53-60). Singapore.

Maher, C. A., Davis, R. B., & Alston, A. (1991). Implementing a thinking curriculum in mathematics. *Journal of Mathematical Behavior, 10*, 219-224.

Maher, C. A., & Martino, A. M. (1997). Conditions for conceptual change: From pattern-recognition to theory posing. In H. Mansfield & N. H. Pateman (Eds.), *Young children and mathematics: Concepts and their representations*. Sydney, Australia: Australian Association of Mathematics Teachers.

Maher, C. A., & Martino, A. M. (1996). The development of the idea of proof: A five year case study. *Journal for Research in Mathematics Education, 27* (2), 194-219.

Maher, C. A., & Speiser, R. (1997). How far can you go with block towers? *Journal of Mathematical Behavior, 16* (2), 125-132.

Maher, C.A., Speiser, R., Friel, S., & Konold, C. (1998). Learning to reason probabilistically. *Proceedings of the Twentieth Annual Conference of the North American Group for the Psychology of Mathematics Education* (Vol. 1, pp. 82-87). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Martino, A. M. (1992). Elementary students' construction of mathematical knowledge: Analysis by profile (Doctoral dissertation, Rutgers University) *Dissertation Abstracts International, 53,* 1833a.

Martino, A. M., & Maher, C. A. (1999). Teacher questioning to stimulate justification and generalization in mathematics: What research practice has taught us. *Journal of Mathematical Behavior, 18* (1).

Muter, E. M. (1999). *The development of student ideas in combinatorics and proof: A six year study.* Unpublished doctoral dissertation, Rutgers University.

Muter, E. M., & Maher, C. A. (1998). Recognizing isomorphism and building proof: Revisiting earlier ideas. *Proceedings of the Twentieth Annual Conference of the North American Group for the Psychology of Mathematics Education* (pp. 461-467), Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Speiser, R., & Walter, C. (1998). Two dice, two sample spaces. In Pereira-Mendoza, Kea, Kee & Wong (Eds.) *Proceedings of the International Conference on the Teaching of Statistics* (ICOTS-5, Vol. 1, pp. 61-66). Singapore.

Vidakovic, D., Berenson, S., & Brandsma, J. (1998). Children's intuitions of probabilistic concepts emerging from fair play. In Pereira-Mendoza, Kea, Kee & Wong (Eds.) *Proceedings of the International Conference on the Teaching of Statistics* (ICOTS-5, Vol. 1, pp. 67-73). Singapore.

Discussion Groups

PROBLEM POSING RESEARCH: ANSWERED AND UNANSWERED QUESTIONS

Lyn D. English

Problem posing occurs naturally in our world. It can be seen in the workplace, in government departments, in sporting bodies, and in family life, to name a few. Unfortunately, problem posing does not occur frequently within the confines of the mathematics classroom. This is in spite of the seminal work of Brown and Walter (e.g., 1993), and of Silver and his colleagues, where it has been clearly shown why problem posing is a desirable (and essential) instructional practice. Research on the topic has not been as prolific or as comprehensive as one would expect. For example, there have been very few studies on the problem posing abilities and cognitive processes of elementary students, and limited research on specially designed classroom problem posing programs. We also have very little knowledge about the long-term developments of students who have been immersed in a mathematics curriculum in which problem posing permeates. Although not denying the contributions of existing studies, there remain a number of significant issues that warrant attention from the mathematics education community. Some of these issues will be explored within the Problem Posing Discussion Group, including:

1. What is the nature of problem posing?

(a) How do we conceive of problem posing? (b) What is actually entailed in Silver's (1994) frequently cited definition of problem posing: *Problem posing refers to both the generation of new problems and the re-formulation of given problems, and can occur before, during, or after the solution of a problem* (p. 19). (c) Have our views of problem posing changed and/or broadened since Silver's definition? If so, how? (d) How might we "define" problem posing for the new millennium?

2. What do we currently know about problem posing, with respect to:

(a) The links between problem solving and problem posing? (b) The links between various reasoning/thinking processes and problem posing? (c) The cognitive and social processes of students and adults when involved in components of problem posing (i.e., those components that have been identified in the literature)? (d) The processes that teachers engage in when (i) posing problems, and (ii) designing and implementing problem posing experiences within the curriculum? (e) The nature of the problem posing programs that have been implemented? What have been the outcomes, in terms of students (and teachers') mathematical growth?

3. What are the research issues that we need to address in the new millennium?

Plan for involvement of participants. Because this will be the first Problem Posing Discussion Group, it is envisioned that a good deal of the time would be spent discussing the issues proposed above. An important component of this discussion time would be the sharing of problem posing studies that have been undertaken by DG participants and others. Directions for future research and development in the area will be documented. Depending on the willingness of the DG participants, a special journal issue devoted to problem posing could be developed.

References

Brown, S. I., & Walter, M. I. (1993). *Problem posing: Reflections and applications.* Hillsdale, NJ: Lawrence Erlbaum.

Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics, 14* (1), 19-28.

THE QUALITY AND ROLE OF CURRICULUM MATERIALS IN MATHEMATICS EDUCATION

Yeping Li

As an outline of school educational activities, curriculum has been the focus of educational reforms in past several decades. In contrast, there are very limited research efforts given to examine the nature of mathematics curriculum and its role in teaching and learning mathematics. Efforts to change curriculum in the past have focused on revising and developing curriculum materials that were used in classrooms. However, previous reform efforts in changing curriculum materials have not been successful as educators might expect. Previous efforts and results in reforming curriculum, in fact, suggest the importance of developing research on mathematics curriculum. As an initial step towards a better understanding of the nature and effects of curriculum materials in teaching and learning mathematics, this discussion group is proposed as a means to organize interested education researchers to develop research on this topic. In particular, this discussion group will focus on the following two issues:

(1) What makes a high-quality curriculum material?

(2) What role of curriculum materials can and should play in teaching and learning mathematics in classrooms? How to examine?

Based on discussions to take place at Snowbird, a list of potential research questions will be generated/selected and interested participants will be organized to develop further research activities on this topic after the meeting. The issues of curriculum materials in mathematics education can and should be examined with a variety of points of view. Collaborative work, based on participants' research interests, can be developed either cross-nationally or within the United States. Participants will be strongly encouraged to share their research and come together again in future PME-NA meetings.

INQUIRY INTO VIDEOTAPE DATA ANALYSIS FOR STUDYING THE GROWTH OF MATHEMATICAL UNDERSTANDING

Arthur Powell

This discussion group will explore and examine analytical models for using as a research tool rich, complex videotape data of learners engaged in developing mathematical ideas. Increasingly, researchers in mathematical education are using videotape data for studying the growth of mathematical understanding. This discussion group would provide opportunities of sustained, in-depth conversations around such questions as methodological approaches, foci of inquiry, theoretical foundations of analytical models, as well as perspectives on task designs and coding.

Participants will be invited to make brief presentations on specific issues on the use of videotape data for studying the growth of mathematical understanding and to raise issues for the group to consider working on. From the deliberations of the discussion group, we intend to discover areas of common concern to participants that could form the basis for continued work together in subsequent meetings of PME. To initiate conversations, the submitters of this proposal will stimulate conversations by presenting aspects of an evolving model that has developed over two decades in an attempt to understand the development of mathematical ideas (Davis, Maher, & Martino, 1992). It rests upon a longitudinal study, currently in its thirteenth year, of the development of mathematical ideas of a focus group of students (Davis & Maher, 1990, 1997; Maher & Martino, 1996a; Maher & Speiser, 1997). To understand how students think and reason about a collection of mathematical ideas, the research and data analysis typically lead to an analysis of individual learners either in the context of clinical interviews or working in-groups, constructing mathematical knowledge (Davis et al., 1992; Maher & Speiser, 1997; Speiser & Walter, 2000).

In this study, we attempt to understand the growth of mathematical understanding by examining temporally the discourse and work of students as they engage in mathematical inquiry. The theoretical underpinnings of this study come from three sources: research on the development of representations (Davis, 1984; Davis & Maher, 1990, 1997; Speiser & Walter, 2000), models of the growth of understanding (Pirie, 1988; Pirie & Kieren, 1989, 1994), and theories about the generation of meaning (Dörfler, 2000). Within this theoretical framework, to study the history, development and use of students' arguments, we identify *events*, i.e., connected sequences of utterances and actions that demand explanation by us, by the learners, or by both. A *trace* is a collection of events, first coded and then interpreted, to provide insight into students' cognitive development (Maher & Martino, 1996a; Maher & Speiser, 1997). The trace contributes to the narrative of a student's personal intellectual history, as well as to the collective history of a group of students who collaborate. An event is called *criti-*

cal when it demonstrates a significant moment of insight from previous understanding (also recorded as events), in the context of an emergent narrative. We may refer to critical events or moments of insight as *conceptual leaps* (See, for example, Maher & Martino, 1996b). They are obvious and striking moments, in which students demonstrate compelling intellectual power, to each other and to us, by having wonderful ideas (Duckworth, 1996) and putting them across in forceful ways (Maher & Martino, 1996b). The paper will elaborate on the many kinds of critical events, including interesting wrong leaps and cognitive obstacles. In large measure, we see education as creating situations that elicit critical events and then supporting reflection on their consequences.

This analytical model for examining videotape data emerges from the research activity of a thirteen-year longitudinal study of the development of mathematical ideas of a focus group of students. At the time of the latest stage of the study, most of the students had been variously together since the first grade. In July 1999, between their junior and senior years in high school, the students participated in an intensive summer institute, funded by a grant from the National Science Foundation. During the institute, students worked on open-ended mathematical tasks related to calculus ideas and designed to allow for exploration and discovery. For two weeks, they worked on tasks for several hours a day. Students organized into three groups of five or six, the students were encouraged and at times invited, to exchange ideas between groups. Videotape and hardcopy data of the students' mathematical work and thinking were collected. The hardcopy data included the written work of students and researcher notes. Each group of students had a dedicated video camera trained on it, while graduate students and principal researchers wrote observer notes. In addition, a fourth, roving camera captured data from each group and student presentations at an overhead. This camera also followed teacher-researchers. After the summer institute, graduate students and researchers watched the videotapes, wrote detailed descriptions of the tapes, identified critical events, transcribed excerpts corresponding these events, and drafted analyses of the events. The descriptions and transcriptions were verified by other graduate students and triangulated with the on-site observer notes. During and after the summer institute, teams of researchers interviewed the students and graduate students wrote notes of the interviews. Through these interviews, they inquired into students knowledge of mathematical ideas deemed prerequisite for the open-ended tasks explored in the summer institute as well as into what student reconstructed when invited to revisit the tasks. These interview data were collected in much the same way as the group work data.

There are four sources of data for the presentations of that will initiate the proposed discussion group: (1) videotapes of the focus group of students engaged in mathematical inquiry and follow-up individual and small-group interviews, (2) corresponding written work of the focus group of students, (3) researcher notes recorded on-site; and (4) descriptions of all videotapes and transcriptions of portions of the videotapes recorded and verified by graduate students.

References

Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education*. Norwood: Ablex.

Davis, R. B., & Maher, C. A. (1990). What do we do when we 'do mathematics'? In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views of the teaching and learning of mathematics* (Monograph No. 4). Reston, VA: National Council of Teachers of Mathematics.

Davis, R. B., & Maher, C. A. (1997). How students think: The role of representations. In L. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 93-115). Hillsdale, NJ: Lawrence Erlbaum.

Davis, R. B., Maher, C. A., & Martino, A. M. (1992). Using videotapes to study the construction of mathematical knowledge by individual children working in groups. *Journal of Science Education and Technology, 1*, 177-189.

Dörfler, W. (2000). Means for meaning. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 99-131). Mahwah, NJ: Erlbaum.

Duckworth, E. (1996). *The having of wonderful ideas, and other essays on teaching and learning*. New York: Teachers College.

Maher, C. A., & Martino, A. M. (1996a). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27* (2), 194-214.

Maher, C. A., & Martino, A. M. (1996b). Young children invent methods of proof: The gang of four. In P. Nesher, L. P. Steffe, P. Cobb, B. Greer, & J. Golden (Eds.), *Theories of mathematical learning* (pp. 431-447). Hillsdale, NJ: Erlbaum.

Maher, C. A., & Speiser, R. (1997). How far can you go with block towers? Stephanie's intellectual development. *Journal for Research in Mathematics Education, 16* (2), 125-132.

Pirie, S. (1988). Understanding: Instrumental, relational, intuitive, constructed, formalised …? How can we know? *For the Learning of Mathematics, 8* (3), 2-6.

Pirie, S., & Kieren, T. (1989). A recursive theory of mathematical understanding. *For the Learning of Mathematics, 9* (3), 7-11.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26* (2-3), 165-190.

Speiser, R., & Walter, C. (2000). *Five women build a number system*. Stamford, CT: Ablex.

DEFINING AND PROMOTING QUALITY RESEARCH IN MATHEMATICS EDUCATION

Martin A. Simon

PME-NA needs to take a regular and proactive role in determining what constitutes quality research and how quality research can be promoted among researchers and doctoral students. This is not a one-time activity; rather it needs to be an ongoing conversation.

Research in mathematics education has changed dramatically in the last 20 years. Twenty years ago qualitative research designs were just beginning to be accepted. As the acceptance increased, methodologies were adapted from other fields. Today, the field is establishing research designs of its own (c.f., Kelly & Lesh, 2000). However, this evolution brings with it a number of problems. I list three of these:

1. As new methodologies continue to be developed, we need to find ways to establish criteria (flexible and changing) for quality research. We cannot afford to either be without criteria nor can we afford to have criteria that stand in the way of innovative methodologies.
2. Top mathematics education research journals such as the *Journal for Research in Mathematics Education* receive a high percentage of articles that are clearly not acceptable for publication.
3. Many of us at research-oriented institutions are experiencing a lack of quality applicants for open positions. One reason for this is that, as a community, we are not adequately preparing doctoral students to do high quality research. This problem suggests a need to both articulate what we mean by "high quality research" and to understand better the development of a mathematics education researcher.

Determining ways to think about these issues across a varied set of research methodologies is inherently problematic. In this first year, we may not get beyond working on articulating what we mean by "quality research."

Participants of the discussion group will work in small groups and then come together to discuss the following:

1. What are the current problems facing our community with respect to the quality of mathematics education research?
2. What are the goals of research in mathematics education?
3. What are the products of research in mathematics education?
4. How can we judge the quality of research in mathematics education?
5. How can we judge the quality of publications and presentation proposals?

An important issue that overlaps with several of the questions listed above is the following: What is the role of theory in research in mathematics education? We will also discuss how to extend the efforts of this group within PME-NA and beyond.

Algebraic Thinking

ON TRANSFORMATIONS OF FUNCTIONS

Bernadette Baker
Drake University
bernadette.baker@drake.edu

Clare Hemenway
University of Wisconsin Marathon
chemenwa@uwc.edu

Maria Trigueros
ITAM (Instituto Tecnologico Autonomo de Mexico)
trigue@itam.mx

Abstract: This study focuses on the analysis of student understanding of transformations. Using APOS theory as a theoretical framework, a genetic decomposition for the concept of transformation was developed. The decomposition was used to analyze class work and interviews from 24 college students who had taken a pre-calculus course based on transformations of functions that included writing as a process and the use of graphing calculators. This paper analyzes students' difficulties related to the concept of transformation and the efficacy of writing and calculators as teaching tools. Results show that students tend to develop a strong dependency on calculators to visualize functions but that when used together with writing assignments they seem to help in the development of the concept. Results also suggest that courses designed on the basis of the use of transformations should be designed with care because this concept proved to be a difficult one for students.

Introduction

The primary focus of a pre-calculus course is an intensive study of functions, as a thorough knowledge of functions is essential for the understanding of concepts encountered in further mathematics courses. The pre-calculus course in some universities has been centered on the teaching of some base functions, called benchmarks, (e.g., linear, quadratic, exponential, logarithmic, rational functions) and their properties and then the student is introduced to more general functions by means of transformations. Many of these courses use calculators as tools to help students in visualizing the transformations on functions and some of them use process writing as a tool to foster students' abilities to construct the necessary abstractions in order to develop a better understanding of the concepts of the course. It is the aim of such courses to help students develop a strong understanding of the concept of function by studying properties in these specific cases. It is hoped that these students will begin to develop a coherent schema of both functions and transformations by seeing connections from working with examples in a variety of settings.

Much research has been conducted on students' difficulties on the understanding of the concept of function, with some studies including the use of calculators and the use of writing. There is not as much knowledge from research about transformations. This is very important if transformations are to be used to help students deepen their understanding of functions and to get useful information about "if" and "how" precalculus courses based on transformations of functions can be successful.

Previous studies about students' understanding of transformations (Eisenberg & Dreyfus, 1994; Block, 2000; Quiroz, 1990) indicate that this is a difficult concept for students. They suggest that students do not easily recognize transformations on functions and that even if they have fewer difficulties when working with transformations in simpler functions such as linear and quadratic, they cannot handle them with ease. They suggest that courses based on transformations don't get the expected results because the learning of transformation requires a deeper understanding of function, maybe at an object level. Thus, they suggest that special care should be taken when teaching these courses to be sure that students have acquired a sufficiently sophisticated knowledge of functions before proceeding to the teaching of transformations.

The act of writing has been described as a problem solving process (Hayes, 1989; Flower, 1985). As such the focus of the writing is the thinking and the writing process rather than the product; and that this process is recursive, meaning that it does not necessarily proceed on a linear path from step to step. One popular type of writing assignment used in collegiate classes is in-class explanatory writing for assessment of student understanding (Keith & Keith, 1985; Le Gere, 1991; Meyer, 1991). This type of assignment enables the instructor to have an ongoing dialogue with each student in the class and to attempt to redirect incorrect thinking.

This study intends to analyze the following research questions:

- Can a pre-calculus course centered on the teaching of benchmark functions and their properties and then generalizing these functions via transformations promote a sufficiently rich understanding of both functions and transformations for the students?
- Are the use of different representations, writing, and the calculator effective tools to promote students' understanding of transformations on functions?
- Does the use of transformations on benchmark functions transfer to other settings, such as domain and range?
- Did the multiple representations of the functions studied contribute to students' understanding of domain and range?

Theoretical Framework

The theoretical framework used as a basis of this study is APOS (Action, Process, Object, Schema) theory as described in Asiala et al. (1996). A genetic decomposition of the concept of transformations was developed.

At an Action level, students are able to perform transformations on functions in an analytical context by substituting values in them one by one and by drawing the graph of the function based on the evaluation of independent points. These students depend on concrete visualization to be able to deduce properties of transformed functions and they have a static conception of transformation at best.

At a Process level, students are able to coordinate the actions of evaluating the function at different points and the change in the properties that follows from the application of the transformation in the case of simple functions, either in a graphical or analytical context. These students have developed a more dynamic conception of transformation and can think of intermediate steps between the initial and the final states based on the properties of the benchmark functions, but only when these are very simple.

Students at an Object level have encapsulated the process of applying a transformation to any function and can tell in advance the properties of the transformed function. They are able as well to identify a more complex function as the result of applying transformations to a basic one independently of the representation of the given information.

Methodology

This study was conducted at a medium sized mid-western university during a single semester using four sections of pre-calculus of about sixty students each. The courses involved two instructors, each of whom taught both a writing section and a non-writing section. A common set of class lecture notes was used, and all sections used graphing calculators extensively during class, for homework assignments, and during quizzes and exams.

The writing students wrote in-class responses to questions or responses to situations that had been discussed in class on previous days. The non-writing students completed additional examples of problems instead of writing in class. After the course was over, audio taped interviews were completed with 24 students. The students were stratified into three groups according to their incoming skill level on a placement test. Two students from each section at each skill level were chosen at random to be interviewed. The analysis included responses to the questions listed below as well as pretest and posttest responses, and writing responses for students in the writing groups. The classification of students according to the genetic decomposition was agreed upon after negotiation between the researchers on the analysis of each student's work. Each student was assigned a pseudonym to protect her or his identity.

The questions:

- Sketch a graph of $f(x) = 2(x-5)^2 + 3$. Explain your graph.
- From your study of algebra, what differences do you expect to see between the graph you drew above and the graph of $g(x) = \dfrac{2}{x-5} + 3$?
- What similarities do you expect to see between the graph you drew above and

the graph of $g(x) = \dfrac{2}{x-5} + 3$?

- Determine the domain and range for each of the following:

  $f(x) = \sqrt{x+2} + 1$

  $g(x) = |2x - 3|$

Some Results

The data showed that only 4 of the 24 interviewed students had a reasonably good understanding of transformations. Even those who did perform well with transformations had difficulty transferring that knowledge to an unfamiliar setting. This student, Andy, whose comments are excerpted here, was able to complete the transformation question but struggled with finding the range of f(x) in the domain and range question.

S: ...what would make the function zero...I always do it from the graph, am I allowed to graph it?

I: right now you're plugging in numbers to see...every value you get

(uses calculator to generate graph)

S: okay, looking at the graph it'd be anything greater than negative two...so negative two to infinity...and range would be all positive numbers...I don't know if it's all or just part of it...the y-values are only above zero...there aren't any negative values in the range as far as I can tell

I: how come the graph doesn't show a value of zero for that? (referring to the point where x = -2)

S: it'd make it undefined...it wouldn't come out...the whole thing would be undefined if was equal to zero...

(more probing)

S: It's like backwards...I don't know...all I know is it can't be in the negatives for the range

This is a case in which Andy was initially checking values individually, then used the calculator to view the graph. He has many of the "right words", but can't apply the concept to a simple problem. Even with probing he can't explain why he should exclude negative numbers and numbers between zero and 1.

Results of this study are consistent with those of previous studies in signaling that students have a better understanding of transformations when applied to functions with which they have more familiarity such as linear or quadratic functions. They have difficulties with other simple but not so familiar functions such as hyperbolas or square roots. The following are examples that show typical responses of students at different levels of the genetic decomposition.

Gwen discusses the transformation question in this excerpt:

S: well…since it's…the 2 is over the x minus five…um, it would be the asymptote type graph because…um…because, uh, I'm not sure exactly why but I graphed it and that's what it looked like…and what similarities do you expect to see between the graphs you drew above and the graph of g of x two over x minus five plus three…uh…well it's the same thing so I guess I'll say the same thing…oh, this one…scratch that…well instead of a parabola it'll be a…um, it goes in a straight line and then goes up and down and then goes in a straight again and it has an asymptote at the um x equals five…well.

Gwen needs to visualize the functions with the calculator to be able to compare the two functions, she has problems identifying the curvature of the hyperbola, and she talks about it as if it is formed by segments of straight lines with asymptotes in between. These elements, taken together with her answers to all the other questions and instruments show that her understanding of transformation is at an action level.

Olivia has more success with the transformation question in this excerpt:

I: now that you are graphing on your calculator then?

S: oh it's not a...it's a hyperbola. I thought it was a line that's what…oh, OK, then they'd both have the same shift.

I: OK do you want to write that down and…OK from what you are writing they both have the same shifts as you know the…what are you saying, the vertical shift

S: they both have a vertical shift of three… and a horizontal shift of five

We consider this student to be at the process level of understanding of the concept of transformation. As we can see in the excerpt, Olivia is able to coordinate the shifts of the benchmark function and she can tell how the function changes but, with additional probing, she is not sure about the meaning of the stretch factor.

Doug displays an object level conception of transformation in this fourth excerpt:

S: That's the benchmark um, two…graph the benchmark and move it ver…to the right five and up three…vertical stretch by two would make it more narrow…then…what similarities do I expect to see between those two graphs um…they'll both have a vertical shift up three…they'll have the same transformations just different initial benchmarks.

Doug's response to both sections of this question shows that he can tell the properties of the transformed function and he can identify them in the case of the two given functions.

It is difficult to conclude from this study how effective the use of the graphing calculator is in fostering students' understanding of the concept of transformation. Most of the interviews show that students develop a strong dependence on the opportunity to "see" the graph of a function in order to tell something about it and that even given that opportunity they do not always succeed in explaining what they see in terms of transformations. However, the use of calculators together with the practice of writing seems to be of some help in developing a better understanding of the concept of transformation. From the 24 students interviewed for this study, 8 of those in classes using both writing and calculators showed a process or object conception of transformation while only 5 students from the non-writing classes showed a similar level of understanding. While students who used writing during their class work did better in terms of their understanding of transformations, we found that these students didn't have any greater success than the non-writing students in their understanding of domain and range.

Students that have a process or object understanding of functions are capable of understanding the transformations of functions in a dynamic way (Edwards, 1992) whether the functions are presented in an analytical or in a graphical context. Monica's interview, excerpted here, demonstrates this.

S: cause…the vertical shift would be…um… the horizontal shift implies…to move right and the vertical… um… shift up two units so the parabola is um…I will go up two…see there…yeah I should probably um… do the stretch first…I'll do that … and move this up… that is stretched by a factor of… two

I: by two, OK

S: now I want to move it over five this way so…I can do that…and we'll shift it up three…

Monica shows a dynamic understanding of the transformation. She talks about the movements of the curve while applying the transformation and draws the different stages the curve goes through in her sketches.

Concluding Remarks

In general, we can conclude from the results of this study that students should have a stronger understanding of function to be able to fully understand the concept of transformation. Calculators are used by these students as visualization tools, and they depend heavily on the graphing calculator to be able to analyze the properties of functions, even in the simpler cases. Additionally, even when they can see the graph of a function either on the calculator screen or on paper, many of these students are not able to recognize the properties of the given function, such as domain and range.

In this study, some students developed an understanding of transformations at a process level. These students benefited more from the use of writing and the calculator and are capable of recognizing transformations on linear and quadratic functions. They show an object understanding of such functions that is the basis of their understanding of how a transformation on a function changes its graph and its properties. Only a few students showed an object understanding of transformation. These students can identify transformations on any of the given functions and are able to describe the effects of the transformation on the graph and properties of the functions. Students having an object conception of transformation don't need the calculator to draw the graphs of the functions and don't rely on it to verify their answers.

The results obtained in this study are similar to those found by other researchers. Eisenberg & Dreyfus (1994) suggested that an object conception of function might be a prerequisite to the effective understanding of transformations although they didn't have empirical support for this assertion. In this study, that seems to be the case. Students who don't have a strong conception of function are not able to recognize transformations on functions even after being taught to use them as a starting point for the analysis of the properties of general functions

As also identified by other researchers, this study shows that students have fewer difficulties when the basic functions are linear or quadratic and that vertical transformations seem to be easier for students than horizontal ones. This result can be explained in terms of the theoretical framework used in this study: vertical transformations are actions performed directly on the basic functions while horizontal transformations consist of actions that are performed on the independent variable and a further action is needed on the new variable to get the result of the transformation.

More research is needed on students' understanding of transformations of functions. Eisenberg & Dreyfus (1994) proposed that transformations could be ordered in terms of their difficulty for students. They left as an open question whether it was possible that strategies for teaching functions based on transformations could succeed. Dubinsky & Harel (1992) and Schwingendorf et al., (1992) had success in getting students to develop process and object conceptions of functions by use of computer programming. The results found in this study suggest that, in the case of vertical and horizontal transformations, pre-calculus courses based on transformations cannot succeed in their goal of helping students have a deeper understanding of the concept of function unless students already have a strong conception of the basic functions. Thus it seems necessary at the beginning of these courses that students be enabled to construct basic functions as objects before proceeding to study the transformations on these functions. Perhaps with more careful design of both writing and calculator activities, such constructions can be accomplished.

References

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In J. Kaput, A. H. Schoenfeld, & E. Dubinsky (Eds.), *Research in collegiate mathematics education II* (pp. 1–32). Providence, RI:

American Mathematical Society & Washington, DC: Mathematical Association of America.

Block, I. (2000). Personal correspondence with M. Trigueros.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 85–106). Washington, DC: Mathematical Association of America.

Edwards, L. (1992). The role of microworlds in the construction of conceptual entities, In G. Booker, P. Cobb, & T. N de Medicutti (Eds.), *Proceedings of the Fourteenth Annual Meeting of the Psychology of Mathematics* (Vol. 2). Mexico.

Eisenberg, T., & Dreyfus, T. (1994) Learning to visualize functions. *Research in collegiate mathematics education I.* CBMS Issues on Mathematics Education (Vol. 4, pp. 45-68).

Flower, L. (1985). *Problem solving strategies for writing* (2nd ed.). New York: Harcourt, Brace & Jovanovich.

Hayes, J. R. (1989). *The complete problem solver* (2nd ed.). Hillsdale, NJ: Erlbaum.

Keith, S., & Keith, P. (1985). *Writing and learning college mathematics*. Paper presented at the annual meeting of the Conference on English Education, Cedar Rapids, IA.

LeGére, A. (1991). Collaboration and writing in the mathematics classroom. *Mathematics Teacher*, *84* (3), 166–171.

Meyer, R. W. (1991). A classroom note on integrating literacy activities in the mathematics classroom. *Mathematics and Computer Education 25* (1), 38–41.

Quiroz, M. (1989). *Instalación de un Lenguage Gráfico en Estudiantes que Inician Estudios Universitarios.* Un Enfoque Alternativo para la Reconstrucción del Discurso Matemático Escolar. Tesis de Maestria. CINVESTAV México.

Schwingendorf, K., Hawks, J., & Beineke, J. (1992). Horizontal and vertical growth of the student's conception of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 133–149). Washington, DC: Mathematical Association of America.

STUDENT ACHIEVEMENT IN ALGEBRAIC THINKING: A COMPARISON OF 3RD GRADERS' PERFORMANCE ON A STATE 4TH GRADE ASSESSMENT

James J. Kaput
University of Massachusetts-Dartmouth
jkaput@umassd.edu

Maria L. Blanton
University of Massachusetts-Dartmouth
mblanton@umassd.edu

Abstract: As part of a long term district-wide study, we report on achievement test results involving students in a 3rd-grade classroom taught by a teacher in a professional development program led by the authors and directed towards the integration of algebraic reasoning with elementary school mathematics. The teacher's practice, reported upon elsewhere, embodied many of the features targeted in the professional development program, and her students' performance on a state-wide standardized test for 4th graders exceeded those of a comparable control class from the same school and those of her district's 4th graders, and approximated the performance of 4th graders state-wide.

Perspective and Purpose of the Study

We are currently engaged in a 5-year district-based project integrating the development of algebraic thinking into elementary school mathematics (Kaput, 1999) in an educationally underachieving school district. While our primary focus is on K–5 teachers and administrators, their professional development is ultimately validated through evidence of student achievement as measured by independent standards—in this case, students' capacity for problem solving and generalization reflecting the Patterns, Functions and Algebra Strand of the NCTM *Principles and Standards for School Mathematics* (2000). Other analyses of the 3rd grade classroom in which this study occurred have shown that the teacher, a participant in the project, supported practices of algebraic thinking and that her students showed an emerging capacity for this as expressed through their activities of generalizing (Blanton & Kaput, 2000a; Blanton & Kaput, 2000b).

Methods/Data Source

We administered a set of fourteen test items (see Appendix) selected from the 4th grade Massachusetts Comprehensive Assessment System (MCAS), a state-wide mandatory exam, to Jan's class of fourteen 3rd grade students, a class with low socioeconomic status (SES) by standard measures. The same items were administered to a second 3rd grade control class with comparable SES from the same school. Analysis of students' responses to the MCAS items, in both individual and partner settings, offers evidence to support the strategy we have adopted with elementary teachers as a means to build classrooms that prioritize understanding mathematics, in particular as it relates to students' abilities to form generalizations in their mathematical thinking and express those generalizations in increasingly formal ways. The test enables us to compare these students' achievement with students across the district and state.

For the experimental class, the test was administered through a 3-day process. Students first completed the exam individually under standard MCAS conditions (first day), then with a partner (second day), and finally, through whole-class discussion (third day). Students were also asked to provide written justifications for their responses to multiple choice items (10 out of 14 items were multiple choice; the remainder were open-response items). The controls took the exam individually only and were not asked to provide explanations of their answers, but used approximately the same amount of time on the test. Their instructor, an experienced teacher, had not participated in our professional development seminars and employed methods similar to those common across the district. We regard the comparison with another 3rd grade class as important due to the literacy demands of the 4th-grade test and the fact that well over half of the experimental class' students were from homes where English was a second language.

Results and Conclusions

Item-Based Comparison with Control Class: State-wide levels of achievement for the MCAS (from which these items were selected) were determined in the Spring 1999 assessment to be (a) advanced—81%; (b) proficient—67%; and (c) needs improvement—41%. Based on these cut-off levels, it follows that Jan's 3rd grade students performed significantly above the 'proficient' level on 33% of the items. They were at or above the 'needs improvement' level for 74% of the items.

An item analysis comparing the results of the experimental group (Jan's classroom) with the control group shows that students in Jan's classroom outperformed the control group on 11 of the 14 test items (see Figure 1). Moreover, Jan's students performed significantly better (at level *alpha* = 0.05) than the control group on 4 of these 11 items. We note that the items on which the control group outperformed the experimental group (although not at a statistically significant level) were multiple choice (items # 3, 9, and 13), and we don't have information from the control group (i.e., written justifications) to determine if their performance was due to chance. Of these 3 items, Jan's students' scored 22% on item 13, much lower than the 41% cut-off level and the control group's score of 50%. In their written justifications for this item, we found that the most common error was that students added numbers given in the problem or the list of possible responses to determine their answer (e.g., 9 + 4 = 13 and 3 + 4 = 7 produced responses of 9 and 7, respectively). Of those 4 students who answered the problem correctly, one described that he had counted and one drew a model of divided oranges. Of the remaining 2 students, one did not give a written justification and the other gave a response that was unintelligible. Only 50% of the students provided a written justification.

Of the 14 items on the assessment, we identified seven as being deeply algebraic in nature (items #2, 3, 6, 7, 8, 11, and 14), requiring students to find patterns generated numerically and geometrically, understand whole number properties (e.g., commutativity), and identify unknown quantities in number sentences. We find it significant that the experimental class outperformed the control class on 6 out of 7 of these items.

We also note that the experimental class performed at the 'proficient' or 'needs improvement' level for 79% of the items according to the state standards for those items. Moreover, they scored at these levels on 72% of the algebra items, which tended to be harder for all groups.

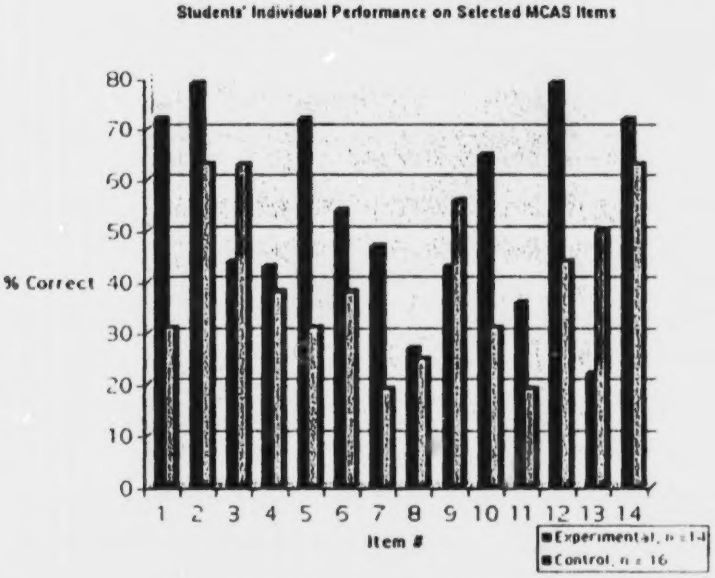


Figure 1. Item comparison with control class.

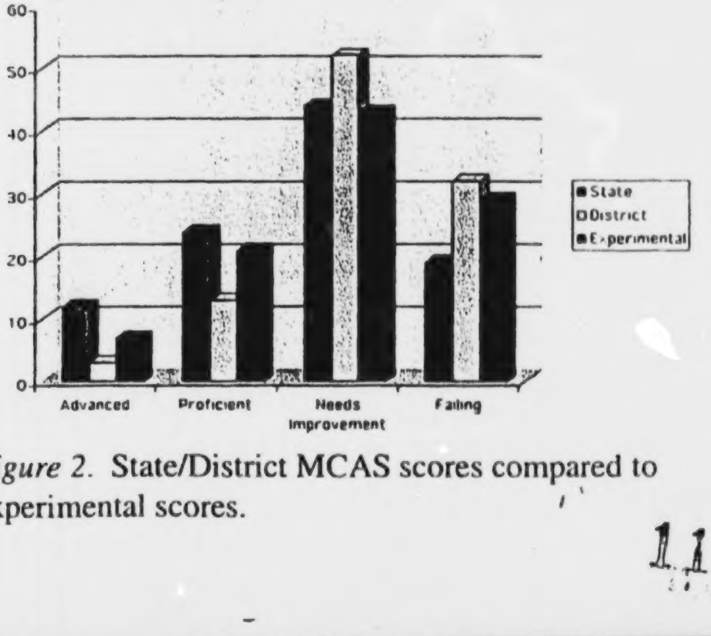


Figure 2. State/District MCAS scores compared to experimental scores.

Overall Comparison with State and District Performance: Jan's students' individual performances, while varied, also support the promising results indicated by the item analysis comparison with the control group (Figure 1) and when compared with State and District results (see Figure 2). The respective State/District achievement levels on the *entire* test are compared with the experimental group's scores by percent per category in Figure 2, with the experimental group's score in **boldface**. (That is, the **boldface** score represents the percentage of students in Jan's class who scored at a particular level.)

Advanced:	12/3/**7**
Proficient:	24/13/**21**
Needs Improvement:	44/52/**43**
Failing:	19/32/**29**

These results indicate that the experimental 3rd-grade class performed approximately as well as the 4th graders state-wide, and significantly better than the district 4th graders (whose performance had improved considerably over the prior year). In our view, given the significant advantage of an additional year's instruction of the 4th grade comparison groups at the state and district levels, and especially given the significant development in the necessary verbal skills during the intervening year relevant to several of the items as well as the low SES factors, especially language, for the experimental class, we regard these as strong results.

Item Analyses for Individuals: In conjunction with Figure 2, Figure 3 provides an item analysis of those MCAS items used in our assessment that were selected from the 1999 MCAS (items 1, 2, 3, 4, 5, 6, 9, 10, 12, and 14). In particular, it shows a comparison of performances by the experimental and control groups with performances at the state, district, and school levels (state, district, and school data for the other items used here were not available). Note that the 14 assessment items we selected came from a variety of MCAS resources, including the 1999 MCAS items, and from multiple strands represented on the MCAS. Again, we see that Jan's students' performance was comparable to that at the state, district, and school levels for most of the items (i.e., # 1, 2, 5, 6, 10, 12, 14), and in some cases her students outperformed 4th-grade results overall (i.e., #10, 12).

Analyses for Students Working in Pairs: Finally, we include in Figure 4 results of an item analysis of Jan's students working in pairs to complete the MCAS items. As expected, students' performed even better when working with a partner. In particular, the analysis showed that students performed at the 'advanced' level on 57%

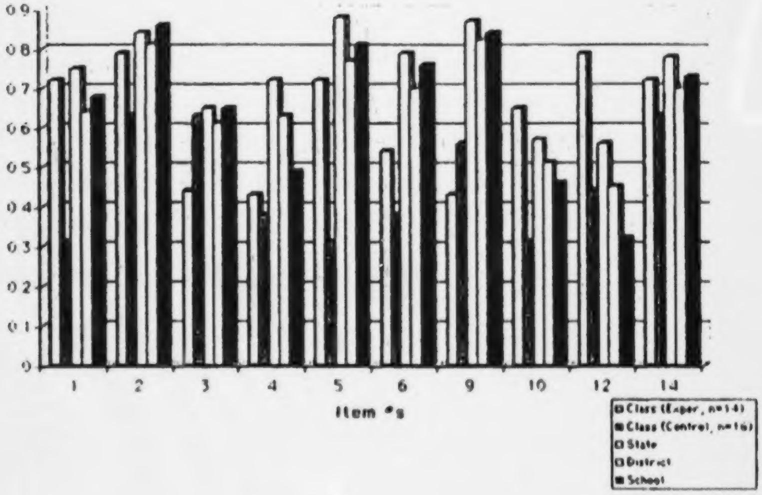


Figure 3. Percentage of correct responses for each comparison group by item.

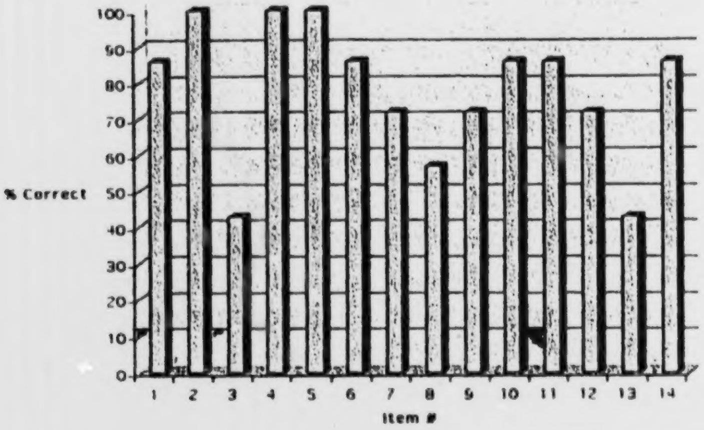


Figure 4. Item analysis of students' performance when working in pairs.

of the items and were 'proficient' or above on 79% of the items. Also, students performed at least at the level of 'needs improvement' on all of the items. Additionally, of the problems for which students scored at the 'proficient' or 'advanced' levels (items # 1, 2, 4, 5, 6, 7, 9, 10, 11, 12, 14), five were problems we identified as strongly algebraic. Thus, working in pairs, students scored at or above the 'proficient' level on 72% (5 out of 7) of the algebra items—scores at this level were achieved in but a few schools state-wide. They scored at least at the level of 'needs improvement' on all of the items we characterized as strongly algebraic.

Overall scores for partner-pairs were quite strong. We believe that, beyond practice effects, this reflects the factoring out of verbal skills as the students were able jointly to interpret the text of problem statements. Out of 7 partner groups, 57% (4 out of 7) performed at the 'advanced' level, 29% (2 out of 7) performed at the 'proficient' level, and one group performed at the level 'needs improvement'. All partner groups ranked under this scheme, scoring at least at the level 'needs improvement', with a maximum score of 90% and a minimum score of 47%. Anecdotally, we note that for the most part students took the assessment quite seriously and, as we observed, were deeply engaged in a process of argumentation and justification with each other during the partner exam and the whole-class discussion. We take this as additional confirmation of the type of socio-mathematical norms that had evolved in Jan's classroom throughout the year, norms that we take to be critical for the development of students' algebraic thinking (Cobb & Bauersfeld, 1995).

References

Blanton, M., & Kaput, J. (2000a). Characterizing generative and self-sustaining teacher change in a classroom practice that promotes students' algebraic thinking. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the Twenty-Fourth Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 144). Hiroshima, Japan: Hiroshima University.

Blanton, M., & Kaput, J. (2000b). Generalizing and progressively formalizing in a third-grade mathematics classroom: Conversations about even and odd numbers. In M. L. Fernandez (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the Psychology of Mathematics North America* (Vol. 1, pp. 115-119). Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

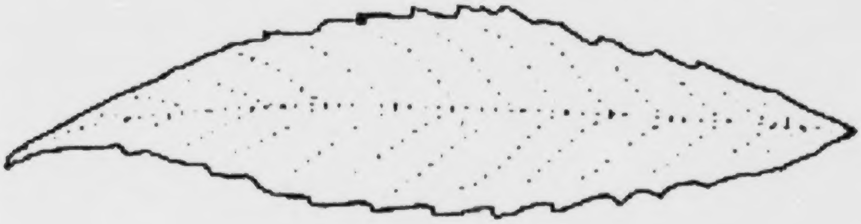
Cobb, P., & Bauersfeld, H. (1995). Introduction: The coordination of psychological and sociological perspectives in mathematics education. In P. Cobb & H. Bauersfeld (Eds.), *Emergence of mathematical meaning: Interaction in classroom cultures* (pp. 1-16). Hillsdale, NJ: Erlbaum.

Kaput, J. (1999). Teaching and learning a new algebra. In E. Fennema, & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133–155). Mahwah, NJ: Erlbaum.

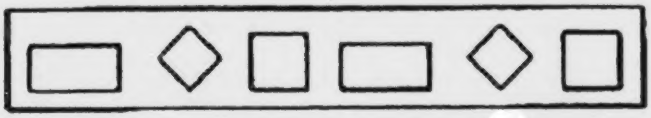
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Appendix

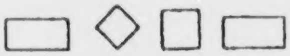
1. How many CENTIMETERS long is the leaf?

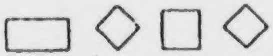


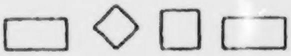
Use the pattern in the box below to answer the next question.

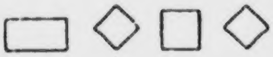


2. What are the next four figures in the pattern above?

A. 

B. 

A. 

B. 

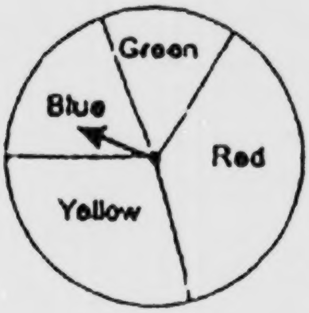
3. What number does *n* stand for in the sentence below?
   $(8 + 2) + 6 = 8 + (n + 6)$

   A. 2 B. 6 C. 8 D. 16

4. Your lunch time begins at 12:40 P.M. If your lunch time is 35 minutes long, what time does it end?

   A. 12:05 P.M. B. 1:10 P.M. C. 1:15 P.M. D. 1:30 P.M.

5. This is a spinner for a game. Which color are you most likely to spin?

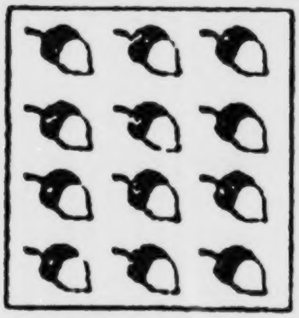


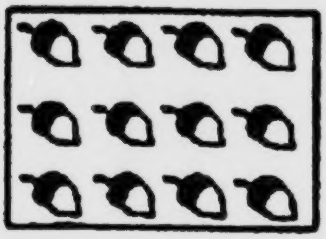
A. blue B. green

C. yellow D. red

6. Melvin collected acorns from the yard.

First he placed them like this: Then he placed them like this:



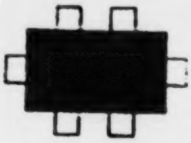


Which number sentence shows the TWO ways Melvin placed his acorns?

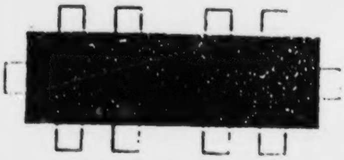
A. 3 x 4 = 4 x 3 B. 3 x 4 C. 3 x 4 > 4 x 3 D. 3 x 4 ≥ 4 x 3

7. Andrew is setting up tables for a birthday party.

He knows that six people can sit about this table:



When he puts two of these tables together end to end, he can seat ten people.



How many people can Andrew seat if he puts three tables together end to end?

8. Write the RULE to find the next number in this pattern.
87, 81, 75, 69, ____

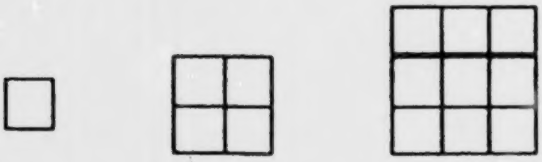
9. There are 60 pieces of art paper and 42 children. If each child gets one piece of art paper, how many pieces will be left for another project?

A. 9 B. 18 C. 27 D. 42

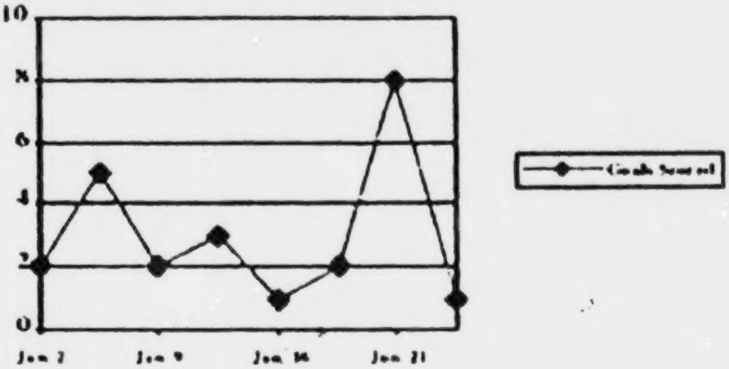
10. What is the GREATEST number of different outfits you can make with 2 pairs of pants and 5 shirts? (Each outfit must have exactly one pair of pants and one shirt.)

A. 5 B. 7 C. 10 D. 25

11. How many of the smallest squares will be in Figure 5 if this pattern continues?



12. How many goals did the Boston Bruins score in their game on January 16, 1999?

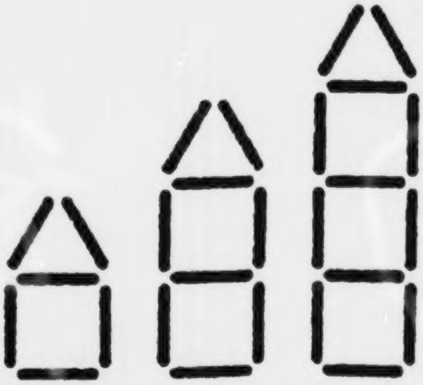


(a) 1 goal (b) 2 goals (c) 4 goals (d) 3 goals

13. Mr. Gillman wants to give apple slices to his 13 soccer players during their game. Each player will receive 3 slices. He plans to cut each apple into 4 slices. How many apples will Mr. Gillman need?

A. 8 B. 10 C. 7 D. 9

14. Donna made this pattern using sticks. Draw the next figure in the pattern.



Explain how you got your answer.

ANALYZING COLLEGE CALCULUS STUDENTS' INTERPRETATION AND USE OF ALGEBRAIC NOTATION

Margaret Tatem Kinzel
Boise State University
kinzel@math.boisestate.edu

Abstract: The study investigated students' interpretation and use of algebraic notation through task-based interviews. Ten calculus students participated in a sequence of four interviews in which they worked on nonroutine algebraic tasks. The analysis of the interviews focused on the ways in which students used and interpreted algebraic notation within the contexts of the tasks. Trends across an individual's work as well as trends across participants were identified. In general, it was found that the participants, while comfortable with notation, lacked sufficient insight into algebraic notation to take full advantage of its potential. That is, the participants seemed able to apply a notational approach to familiar or routine tasks but lacked the tendency and skill to apply a notational approach to a less familiar situation or to interpret notational expressions within that situation.

Introduction

Advances in technology have made it possible to place powerful computer algebra systems in the hands of students, systems that are capable of performing quickly and accurately most of what has traditionally constituted the school algebra curriculum. Combined with initiatives proposed by the National Council of Teachers of Mathematics (1989, 1995), mathematics educators are pushing for a new vision of school algebra that "shift[s] emphasis away from symbolic manipulation toward conceptual understanding, symbol sense, and mathematical modeling" (Heid, Choate, Sheets, & Zbiek, 1995, p. 1). In its most recent document, *Principles and Standards for School Mathematics*, the Council articulates a vision of school algebra as "more than moving symbols around" (NCTM 2000, p. 37), that students would build conceptual foundations to guide the manipulation of symbols. The current study contributes to an articulation of "symbol sense," which may be defined as a coherent approach to algebraic notation that supports and extends mathematical reasoning (Kinzel, 2000). Arcavi (1994) suggested that such an approach would allow students to see algebra as a "tool for understanding, expressing, and communicating generalizations, for revealing structure, and for establishing connections and formulating mathematical arguments" (p. 24). This study investigated the extent to which informed users of algebraic notation implemented such an approach.

A constructivist perspective on knowing assumes that individuals develop coherent ways of operating based on current understandings of previous experiences. The current study postulated that individuals who have participated in formal school algebra instruction have developed ways of operating with respect to algebraic notation, and that these ways of operating can be inferred through close analysis of individuals' work. Nonroutine algebraic tasks were chosen to engage individuals in a range of activities related to algebraic notation. Such tasks may require the individual to introduce and select a form of notation, construct a representation of mathematical relationships using that form, manipulate the resulting notational representation, and interpret the representation at various points within his or her work. Analyzing the individual's work and articulations across these interconnected activities allowed for a characterization of that individual's way of operating with respect to algebraic notation. Further, analysis across individuals allowed for the identification of trends and interactions.

Methods

Ten college calculus students participated in a sequence of four task-based interviews. The participants were asked to think aloud and respond to questions as they worked on nonroutine algebraic tasks. All participants earned a grade of C or higher in a calculus course at a major university, and thus can be considered informed users of algebraic notation, capable of addressing the tasks and likely to use algebraic notation within their work. All interviews were videotaped and transcribed, and the transcripts annotated to coordinate the individual's articulations and inscriptions. The sequence of four interviews allowed the researcher to conduct ongoing analyses of each individual's work, construct working hypotheses, and adapt future interview schedules to explore these hypotheses. Analysis of the data focused on characterizing an individual's way of operating with algebraic notation as well as identifying trends or similarities across participants.

Analysis and Results

The researcher's framework on algebraic notation guided the analysis of both individual participants' data and data across participants. Sierpinska's *acts of understanding* (1994) influenced the development of this framework by distinguishing between instances in which the notation is the *focus of* an individual's activity and instances in which the notation is the *basis for* that activity. It is reasonable to assume that an individual's attention may shift between notation as object and notation as basis within work on a single task, and thus the shifts and possible interactions within that work are significant. The researcher's framework anticipated that the activities of selection, representation, manipulation, and interpretation would occur and overlap within an individual's work on a task. Applying algebraic notation in a particular instance involves (at least implicitly) (a) selecting the form of notation to be used, (b) representing the situation in terms of that form, (c) manipulating the notation, and (d) interpreting the results. Throughout this activity, the individual may engage in reasoning about the notation, about the connection between the notation and the underlying situation, about the manipulative procedures being used, or any combination of these aspects. Reflecting on these reasoning activities can support an individual's development of an awareness of and appreciation for algebraic structure. (The individual may not be explicitly aware of these categories of activity, nor of possible interactions between them.) In some tasks, the notation may be given as part of the task statement. In these instances, the activity of selection may not be required; that is, the problem-solver is not required to choose to use notation nor to choose the form of that notation. "Selection" in these instances may consist of the individual choosing the focus of attention and (through interpretation) establishing a representational connection to the given context. These categories of activity guided the analysis and supported the characterization of interactions between the categories; that is, ways in which particular choices or tendencies within one category of activity significantly influenced activity within other categories.

Loose Attention to Definition of Symbols

A loose attention to the definition of symbols was observed across participants. This finding is consistent with Rosnick's (1982) earlier work. The context of task-based interviews revealed the ways in which this loose attention constrained the representational integrity of an individual's work. Definitions, either introduced by the task statement or by the individual, were not integrated into the representational activity, and thus did not sufficiently guide the construction and interpretation of the notation. For example, on the Alpha/Beta Travel Task, Lisa introduced *d* as "number of days"; in this particular task, there are at least two quantities measured in days.

> *Alpha/Beta Travel Task Statement*: Alpha travels uniformly 20 miles a day. Beta, starting from the same point three days later to overtake Alpha, travels at a uniform rate of 15 miles the first day, at a uniform rate of 19 miles the second day, and so forth in arithmetic progression. When will Beta overtake Alpha? (Posamentier & Salkind, 1988)

Lisa did not specify to which quantity her "*d*" referred, and proceeded to construct expressions that involved more than one interpretation. For Beta's rate, Lisa wrote "15 + 4*d*," which will give the number of miles Beta travels on a given day, but does not give the cumulative total distance. Alpha's rate was expressed as "20 miles/day," which Lisa simplified to "20/*d*." In her expression for Beta, *d* most appropriately refers to one less than the number of days that Beta has traveled (with 15 miles traveled the first day). In her expression for Alpha, *d* is acting simply as a label for the unit of days. Using her two expressions involving *d*, Lisa constructed an inequality: $15 + 4d - 3\left(\frac{20}{d}\right) > \frac{20}{d}$. Lisa intended this inequality to represent Beta's travel, minus Alpha's headstart, being greater than Alpha's travel. Lisa solved this inequality and arrived at a solution of three days ($d > 2.975$). Her initial loose definition of *d* seemed to allow Lisa to work in this manner, leaving her interpretation of *d* unchallenged. By defining and treating *d* loosely as "days," the integrity of Lisa's notation was compromised; she did in fact create and solve an inequality, but her representation did not relate appropriately to the situation as presented in the task. Similar examples across participants indicate that the specific, unambiguous definition of symbols may not be integrated into participants' approaches to algebraic tasks, and further that a loose definition may seriously compromise the integrity of the individual's intended representation.

Ad Hoc Interpretation of Notation

Prior to the study, the researcher assumed that the activity of interpretation would be interwoven throughout an individual's work on an algebraic task. Analysis of the data collected revealed local, *ad hoc*, interpretations of notation that were not integrated into the larger context of the individual's work. For example, the Treasure Hunt Task statement presents information about distances between towns as follows:

> A treasure is located at a point along a straight road with towns A, B, C, and D on it in that order. A map gives the following instructions for locating the treasure:
>
> a. Start at town A and go half of the way to C.
>
> b. Then go one-third of the way to D.
>
> c. Then go one-fourth of the way to B, and dig for the treasure.
>
> If AB = 6 miles, BC = 8 miles, and the treasure is buried midway between A and D, find the distance from C to D. (Charosh, 1965)

In working on this task, Aaron interpreted the distance statements on two separate occasions. First, Aaron interpreted these statements as indicating distances between towns along a straight road, the intended interpretation within the task statement. As he continued to work on the task, Aaron constructed an equation: $a + \frac{1}{3}(D - a) + \frac{1}{4}\left[\frac{1}{3}(D - a) - B\right] = \frac{1}{2}(A + D)$; *a* represents the distance traveled in step (a), A, B, and D refer to the towns as if coordinates on a number line. Later, Aaron wanted to eliminate variables from this equation, and interpreted the given distance statements as products ("A times B equals 6") to facilitate his manipulations. Interpreting "AB = 6" as a product allowed Aaron to solve for B and substitute into his equation. When challenged with his earlier distance interpretation, Aaron chose his second, inappropriate interpretation. His goals for manipulating his notational forms (i.e., wanting to eliminate variables from an equation) overrode the interpretation of the statements more in line with the situation proposed by the task. Aaron's example illustrates a trend observed across participants, that notational expressions were interpreted as needed (e.g., Aaron needed to interpret "AB = 6" in order to move forward with his manipulations), and that the interpretations may not have been integrated into the total problem-solving activity (e.g., Aaron was not bothered by the contradictory interpretations). Such lack of integration would limit an individual's ability to evaluate the effectiveness or appropriateness of a notational approach while engaged in implementing that approach.

Undue Focus on Manipulations

Perhaps the most striking observation concerned the participants' focus on manipulations of the notation. Across tasks and participants, activity seemed to be driven by goals for manipulation of specific expressions, and these goals often overshadowed other aspects of operating with the notation. As discussed in the previous section, Aaron allowed his goal of eliminating variables to override an appropriate interpretation of given statements; similar examples can be found in other participants' work (Kinzel, 2000). A larger trend, however, involved the way in which manipulative concerns constrained individuals' notational approaches. That is, participants seemed to consider in advance if a notational approach could be fully implemented (such as the feasibility of constructing and solving an equation). While on the surface such advance consideration may seem appropriate and even desirable student behavior, the analyses revealed limited criteria for these considerations. In some cases, the participant's limited manipulative skill may have constrained the range of expressions that could be considered. Eric, for example, would not commit to an equation that he could not mentally imagine solving. Since he demonstrated limited manipulative skill, this reluctance limited his ability to construct partial representations and reflect on possible interpretations of these expressions. Such constraining influence was observed also in participants with more advanced manipulative skill. Tina's confidence in her equation-solving skill seemed to promote a belief that any of the tasks could be solved through the direct solution of an equation. In fact, many of the tasks used in the study cannot be solved directly, and require interpretation of notational expressions. When faced with such an instance on the Greeting Cards Task, Tina concluded that she would not be able to solve the task, even though an interpretation of her expressions within the context of the task could have led to a correct solution.

> *Greeting Cards Task Statement*: A greeting card dealer has three kinds of packets. Packet A contains 3 get-well cards, 1 birthday card, and 7 anniversary cards. Packet B contains 2, 3, and 1 respectively. Packet C contains 5, 4, and 2, respectively. He wishes to merge these so as to form new packets, each containing *one* get-well card, *one* birthday card, and *one* anniversary card. What is the least number of packets A, B, and C that must be used to accomplish this? How many new packets will be made? (Charosh, 1965)

Tina constructed three equations to represent the number of get-well, birthday, and anniversary cards. At this point, she stated that she needed another equation, since she had four unknowns and only three equations. (Tina was unable to articulate a reason for this requirement.) In the context of the interview, Tina determined the relationships between the numbers of packets needed (e.g., open twice as many of Packet A as Packet C) but held onto the belief that her three equations should have led to a unique solution, in spite of her own stated preference for a fourth equation. Tina, as did other participants, focused an undue amount of attention on the manipulative aspect of a notational approach, and in so doing may have limited the representational power available through the notation.

Conclusions

Analyses of particular instances of participants' work led to the identification of trends across participants in terms of interactions between categories of activity. The characterization of these trends seems to indicate a limited sense of symbols, or lack of "representational insight." This term is borrowed from cognitive psychology studies investigating young children's early development of symbolic thought (DeLoache & Marzolf, 1992) and refers to coming to recognize that something can stand for something else. In terms of algebraic notation, the term can refer to recognizing the representational role of expressions, that the symbols represent quantities within the task situation. Sfard (2000) emphasizes the essential role of symbols in mathematical activity by articulating how mathematical entities are "symbolized into being." The results of this study indicate that, at least for these participants, such a representational awareness was not fully incorporated into their work. That is, although the participants demonstrated varying levels of skill in terms of constructing, manipulating, and interpreting notational expressions, all participants seemed to fall short of an approach which took full advantage of the representational power of algebraic notation. Such an approach would include an intention to create a notational representation of the relevant mathematical relationships, indicating a recognition of and appreciation for the applicability of algebraic notation within a given situation. Second, such an approach would include the technical skill to actually construct and manipulate such representations.[1] Finally, such an approach would include the tendency to attend to interpretation of the notation throughout the solution process, an awareness that the construction and manipulation of notational expressions takes place always in light of an interpretation with respect to the problem context. ("Context" is not meant to imply a physical setting; the task context could be an abstract mathematical situation.) Such attention maintains the representational integrity of the expressions, and avoids the *ad hoc* interpretations discussed earlier.

A redefinition of the content of school algebra must address goals for students with respect to the interpretation and use of algebraic notation. The participants in this study can be considered successful algebra students in that they each had passed an algebra course in either high school or college. Further, these students were succeeding in calculus courses; all participants earned a C or better in a calculus course. These results demonstrate that it is possible for students to successfully complete an algebra course and progress on to calculus courses without developing a robust concept of representation with respect to algebraic notation. Such a robust concept would include the ability *and tendency* to construct and analyze notational representations for appropriate mathematical situations. Understanding the ways in which students operate with notation in the course of problem-solving activity allows teachers and researchers to contrast students' activity with desired notions of symbol sense. Interpreting students' articulations related to notation within problem solving contributes to defining goals for students' use of notation as well as contributing to understanding potential obstacles to achieving those goals.

References

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the Learning of Mathematics, 14* (3), 24-35.

Charosh, M. (1965). *Mathematical challenges*. Washington, DC: National Council of Teachers of Mathematics.

DeLoache, J., & Marzolf, D. (1992). When a picture is not worth a thousand words: Understanding the symbolic function of pictures and models. *Cognitive Development, 7*, 317-329.

Heid, M. K., Choate, J., Sheets, C., & Zbiek, R. M. (1995). *Algebra in a technological world*. Reston, VA: National Council of Teachers of Mathematics.

Kinzel, M. T. (2000). *Characterizing ways of thinking that underlie college students' interpretation and use of algebraic notation.* Unpublished dissertation, The Pennsylvania State University, University Park.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Posamentier, A., & Salkind, C. (1988). *Challenging problems in algebra*. Toronto: Dover.

Rosnick, P. (1982). *Students' symbolization processes in algebra.* (Technical Report) Amherst, MA: University of Massachusetts.

Sfard, A. (2000). Symbolizing mathematical reality into being—Or how mathematical discourse and mathematical objects create each other. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design*. Mahwah, NJ: Erlbaum.

Sierpinska, A. (1994). *Understanding in mathematics*. Bristol, PA: Falmer Press.

Note

[1] Note that this technical skill may involve the use of technology to support the actual manipulations.

STUDENTS' REACTIONS AND ADJUSTMENTS TO FUNDAMENTAL CURRICULAR CHANGES: WHAT ARE "MATHEMATICAL TRANSITIONS?" HOW CAN WE STUDY THEM?

Jon Star, Dawn Berk, Carol Burdell, Amanda Jansen Hoffmann,
Violeta Lazarovici, Gary Lewis, & Jack Smith
Michigan State University
jsmith@msu.edu

Abstract: The "Navigating Mathematical Transitions" project is a 3-year study that examines students' experiences as they move between "traditional" mathematics curricula and those inspired by the NCTM standards (1989). In this paper, we focus on our method of analysis, particularly the ways that we have conceptualized "mathematical transitions". We describe our analytical framework, which consists of four categories that summarize changes that students' experience during the curricular shift: (1) student achievement, (2) significant differences students notice and report, (3) changes in disposition towards mathematics, and (4) changes in learning approach. We give examples and raise issues with each of these categories, in an attempt to explore what aspects of students' experiences our framework does or does not allow us to capture.

Introduction

The release of NCTM's *Curriculum and Evaluation Standards* (1989) ushered in lively debate in U. S. mathematics education, and after a decade of curriculum development, assessment studies have begun to examine the effectiveness of Standards-based materials (e.g., Schoen, Hirsch, & Ziebarth, 1998). But very little attention has been given to *students' experiences* as they move between programs of curricula and pedagogy that differ dramatically in conceptions of thinking, knowing, and doing mathematics. These differences create boundaries that students must cross to achieve their academic and career goals.

This research report will present emerging analyses from the "Navigating Mathematical Transitions" project, a three-year study of how high school and college students cope with changes in mathematics curriculum and teaching. We are studying students as they move between relatively traditional and substantially different mathematics programs inspired by NCTM Standards (1989) at two junctures: junior high to high school and high school to college. We have completed two years of data collection on approximately 80 students, and we now report on our emerging analytical framework and present outlines of specific individual cases of students' experiences and adjustments. This work follows closely upon our presentation at PME-NA 2000 (Smith et al., 2000) with a stronger focus on our methodology.

We draw upon both cognitive and situated perspectives on learning and development to conceptualize and study students' mathematical transitions. We assume that students carry a diverse body of conceptions and feelings from their prior mathematical experiences in and out of school. Specifically, we presume that students bring and are oriented by the following sorts of cognitions: (1) emergent goals for their future; (2) mathematical knowledge (e.g., procedural skills and understandings of basic concepts); (3) beliefs and attitudes about mathematics and themselves as learners; and (4) strategies and plans for achieving personal goals. However, students' experiences are not solely individual, and thinking and learning are not located exclusively inside their heads. What students see as salient, and what they do to learn mathematics (or not) depends on a wide range of social, cultural, and institutional factors.

We distinguish between *general developmental changes*, such as increased freedom and responsibility for learning that typically accompany the move from junior high to high school and high school to college, and mathematical discontinuities and transitions. We use the term *mathematical discontinuities* to refer to marked differences between students' prior notions and their current perceptions of how they are expected to think and act mathematically. *Mathematical transitions* are students' responses to those discontinuities: How they consciously experience and understand the difference(s), how they respond (or not), and how they understand the results of these responses.

Methods

This project was designed to study these mathematical transitions in one geographical context (south central Michigan). Two Standards-based curricula, Connected Mathematics Project (CMP) materials for junior high (Lappan, Fey, Friel, Fitzgerald, & Phillips, 1995) and the Core-Plus Project materials (CPMP) for high school (Hirsch, Coxford, Fey, & Schoen, 1996), were developed in Michigan and have been adopted widely throughout the state. The University of Michigan has implemented Harvard Consortium materials (Hughes-Hallett, Gleason, et al., 1994) in all calculus and precalculus classes, and Michigan State University has retained a more traditional calculus and pre-calculus curriculum. So the region provides a promising context for studying students' mathematical transitions. We "follow" high school and college students across 2.5 years of mathematics work in fundamentally different curricula—assessing their performance, learning of key content, daily experience, beliefs, and goals.

We examine mathematical transitions at two high schools and two universities. At one high school and university, students move from traditional curricula to those associated with current reforms. At the other high school and university, the move is opposite, from reform to more traditional curricula. We are following approximately 20 volunteers at each site. This 2 x 2 research design is summarized below.

We combine a systematic program of classroom observation (to assess the enacted curriculum) with a broad program of assessment of students' experiences, learning,

Table 1. Research Design of the Mathematical Transitions Project.

	Type of Curricular Shift	
"Location" of Curricular Shift	**Reform to traditional**	**Traditional to reform**
Junior high to high school	Prescott High School (PHS) CMP -> various texts	Logan High School (LHS) Various texts -> CPMP
High school to college	Michigan State University (MSU) Core-Plus -> Thomas & Finney Calculus	University of Michigan (U-M) Various texts -> Harvard Calculus

and personal goals. We assess each student on six dimensions: (1) achievement (course grades and grade point averages [GPA]), (2) learning of key ideas, (3) daily course-related experience, (4) career and educational goals, (5) beliefs about self and mathematics, and (6) strategies for coping with changes and challenges. Fundamental tools are self-report (journals and e-mail), survey instruments, and individual interviews (2-3 times each semester). This rich and diverse corpus of data is compiled into individual case reports describing and summarizing each student's experience.

Results

In our analyses of the data thus far, individual differences have predominated over easily recognized general patterns, even within a particular site. If we had to summarize our findings in one sentence, our choice might be, "Different students react to curricular shifts differently in each school context." In looking at the data collected so far, we are in the process of formulating a *decision rule* to determine which students have or are currently experiencing a mathematical transition. This decision rule includes four categories that summarize changes students have experienced during the curricular shift: (1) student achievement, (2) significant differences students notice and report, (3) changes in disposition towards mathematics, and (4) changes in learning approach. Our challenge has been to determine what levels of change in these four areas constitute a mathematical transition. At present, our decision rule states that we will characterize a student as having experienced a mathematical transition if we find significant change in two or more of these four categories.

In previous papers, we have written more extensively about the results from our first year of data collection. In this paper, we have chosen to step back from our results and consider our methods a bit more closely. As mathematical transitions have not been previously studied in the educational literature, we have had to define and operationalize constructs necessary for our analyses. In this paper, we focus particularly on the way in which have answered the question, "What is a mathematical transition?" We look closely at the four components of our decision rule, giving examples and raising issues about what our framework captures and perhaps obscures. We then bring the four components together and apply our decision rule to a small number of students' cases, and again ask what our framework allows us to capture and what it perhaps fails to illuminate.

Mathematics achievement

In characterizing students' mathematical transitions, one obvious place to start would be to look at students' grades in their mathematics courses. Since achievement is one measure of students' learning, it is reasonable to consider that this construct should provide some sense of whether a mathematical transition has been experienced. However, our data suggests two trends of student achievement, one emerging from the college site and one from the high school site, which question whether (and how) grades should be used as a measure of mathematical transitions.

The first trend that we noticed is at the college sites: students' grades often drop in the move from high school to college (Star, 2001). For example, at the U-M site, 17 of the 19 students in our first year sample had drops in their overall GPAs in their first semester. At the college sites, it seems likely that a drop in grades may be more related to general developmental changes than those associated with mathematics. Thus, if a drop in grades were considered to be an indicator of a mathematical transition, it is likely we would falsely identify many college students as having experienced a transition (e.g., "false positives"). A second trend that we noticed at the high school sites is that there is typically very little change in students' grades as they move from middle to high school (Jansen & Herbel-Eisenmann, 2001). High performers in middle school tend to remain high performers in high school, and vice versa. Thus, using achievement as an indicator of mathematical transitions might lead us to fail to identify many high school students who did in fact experience a mathematical transition (e.g., "false negatives"). These two observations suggest that achievement should be approached with caution as an indicator of mathematical transitions. While we feel that grades do indicate something, achievement data do not portray as clear a picture as one might initially expect. Further support for caution in the use of achievement has emerged at our high school sites, where we have found that teachers' grading practices vary from classroom to classroom and often incorporate features which are only marginally related to student learning of mathematics (e.g., organization and neatness; Jansen & Herbel-Eisenmann, 2001).

With these two observations in mind, we devised a strategy that enables us to use achievement as an indicator of potential mathematical transitions but at the same time minimizes the number of false positives and negatives. This strategy involves comparing students' math grades with their overall GPAs, across multiple time periods, in order to identify students who experienced "significant" change in their math grade as compared to their overall grade, where "significant" will be defined below. The importance of this type of comparison between math grade and semester GPA is illustrated by the example of two U-M students, Jack and Teresa. Both Jack and Teresa did very well in high school, both in math classes (grades of 4.0 for both) and in their overall GPA (4.0 and 3.9, respectively). And both struggled in their first semester of U-M math, with each earning a 2.3, or a drop of 1.7 and 1.6 points, respectively. However, Jack's overall GPA fell to 2.6, while Teresa's GPA only dropped to 3.5. From our perspective, Jack represents a case of someone who experienced a general developmental change: Upon coming to college, all of his grades (including math) suffered a major drop. In contrast, in Teresa's case, something unusual seemed to be happening in math class; she did relatively well in all of her classes except for math. Cases such as Teresa, where grades indicate that something noteworthy may have been happening around her math class, are worth further investigation. In cases such as Teresa, there is a *mismatch* between the pattern of achievement in overall GPA and in mathematics; this mismatch is the criterion that we will use to indicate which students should be flagged. More specifically, a "significant" achievement change occurred when the change in students' math grade differed from the change in overall GPA by more than 0.5 grade points (on a 4-point scale). In the example above, Jack's GPA dropped 1.7 points (from 4.0 to 2.3), and his math grade dropped 1.3 points (4.0 to 2.6), so the difference between these two grade changes is 0.4 points. Teresa's GPA dropped 0.4 points (3.9 to 3.5), while her math grade dropped 1.7 points (4.0 to 2.3), for a difference of 1.3 points. By our definition, Teresa has experienced a significant change in achievement, while Jack has not.

In addition, our extensive conversations about grades have lead to a search for different ways to measure students' learning of mathematics content. One alternative that we are currently considering makes use of the problem-solving interviews that were conducted at each site. These interviews give us a window into the ways that students come to understand the mathematics that they are studying. At present, our thinking into the ways in which problem-solving interviews can be analyzed is preliminary.

Significant differences

In order to understand students' experiences as they moved between traditional and standards-based curricula, we asked them in interviews to describe what they noted as different between their previous and their current mathematics classes. Often the interviewer began with an open prompt like, "What, if anything, do you feel is different in math class this year?" Some students had a lot to say in response to this question, while others simply reported, "It's just different." So we often probed with questions that prompted students to consider possible differences in specific categories, such as the textbooks or the homework problems.

As we began our initial analysis of the differences reported by students during the first year of the project, we found that the differences could be classified based on their origin. Three categories emerged from this initial analysis: teachers/teaching, curriculum, and site policy. We define *Curricular differences* as those differences that had their origin in the written or intended curriculum. For example, differences noted by students in the types of problems presented in the textbooks were coded as Curricular differences. These differences also included observations about the difficulty of the content. For example, Kevin, a student at PHS, noted that the mathematics expected of him in moving from a reform to a traditional classroom was more complex, but focused on computation over connections between representations in mathematics: "This year is more about multiplying the numbers and last year was more about inserting them into the equation and getting them to where they like bond between two ways of doing it." *Teachers/Teaching differences* represent those differences that had their origin in the teaching or the teacher's personal choices. These differences included those reported by students like Pablo at MSU, who remarked that one of the instructors "just didn't care" about students or his teaching. As another example, a PHS student, Bethany, did not like that her 9th grade teacher presented a single method for solving problems; she liked choosing the way to solve the problem that she preferred. *Site Policy differences* represent those differences that had their origin in the decision of the sites' mathematics departments rather than individual teachers. For example, at U-M, the Mathematics Department, not the curriculum (the Harvard Calculus materials) or the instructors, developed the course component of group homework. Thus, when students noted this as a difference from high school, it was categorized as a difference in Site Policy.

Some differences by their nature consisted of complex interactions between categories, and, as such, were classified in the intersection of two or more of the categories. For example, at PHS, the use of graphing calculators depended on both the curriculum and the teacher's classroom decisions. As a result, differences in the use of the graphing calculators at PHS were classified as an interaction between Curriculum and Teachers/Teaching. However, at MSU, the Mathematics Department determined where and how graphing calculators were used; in some courses, graphing calculators were not allowed on the exams. Thus, noted differences in the use of graphing calculators at MSU were coded as differences in Site Policy.

Students at every site were able to report at least a few differences that they observed. In fact, participants may have simply reported differences because they were asked to do so, rather than because those differences were particularly salient. Thus, our next step was to determine for which students the differences between curricula had been significant. Differences were deemed to be significant to a particular student if he/she (a) reported them spontaneously, or (b) repeatedly mentioned them, or (c) gave them particular emphasis or attributed particular impact to them. Thus, the principal objective here was to require some indication of importance for and/or impact on the student.

Determining whether or not a student had reported a difference with sufficient emotion or emphasis to be deemed "significant" proved to be challenging. The emotion with which students communicated differences could not be easily determined from the transcripts. Often we had to rely on the interviewer's memory of students' responses during the interview, which made reliability analyses, both within and across sites, difficult to conduct. In addition, since interviewers both had a relationship with the participants and regularly requested information on the differences they noted, there is a potential for bias in judging the emotion and intensity of students' statements. Due to these problems, the frequency of reported differences, rather than the emotion accompanying statements, often determined the significance of the differences in our analysis.

Disposition towards mathematics

Another factor in our model of mathematical transitions is whether students have experienced a change in their disposition towards mathematics. The word "disposition" is used colloquially to refer to an attitude; thus we are interested in capturing changes in students' attitudes toward mathematics. More specifically, we use this factor to refer to students' interest in, attitudes about, beliefs toward, motivation to succeed in, and enjoyment of mathematics. Data on attitudes has come primarily from interviews; data on beliefs has come from survey data. Changes in attitude or belief may be accompanied by changes in actions, such as a decision to take more mathematics, but such changes are neither necessary nor sufficient. At present, our criteria for "significant" change in attitudes and beliefs are not clearly and objectively defined but rather are decided through discussion of individual cases by the project team. In the absence of convincing data, we score the disposition factor as "no change." Thus, our current scheme attempts to provide a label (yes or no) as to whether a student experienced a change in mathematical disposition. In addition, for those students labeled "yes", we try to characterize the disposition change as either becoming more positive or more negative.

For example, consider two students' mathematical dispositions from PHS: Kevin and Stacy. Both of these students exhibited a change in their attitudes toward their mathematics courses, but in different ways. Kevin was in the advanced track of mathematics courses. His disposition became more negative in high school: he found his high school mathematics classes "more boring" than middle school classes, and he wanted to be invited by the high school teachers to become more involved in class. In contrast, Stacy, who was in the lower track of mathematics courses, had a positive change in her disposition. She expressed a preference for high school mathematics due to fewer story problems, more equation solving, and a more "direct" approach to the mathematics.

At present, we are grappling with two issues related to how we operationalize disposition toward mathematics: (1) differences between our conceptualization and that of the literature on "mathematical disposition;" and (2) the challenges of rigorously assessing students' beliefs about the discipline of mathematics. With respect to this first issue, other characterizations of mathematical disposition make more explicit connections to the discipline of mathematics and also articulate ways in which a mathematical disposition has features that are unique to mathematics, as opposed to dispositions in other content areas (e.g., Yackel & Cobb, 1996). What would these alternate conceptualizations of mathematical disposition afford our analysis? A second challenge to our notion of mathematical disposition is methodological: how do we assess students' dispositions? We are currently working on the development of a more explicit method for qualitatively assessing students' beliefs about mathematics as expressed in interview data, and we hope that it can supplement what we have already learned about disposition from our existing, less formal methods and from survey data. Clearly, the construct of "mathematical disposition" is a valuable one in understanding students' transitions; however, we have come to realize that assessing disposition will require more careful work.

Learning approach

The fourth and final category that plays a role in our decision rule for determining mathematical transitions is the student's approach to learning mathematics. By learning approach we mean the autonomous actions that a student undertakes to learn mathematics. Thus a change in learning approach means that the student changes the kinds of actions they undertake to learn mathematics. The qualifier "autonomous" is included to distinguish actions that a student freely undertakes from those that are more or less mandated by teachers' decisions in the classroom. Students' approach to learning mathematics is how they organize themselves within the zone of their own autonomous activity; this zone may include actions undertaken in the mathematics classroom, but it should also include actions undertaken on the students' own time (e.g., individual study strategies) and activities that involve others as resources (e.g., going to see the teacher or professor away from class time, seeking tutorial help, and studying with peers). Some commonly reported changes in students' learning approach include changes in how the textbook was used as a course resource; the use of discussions outside class with the teacher, classmates, and friends as resources; and the sustained use of help-rooms and math help centers, particularly at the college level.

Determining what constitutes a "significant" change in a student's learning approach has been a difficult decision to make. We have decided that a significant change in this category is indicated by the presence of either (1) experimentation with new learning strategies, (2) laying aside old learning strategies, or (3) the use of old strategies in new ways. Because our insight into students' approaches to learning is dependent on their self-report, we look primarily to interviews and the journals for evidence of significant change.

One issue that potentially complicates our analysis is the relationship between learning approach and changes in achievement. For most students (e.g., those who are concerned about earning high grades) autonomous changes in learning approach tended to be initiated in response to a drop (or a perceived potentially imminent drop) in achievement. When a student is dissatisfied with her math grade or is worried that her grade might drop in the near future, she may attempt new learning strategies in order to remedy the situation. Thus, a change in a student's learning approach may only be indirectly linked to a change in curriculum, in that the curricular changes resulted in the grade drop. Other factors share this indirect link, such as a more demanding teacher or additional extracurricular activities, and these alternative explanations could be just as responsible for the student's grade drop as the curricular shift. As a result, in some cases, we have struggled to find evidence that students' changes in learning approach came as a direct result of the curricular discontinuities.

A second issue with the learning approach category is in our requirement that changes in actions are "autonomous"—that is, freely undertaken by the student rather than mandated by the teacher or by other outside influences. Particularly at the high school sites, we have found that changes in students' learning approaches are almost always affected (and in some cases, initiated) by factors other than the student. For example, Stacy, a ninth grade student at PHS, used a strategy in her high school math class that was new to her: reading the textbook before attempting homework problems. She began using this strategy because her high school textbook provided worked examples within each section, and she found reading through the worked examples before starting her homework to be helpful. Stacy did not use this strategy in middle school, most likely because her middle school math textbook did not provide worked examples. Should this change in learning approach be considered autonomous when she could not have used this strategy in the past, even if she wanted to, due to the different structures of the texts?

As was the case with mathematical disposition, we feel strongly about the importance of the category of learning approach in understanding students' experiences during a mathematical discontinuity, but we continue to search for ways to carefully and rigorously determine the "significance" of the changes that students in our sample report.

Illustrative cases

With the preceding description of the four categories in our mathematical transition decision rule, we now briefly describe 4 cases of mathematical transitions to illustrate some of the diversity in our participants' experiences. We have selected them from the nearly 80 in our corpus, not because they were typical or representative of experience at any site, but to illustrate different senses or "types" of transition. We include them to flesh out our basic claim in greater detail: Mathematical transitions in the context of current curricular reforms take different forms. Recall that our decision rule states that a student who had "significant" change in at least two of the four categories is considered to have had a mathematical transition. Each of the four students described below meet this criterion; in fact these four students could be considered to have had a "positive" transition, as each is happier in his or her current curriculum as compared to the previous one. Yet, despite these surface similarities, these four students represent very different flavors of mathematical transitions.

Stacy, a student at PHS, experienced a mathematical transition by virtue of showing significant change in her disposition toward mathematics and also by noting significant differences between her previous and current math class. She showed no change in her achievement in math class, as she was a consistent "A" student in both her reform (CMP) junior high and more traditional high school mathematics classes. But despite her success, she lacked confidence in her ability. The elements she valued in her high school math classes reflected this: She preferred clearly stated procedures for solving problems, tests with content clearly specified in advance, "notes" to guide work on those tests, and teachers who kept "order". The "story problems" in junior high were "harder" because the solution method was often unclear. Stacy's 9th grade experience in Algebra I increased her interest in mathematics; she was drawn to the explicit structure of equation solving. Stacy recognized key differences between CMP and Algebra I (e.g., non-routine, contextual problems and greater student ownership for mathematical thinking in the former). Thus, movement into a more traditional content improved her disposition toward the subject -- perhaps because she felt more secure and confident in her new setting.

Mimi, a student at LHS, experienced a mathematical transition since she noted significant differences and also changed her approach toward learning math. As with Stacy, she showed no change in her achievement, as she was an above average student in junior high and high school, in math and in other subjects. She found that the CPMP curriculum required her to think about what problems were asking for, rather than to remember a formula or general solution. This change was challenging at first but, in her first year of high school, she came to feel that she could understand math—a state that did not generally follow from prior learning via memorization and practice, as was the case in junior high. What she felt she needed to do in order to succeed in math changed; before, memorization and practice allowed her to earn As, but in high school, it became much more important to read and try to understand what problems (particularly word problems) asked her to do. So her important differences were closely tied to changes in how she tried to learn and succeed in mathematics. She preferred understanding what she learned, but it remains unclear whether her disposition toward mathematics has changed significantly in high school.

Although **Matt** looks identical to Mimi from the standpoint of our decision rule (he noticed significant differences and also had a change in learning approach), his experience is quite different from hers. Matt came to MSU from a CPMP high school program and placed into the first semester calculus. He described himself as an undisciplined high school student, doing only what was required to get good grades. He liked elements of his 3-year experience with CPMP but found the pace too slow. When his high school teacher slowed down for other students, he used class time to do his homework. When Matt landed in Calculus I at MSU, he felt lost in the "foreign language" of the traditional curriculum and failed the first test. Shaken, he went to see his faculty instructor who assured Matt that he had been placed in the right class and diagnosed Matt's problems in terms of some weak content and poor learning practices. Under his instructor's guidance, Matt dramatically changed the way he approached the work of Calculus and was eventually successful, earning a grade of 3.5 (on a 4-point scale). Matt saw differences between CPMP and MSU Calculus, both mathematically (e.g., contextual vs. purely symbolic problems) and more generally (e.g., faster pace in college). His approach to learning mathematics changed radically in Calculus and, in fact, led him to expect mastery of the content of his other classes in a way that he never had in high school. Matt's case illustrates that "difficult" transitions can have very positive effects.

Lissie, a student at U-M, experienced a mathematical transition by virtue of a significant change in achievement and in her approach to learning, as well as noticing significant differences. Lissie was an "A" student in all of her classes in high school. Although math was not her favorite subject in high school, she enjoyed considerable success in her math classes and particularly valued the individual attention from her teachers that came with attending a small school. Upon her arrival at U-M, she immediately disliked the reform calculus program. She had strong, negative feelings about her instructor; she felt the course moved too quickly; and she disliked the writing and the group work that were integral to the approach. Lissie's grades suffered, yet she was determined to succeed. She hired a tutor and, with his assistance, began devoting a tremendous amount of time and energy to her math class. The changes Lissie implemented in her learning approach ultimately allowed her grade to stabilize. Lissie eventually came to feel that she understood Calculus much more thoroughly and deeply than she did in high school, and she attributed her greater understanding in part to those features of the course that she had initially hated: having to work in groups and do a lot of writing.

Discussion and Conclusion

The goal of the work presented in this paper is to understand and describe students' perspectives and experiences as they move between traditional and reform mathematics curricula. We have hypothesized that such fundamental shifts in mathematics curricula serve as potential sites for students to experience mathematical transitions. In this paper, we provide our current answer to the question, what is a mathematical transition? Our conceptualization depends on four components: achievement in mathematics, significant differences that students notice and report, disposition towards mathematics, and approach to learning mathematics. While we feel each component lends strength to our framework, we have also identified challenging issues involved in determining each component's significance for our students. It is our hope that being explicit about the challenges we have faced in our analysis will inspire some feedback about the framework itself. Are these the "right" four factors to use? Are there other factors that we should consider? What do these four factors lend us in conceptualizing a mathematical transition? And what, if anything, does our choice of factors obscure? These are some of the issues we are grappling with as we continue our data collection and analyses.

References

Hirsch, C., Coxford, A., Fey, J., & Schoen, H. (1996). *Core-Plus Mathematics Project*. Everyday Learning.

Hughes-Hallett, D. H., Gleason, A. M., et al. (1994). *Calculus*. New York: Wiley.

Jansen, A., & Herbel-Eisenmann, B. (2001, April). *Moving from a reform junior high to a traditional high school: Affective, academic, and adaptive mathematical transitions*. Paper presented at the Annual Meeting of the American Educational Research Association, Seattle, WA.

Lappan, G., Fey, J., Friel, S., Fitzgerald, W., & Phillips, E. (1995). *The Connected Mathematics Project*. Palo Alto, CA: Seymour.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Schoen, H. L., Hirsch, C. R., & Ziebarth, S. W. (1998, April). *An emerging profile of the mathematical achievement of students in the Core-Plus Mathematics Project*. Paper presented at the Annual Meeting of the American Educational Research Association. San Diego, CA.

Smith, J., Herbel-Eisenmann, B., Breaux, G., Burdell, C., Jansen, A., Lewis, G., & Star, J. (2000, October). *How do students adjust to fundamental changes in mathematics curricula?* Paper presented at the Twenty-Second annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA), Tucson, AZ.

Star, J. (2001, April). *"Reform" at the collegiate level: Examining students' experiences in Harvard Calculus.* Paper presented at the Annual Meeting of the American Educational Research Association, Seattle, WA.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27, 458-477.

ALGEBRAIC REASONING DEMONSTRATED BY GRADE 4, 5, AND 6 STUDENTS AS THEY SOLVE PROBLEMS WITH TWO AND THREE UNKNOWNS

Galina Dobrynina
Wheelock College
gdobrynina@wheelock.edu

Mathematics educators recognize that algebraic thinking is a complicated higher-order process that cannot been mastered quickly. For this reason, they recommend that elementary school students begin the study of algebraic ideas early and build meaning of abstract concepts and processes gradually (Davis, 1985; Davydov, 1991; Greenes & Findell, 1999; NCTM, 2000; Usiskin, 1997). Although educators identify specific algebraic concepts and processes young students should learn in elementary school, there is a paucity of research that has examined students' knowledge of these concepts and processes prior to instruction.

The purpose of this study was to gain insight into the reasoning processes employed by grades 4 through 6 students prior to their formal study of algebra. Students solved six problems with two or three variables (unknowns) and recorded their solution methods. Solution methods were identified as arithmetic (guess, check and revise values of unknowns) or algebraic (identify relationships, compare equations, and replace unknowns with numbers.) The problems were presented using three problem settings: weight scales, advertisements, and frame equations.

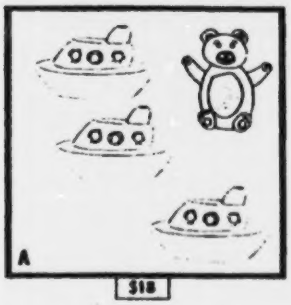
After a completion of the problems, students were interviewed to permit them to elaborate on their recorded solution steps. Students chosen for the individual interviews were those who successfully solved more than 50% of the problems and who demonstrated different solution methods. Of the 25 interviewed, 11 subjects used algebraic solution methods exclusively, 10 subjects used a combination of algebraic and arithmetic methods, and 4 subjects used arithmetic methods only.

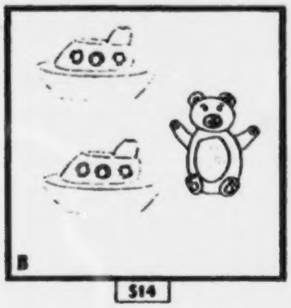
Solution methods employed by subjects who successfully solved the problems exhibited common approaches. For solving two-variable problems, subjects compared pictures A and B to find a value of an unknown and then replaced the value of the unknown into A or B to find the second unknown. For example, Al described these steps in the solution of the following two-variable advertisement problem:

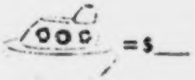
> *What is the price of each toy?*
> *- Same toys have same prices.*
> *- Different toys have different prices.*

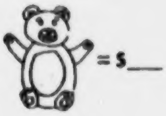
Interviewer: Tell me how you solved the problem. Start with what you did first.

Al: First I looked at these pictures and I saw that the difference was one of this.









Interviewer: One of what?

Al: The difference is one of the boats. And the difference is four dollars. That would mean that the boat is four dollars. I knew that this three boats is 14 dollars, so added two boats that would be 8 dollars. Fourteen minus eight equals six. Now I know the bear.

Interviews showed that more than 50% of the interviewed subjects were able to identify similar structures among two- and three-variable problems in different settings and recognized that the problems could be solved using the same strategy.

The study demonstrated that grade 4-6 students could use algebraic reasoning and successfully solve two- and three-variable problems despite the fact that they had not had any formal training in the use of algebraic solution strategies. Furthermore, algebraic solution strategies produced a 90% success rate. This suggests that: (1) students are ready to study and apply algebra earlier than has been expected, and (2) students in grades 1 through 3 may possess similar understanding.

References

Davis, R. B. (1985). ICME-5 report: Algebraic thinking in early grades. *The Journal of Mathematical Behavior, 4*, 195-208.

Davydov, V. V. (1991). *Psychological abilities of primary school children in learning mathematics* (Soviet ed., Vol. 6). Reston, VA: National Council of Teachers of Mathematics.

Greenes, C., & Findell, C. (1999). Developing students' algebraic reasoning abilities. In L. Stiff (Ed.), *Developing mathematical reasoning in grades k-12* (pp. 127-137). Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Usiskin, Z. (1997). Doing algebra in grades k-4. *Teaching Children Mathematics, 3* (6), 346-356.

EARLY ACCESS TO ALGEBRAIC IDEAS: THE ROLE OF REPRESENTATIONS AND MATHEMATICS OF CHANGE

Teresa Rojano
Cinvestav, México
mrojanoa@mailer.main.conacyt.mx

Elvia Perrusquía
Cinvestav. SEP, México
eperrusq@sep.gob.mx

For the teaching-learning processes of algebra to lead students to understand and use of the basic ideas of this mathematical field, must they receive early instruction on the principles underlying algebraic symbolic language? This is probably one of the questions that research groups (e.g., Roschelle,1998; Kaput, 1998a & 1998b; Carraher, et al., 1999 & 2000) interested in new proposals to the meaningful learning of algebra have more discussed in recent years. This research work aims to obtain related evidence and expanding on previous contributions to this issue.

Our study is based on the following guidelines: a) early introduction to algebra when students (10 to 11 years old) have not yet received formal instruction on algebra and its symbolic language; b) the use of computational environments (Spreadsheet and Math Worlds) as mediating tools between algebraic ideas and learning processes; c) exploiting the possibilities of early learning of algebra, through analysis and exploration of diverse representations of these ideas, produced simultaneously within computational environments; d) resorting to situations and scientific phenomena as contexts where algebraic knowledge may be applied.

The contributions that this study intends to make are oriented towards 10 and 11 year olds that have not yet received any formal algebraic instruction. Motion phenomena contexts are used to introduce algebraic ideas on the basis of the mathematics of change and variation, producing simulations and modeling activities in computational environments such as the Spreadsheets and Math Worlds.

Activity design is carried out by use of different kinds of representations of algebraic ideas in computational environments (charts, graphs, symbolic, numerical) where students can explore and manipulate abstract algebraic concepts oriented towards the development of strategies and abilities needed in situations that would presumably require mastering formal algebraic language. We argue that with this teaching approach, elementary school pupils may have access to the powerful ideas of mathematics of change and variation and, at the same time, start to build algebraic notions related to this mathematical domain (such as slope) without having to deal with algebraic formal symbolic language.

In this paper we discuss issues arising from two sources: a) a Pilot Study, which consists in the design or choice of activities in two versions (Spreadsheets and Math

Worlds), and trying these activities with two pupils (a girl of 11 years of age and a boy 11 years old); and b) an Experimental Study with two groups of elementary school students (10-11 year olds), one working with Spreadsheets and the other working with MathWorlds activities. Methodology is basically longitudinal case study. From the pilot study we report on cases in which evidence of basic algebra notions were found as they evolved in their work with Math Worlds and Spreadsheet activities. Elements from Duval's theory on representation systems in Mathematics (Duval, 1999) were used both for the activity design and in the analysis of children's productions, during the experimental sessions and in the individual interviews.

References

Carraher, D., Schliemann, A., & Brizuela, B. (1999). Bringing out the algebraic character of arithmetic. In *Proceedings of Annual Meeting of the American Educational Research Association*, Montreal, Canadá.

Carraher, D., Schliemann, A., & Brizuela, B. (2000). Early algebra, early arithmetic. In *Proceedings of the Twenty-Second Annual Meeting of the American Educational Research Association*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Duval, R. (1999). *Semiosis y pensamiento humano. registros semióticos y aprendizajes intelectuales.* Universidad del Valle, Colombia.

Kaput, J. (1998a). *Teaching and learning a new algebra with understanding.*

Kaput, J. (1998b). *Technology as a transformative force in education: What else is needed to make it work?*

Roschelle, J., Kaput, J., & Stroup, W. (1998). *SimCalc: Accelerating student's engagement with the mathematics of change.* Available online at: http://www.simcalc.umassd.edu.

A STUDY OF STUDENTS' TRANSLATIONS FROM EQUATIONS TO WORD PROBLEMS

Ana C. Stephens
University of Wisconsin-Madison
acstephens@students.wisc.edu

The National Council of Teachers of Mathematics (2000) calls for an increased focus on problem solving in the K-12 curriculum and suggests that algebra should play an especially important role, asserting that students in the middle and high school grades should "use symbolic algebra to represent situations and to solve problems" (p. 222). While much research has concluded that students have difficulties with word problems, this work has tended to focus on students' abilities to translate word problems into equations—in other words, to translate from verbal representations to algebraic representations.

This study took a different approach in that it examined students' translation work in the opposite direction, that is, from algebraic representations to verbal representations. Ninety participants (Algebra I and Algebra II students) were given three linear, one-variable equations and were asked to write word problems that these equations could be used to represent. Students' written work was analyzed and initially classified as either correct or incorrect. Incorrect responses were further classified based on the nature of the error. For example, given the equation $36 = 2r$, a student whose work was classified as incorrect and later categorized as "changing the unknown" wrote, *"There are 2 rows of 18 eggs in each row. Both rows put together make how many eggs?"* Eleven of the participating students were later interviewed and asked to elaborate on their thinking as they took part in the task so that further insight could be gained about the various categories of student responses.

The purpose of this exploratory study was first to examine the nature of students' responses as they translate algebraic equations into word problems and second to investigate students' understandings of variable, equality, and the structure of algebraic equations. Hiebert and Carpenter (1992) assert that a mathematical idea is understood if it is part of an internal network—that is, if its mental representation is part of a network of mental representations. Unfortunately, most curricula treat mathematical representations as if they were ends in themselves (NCTM, 2000) and treat translations between representations in a strictly unidirectional manner (e.g., by having students translate from word problems to equations and never in the opposite direction). It was therefore hypothesized that the students participating in this study would be unlikely to provide rote, proceduralized responses in response to the given task and that their understandings could thus be more accurately assessed. This opposite-from-traditional approach also addresses a shortcoming in the research on the algebraic equation-word problem connection, as the thinking required to get from equation to word problem may not be the same as the thinking required to get from word problem to equation. The assessment of students' algebraic understandings and misunderstandings draws heavily from Kieran's (1989) procedural/structural understanding framework.

A limited number of categories of student responses emerged in this study, indicating that student thinking was not inconsistent and incoherent but rather could be analyzed in light of findings of previous research on word problems and algebra in general. The results of the study suggest that while a fair number of students experienced success translating from equations to word problems, many others may have lacked the structural understanding of equations, as described by Kieran (1989), necessary to succeed with the task. These findings in combination with previous research support the idea that students should have bi-directional experiences translating between equations and word problems in order to deepen their understanding of the equation-word problem connection.

References

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 35-36). Reston, VA: Erlbaum.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: The National Council of Teachers of Mathematics.

SCHOOL ALGEBRA AS LANGUAGE: THE FIRST ELEMENTS OF ALGEBRAIC SYNTAX AND SEMANTICS

Armando Solares Rojas
Department of Mathematics Education-CINVESTAV, Mexico
asolares@mail.cinvestav.mx

This presentation describes a piece of research of the project "School algebra as a language. The first elements of algebraic syntax and semantics" which is being carried out in the PhD program of the Department of Mathematics Education at CINVESTAV, Mexico.

In the perspective of the research program proposed by Filloy, Puig and Rojano (Rojano, 1994) it is set out to probe the mutual influence between semantic and syntactic aspects of algebra at different moments of the algebra learning process. In this scope, the present study intends to investigate the "first meanings" attributed to the algebraic expressions and their transformations, and the way these meanings evolve when novices use the algebraic language to solve linear equations with one unknown, linear systems with two unknowns and quadratic equations.

We discuss a number of epistemological obstacles (in the sense of Bachelard, 1970) that are present in the historic development of equation solving methods. On the basis of a previous analysis, carried out by Filloy and Rojano in the study *Acquisition of algebraic language: the operation of the unknown* "(1984, 1985, 1989), conjectures on didactic obstacles that children may encounter when learning *to use* the algebraic language will be formulated. For this purpose, we use the idea of Local Theoretical Models, developed by Filloy (Filloy, 2000) as our theoretical background. Specifically, we will use the concept of *Mathematical Sign System,* in order to reformulate some unanswered questions from the study of Filloy & Rojano, concerning student's procedures, *cognitive tendencies* (Filloy, 1991) and personal codes that appear when they solve algebraic tasks, as those mentioned above.

Later on, within the study "School algebra as a language", a clinical study with 11-15 year olds will be carried out to investigate both, the conjectures formulated upon the basis of the epistemological analysis, and the questions reformulated under the new theoretical approach.

References

Bachelard, G. (1970). *La formation de l'esprit scientifique*. VRIN, Paris.

Filloy, E. (1991). Cognitive tendencies and abstracction processes in algebra learning. In F. Furinghetti, (Ed.), *Proceedings of the Fifteenth Annual Meeting of the Psychology of Mathematics Education*, (Vol. 2, pp. 48-55). Italy.

Filloy, E. (2000). *Aspectos teóricos del albgebra educativa*. Grupo Editorial Iberoamericana, México.

Filloy, E., & Rojano, T. (1984). From an arithmetical to an algebraic thougth (A clinical study with 12-13 year olds). *Proceedings of the Sixth Annual Meeting of the Psychology of MathematicsEducation-North America.* (pp. 51-56). Madison, Wisconsin.

Filloy, E., & Rojano, T. (1985). Obstructions to the acquisition of elemental algebraic concepts and teaching strategies. *Proceedings of the Ninth Annual Meeting of the Psychology of Mathematics Education,* (pp. 154-158). Utrech, Holand.

Filloy, E., & Rojano, T. (1989). Solving equations: the transition from arithmetic to algebra. *For the Learning of Mathematics*, *9* (2), 19-25.

Rojano, T. (1994). The case of pre-symbolic algebra and the operation of the unknown. In J. da Ponte, & Matos, J. F. (Eds.), *Proceedings of the Eighteenth Annual Meeting of the Psychology of Mathematics Education* (Vol. 1, pp. 125-128). Lisbon, Portugal.

Learning and Cognition

REMNANT EFFECT OF BASIC SCHOOL MATHEMATICS IN MEXICO

Silvia Alatorre
National Pedagogical University, Mexico City
alatorre@solar.sar.net
Contributors: Natalia de Bengoechea, Elsa Mendiola, and Ignacio Méndez

Abstract: This paper reports a survey study carried out with adults in Mexico City, to which 20 Basic-School mathematics questions were posed, directed chiefly to Arithmetic. The results are analyzed both qualitatively (specially regarding the kinds of mistakes encountered) and quantitatively (relation of overall results with variables such as age, sex, schooling, primary school curriculum, etc.). This study can be considered as an evaluation of the long-term effect of Mathematics Education in the basic levels.

Purposes and Background

The main purpose of this study is to know how much of the mathematical knowledge acquired in Basic School remains in adult Mexican subjects. Secondary purposes are the study of the kinds of mistakes made by subjects with the possible identification of errors that can be attributed to school environment, and of the relationship of each person's performance with exogenous variables such as age, sex, income, primaryschool curriculum, schooling, etc.

In Mexico, serious efforts have been made to improve the quality of basic education and to cover a large percentage of the population. National curricula have existed since the beginning of the XX century, and since 1960 all children in public and private primary schools receive free textbooks, which are the same for all. There has been nevertheless no investigation about how much of the mathematical knowledge acquired in Basic School remains in adult subjects, nor to find out the differences in this remnant effect among the national curricula over the years. The research reported in this paper aims at filling this gap, investigating the knowledge adults have of the mathematics taught at the first six or eight years of schooling.

Several considerations have to be made about the meaning of the word "knowledge" in the last sentence. Although one of the purposes of all the curricula has been to educate individuals to be able to solve the mathematics involved in daily activities, some researches have shown that this ability bears little relationship with the amount of years of schooling or the success at school, but is more related to the experience of the individuals in the areas where mathematics must be used and applied (see for instance Lave, 1998). Carraher et al., (1991) have remarked that most of the time the mathematics taught at school are of no use in daily life, because they do not involve the solution of real problems in real life. Students work to satisfy the teacher's need to have them use whatever he or she is teaching, so that the problem is "solved" when the procedure is used and a number is produced. In this training, students lose all interest in the solution of the problems and most often they do not even attempt to verify the plausibility of the result. "Solutions" of school problems are different from solutions of real-life problems, because in the latter the meaning has a fundamental role in decision making and in the former it has not.

This consideration might lead to a case study research in which a qualitative analysis of the use of mathematics in adults' daily or professional life would be carried out, preferably *in situ*. However, in order to conduct such a study it is necessary to have an overview of the school mathematical knowledge that has remained over the years in adults. This overview is what this study aims at.

A second consideration arises when one must precise the meaning of "school mathematical knowledge". Different primary school curricula over the years have emphasized different aspects. For instance, regarding multiplication, for some the most important aspect was the skill in multiplication tables, for some the knowledge of the formal aspects (commutativity, associativity, etc.), for some the concept of multiplication as an iterative sum, for others the ability to recognize in a problem when a multiplication must be performed. Since this research included subjects having had their basic school formation in different curricula, it became necessary to choose a meaning for "knowledge" that was covered by all of them; this was the ability to do basic arithmetic operations and to solve simple word problems, thus covering a basic numeracy.

Method of Inquiry and Data Sources

A survey was conducted in Mexico City. Two mathematics questionnaires were designed, each consisting of twenty questions conceived in related pairs in order to test additional hypotheses, such as the effect of a context, of the type of numbers involved, of the position of the unknown, etc. The twenty pairs of items were either questions about operations with simple natural, fractional or decimal numbers (for instance: "How much is 9.25 × 10?", or "How much is one half of ½?"), or simple problems of the kinds traditionally used in schools (for instance: "If four ham sandwiches cost 60 pesos, how much do three ham sandwiches cost?"). 90% of the items relate to contents taught in primary school, and 10% to the first years of secondary school. The questionnaires asked questions in an open fashion (no predetermined categories; these were constructed after the answers were obtained).

Each of the questionnaires was applied to a sample of 792 subjects aged 25-60, who were approached in the streets of Mexico City in December 1998, and asked to participate in a study about mathematical knowledge. They also answered to a questionnaire of general data; the interviews lasted an average time of 40 minutes. The sampling handled quotas for schooling and socio-economic levels (the latter through city zones). Both samples were then tested and approved for representativeness for these and other variables such as age, against the data provided by the National Institute of Statistics for the city's population. (INEGI, 1990 and 1995).[1]

The 25-60 age range allowed to have people who started their primary school from 1944 to 1979, and three national curricula were distinguished in that span: 1944, marked by a institutionalization epoch, 1960, characterized by the beginning of the free national textbooks, and 1972, distinguished by the "new maths" approach (see Meneses, 1997-1998). The newest curriculum, which was introduced in 1993, was not represented, since subjects who have been following it have not reached the age of 25.

Results

Analysis of Answers

The overall percentage of correct answers in the mathematics questionnaires was 47%. This is a low average, since most of the items belonged to the primary level of schooling and 82% of the subjects had at least finished this level; the worst results were obtained in items about decimal numbers. For most of the questions, people with increasing levels of schooling had increasing probabilities of giving the correct answers. This behaviour was not only for the levels going from no schooling to secondary school, but also for further levels. People with no schooling had very satisfactory success rates in 21% of the questions, showing that some contents are learnt without the need of schooling, through the teachings of daily adult life. Generally speaking, people with no schooling had worse results than people having done at least a few years of primary school. However, in 13% of the questions people with no schooling had higher percentages of success than people with incomplete primary schooling. These were questions in which little schooling is worse than no schooling at all; here is an example of them: "Yesterday these trousers cost 200 pesos; today they cost 250 pesos. In what percentage were they increased?"

Several common mistakes give insights about some phenomena linked to the teaching of mathematics in basic school. They corroborate that mathematics taught in school have little relation to reality, and that students learn to use algorithms without checking out the plausibility of the results. In some questions some incorrect answers showed the following behaviour: People with increasing levels of schooling had increasing probabilities of making these mistakes (i.e., they are more often committed by people who have had a longer exposure to school). These were interpreted as *school mistakes*, that is, either originated in school or not corrected by it. In some cases school mistakes could be identified as incompletely learned algorithms, or memorized formulae without the knowledge of when to apply them, or incomplete procedures. An example of school mistake is the answer 1.5 > 1.30 > 1.465, probably guided by the idea that the more decimal digits a number has, the smallest it is; the probabilities of giving this answer rise from 0.03 for people without schooling to as much as 0.20 for college educated people.

Relationship Between Total Scores and Exogenous Variables

For each subject the total score (amount of correct responses among the 20 items) was computed. This score for all 1584 subjects interviewed was taken as a response variable in a statistical model with several exogenous variables. The main results are as follows.

Age had no statistical significance. *Sex* had a statistical significance: men showed a slightly higher mean response than women; this must be related to the latter's low self-esteem towards science and mathematics. *Per-capita family income* had a strong quadratic effect, the best results being reached by middle-class people. *Primary school curriculum* that subjects followed in their primary school had no statistical significance. *Schooling* had a high statistical significance. *Wearing off*, which was defined as the rate between the number of years after school and age, and as 100% for subjects who had no schooling, had a high significance. *Daily use of mathematics* was measured through several variables: frequently measuring, weighing and handling intermediate and large amounts of money had statistical significance, and doing arithmetic operations and handling small amounts of money did not.

Conclusions

The effect of schooling is a double face coin. On the one hand, the fact that subjects having done secondary schooling answer better than those who only did complete or incomplete primary studies, and that these in turn answer generally better than people with no schooling whatsoever, indicates that school did do its job: The contents of mathematics in basic education were incorporated (albeit relatively) to the students' knowledge and remain there even after several years. On the other hand, notwithstanding, the fact that this increasing effect appears also in the levels of high school and college has two interpretations, which are not mutually contradictory. The first one is that the basic education levels were not enough to acquire and consolidate this basic knowledge, and that it was finally learnt at levels where it should be treated as obvious. The second one is that only those subjects who did acquire and consolidate this knowledge in basic levels gained access to superior levels.

In the analysis of the effect of schooling, the subjects with no schooling constitute a particularly interesting group. The analysis shows that when daily life and school compete to be the stage where knowledge acquisition takes place, the latter is not always the best, and that low levels of schooling can even be damaging, in the sense that they hinder the development of intuition and common sense for the solution of mathematical problems, and impose upon them algorithms that are ill-learnt and applied. This is also the case, although more dramatically, for the mistakes which were identified as school mistakes, where even high levels of education act as stimuli for making them.

Another interesting finding is the non-effect of primary school curriculum. This might mean that the curricular changes were fruitless, or that curricular changes alone do not solve the problem of mathematics teaching. There is notwithstanding a related aspect that must be considered. Although there are no statistically significant differences in the performance in basic mathematics of adults who followed their primary school with different curricula, these do differ from each other by the size of the population they attended: with each new curriculum the attended population was practically duplicated, the percentage of children who at least started primary school grew and so did the percentage of students who finished it. That is, curricular changes may have had no influence on the mathematical knowledge that adults keep after years go by, but on the other hand the system attended more and more students, and in a more and more complete way. Given these quantitative results, the quality of education could have been expected to decrease, and yet, at least according to the findings of the research reported in this paper, it did not happen. This is an achievement of the Mexican educational system.

References

Carraher, T., Carraher, D., & Schliemann, A. (1991). *En la vida diez, en la escuela cero.* México: Siglo XXI

INEGI. (1990). *Resumen general del XI censo general de población y vivienda, 1990.* México: Instituto Nacional de Estadística, Geografía e Informática.

INEGI. (1995). *Conteo 1995.* México: Instituto Nacional de Estadística, Geografía e Informática.

Lave, J. (1998). *Cognition in practice.* Great Britain: Cambridge University Press.

Meneses, E. (1997-1998). *Tendencias educativas oficiales en México 1934-1964, 1964-1976, 1976-1988.* México: Centro de Estudios Educativos A. C., Universidad Iberoamericana.

Note

[1] The distribution of schooling for Mexico City in this age group is: no schooling, 5%; incomplete primary school (1-5 years), 13%; complete primary school (6 years), 18%; secondary (9 years), 21%; high school (12 years), 19%; college or more (more than 12 years), 23%. (INEGI, 1990 and 1995).

AN INVESTIGATION OF COVARIATIONAL REASONING AND ITS ROLE IN LEARNING THE CONCEPTS OF LIMIT AND ACCUMULATION

Marilyn P. Carlson
Arizona State University
marilyn.carlson@asu.edu

Sean Larsen
Arizona State University
larsen@math.la.asu.edu

Sally Jacobs
Arizona State University
sally.jacobs@mcmail.maricopa.edu

Abstract: This study investigated the effect of research-based curricular materials on the development of first semester calculus students' covariational reasoning abilities. It also explored the role of covariational reasoning in the development of the concepts of limit and accumulation. We found evidence that the covariation curriculum was effective and that the reasoning abilities promoted in the activities were useful for completing select limit and accumulation tasks.

Introduction

Concepts of rate of change and changing rate of change are central to the study of calculus, but it is not clear how students acquire the ability to interpret and represent continuously changing rate for a function's domain. Further, little is known about the importance of covariational reasoning (coordinating images of two varying quantities and attending to the ways in which they change in relation to each other) in students' study of other concepts of calculus (e.g., limit and accumulation). This paper reports results from investigating calculus students' understanding of covariation. It extends the first author's (Carlson, 1998) and others' (Saldhana & Thompson, 1998) study of covariation and builds on previous research investigating the concepts of limit (e.g., Sierpinska, 1987a; Cornu, 1991; Tall, 1992; Cottrill et.al., 1996; Szydlik, 2000) and accumulation (e.g., Thompson, 1994). Covariational reasoning has been shown to be an important ability for interpreting, describing and representing the behavior of a dynamic function event (Saldhana & Thompson, 1998; Carlson, 1998). In addition, Thompson (1994) has observed that images of rate are fundamental for reasoning about accumulation. However, no previous research has investigated the role of covariation in the development of students' understanding of the major conceptual strands of calculus. This paper reports on the covariational reasoning abilities of first semester calculus students and provides initial insights regarding the role of covariational reasoning in responding to specific limit and accumulation tasks.

Theoretical Model

The author's past study of function and covariation (Carlson, 1998) produced a framework for describing students' covariational reasoning abilities. This framework includes five categories of mental actions (Table 1) that have been observed in students when applying covariational reasoning in the context of representing and interpreting a graphical model of a dynamic function event. The initial image described in the framework is one of two variables changing simultaneously. This loose association undergoes multiple refinements as the student moves toward an image of increasing and decreasing rate over the entire domain of the function (Table 1). Describing these actions in the form of a framework provides a powerful tool with which to analyze covariational thinking to a finer degree than has been done in the past. It also provides structure and research based information for building curricular activities, as are described later in this paper.

Table 1. Covariation Framework

MA1)	An image of two variables changing simultaneously
MA2)	A loosely coordinated image of how the variables are changing with respect to each other (e.g., increasing, decreasing)
MA3)	An image of an amount of change of one variable while considering changes in discrete amounts of the other variable
MA4)	An image of rate/slope for contiguous intervals of the function's domain
MA5)	An image of continuously changing rate over the entire domain

Methods

The subjects in this study were 24 first-semester calculus students at a large public university in the United States. The classroom was student-centered involving groupwork and interactive lectures. The instructor for the course was the first author, with the second author observing students' thinking and noting curricular effectiveness. A traditional text was supplemented with a set of curricular modules developed (by the authors) for each of the course's major conceptual strands. The development of each curricular module was guided by insights gained from the research literature on function, covariation, limit and accumulation. Each module included a collection of activities (both in-class and take-home) designed to engage students in sense making while building their understandings of these concepts. The data collection consisted of pre- and post- test assessments of students' covariational reasoning abilities. Follow-up interviews, designed to assess students' covariational reasoning and concept development, were conducted with 8 students after completing each curricular module and at the end of the semester. Interview subjects were selected to represent the diversity of the class' covariational reasoning abilities. Each interview subject had also been videotaped while completing the in-class activities in their individual groups.

The data analysis involved a comparison of pre- and post-test results. Interview and video-taped data were coded, categorized and classified using the covariation framework and general trends were observed and noted.

A Description of the Covariation Module

This module consisted of five separate activities designed to promote students' ability to attend to the covariant nature of dynamic functional relationships. Each activity contained a collection of prompts that encouraged students to coordinate an image of the two variables changing and to attend to and represent the way in which the independent and dependent variables changed in relationship to one another. Since previous results (Carlson, 1998) have reported that a process view of function is needed for applying covariational reasoning, each activity also included prompts that encouraged students to imagine a set of input values being acted upon by a process (or function-machine) to produce a set of output values. The first of the four activities prompted students to construct a maximal volume open-top box from an 8.5 x 11 piece of paper by cutting equal squares from the corners and folding up the sides. After construction of an algebraic model for this situation, students were prompted to determine both the box size for various values of x, and values of x for various box sizes. Also included in the set of writing exercises for this activity were prompts that focussed students' attention on the varying nature of the input and output variables. Students were asked to determine how changing the size of the square affected the size of the volume, and to specifically determine the range of values of x (the length of the side of the square) for which the box's volume was increasing. Written responses were required of each student, with class discussions emphasizing representational connections, in particular the connection between various boxes and corresponding points/values on the graph.

The second covariation activity presented a variety of distance-time graphs, followed by prompts for students to model, through their own motion, the behavior represented by these graphs and to produce a written description of their motion. With the use of motion detectors and a real-time display of the distance-time graph, students were able to acquire immediate feedback of the effect of their motion, and were encouraged to make adjustments as needed until their motion produced the desired graph. (Even though the terms concave up, concave down and inflection point had not been introduced, students were given graphs that changed from concave up to concave down and were expected to draw on their informal intuitions about speed and acceleration to determine how to walk these graphs).

The third activity took the form of a model-eliciting activity (Lesh, Hoover, Hole, Kelly, & Post, 2000), with the six principles of model-eliciting activities guiding its development. The activity was based in a realistic situation that required the construction of a mathematical model, while also requiring students to generalize and document their thinking. Students were also asked to produce a strategy guide for a physics student preparing for a lab exam in a physics class. They were instructed to produce a detailed set of general instructions to explain how to produce a "walking motion" for any possible graph.

The fourth activity was based on the familiar sliding ladder problem that asks students to determine the relative speed of the top of a 10-foot ladder as the base of the ladder is pulled away from the wall. Since these activities were completed early in the semester, the students did not have access to the tools of calculus, and therefore relied on algebraic and covariational reasoning when thinking about this task. In addition to answering the question about the behavior of the top of the ladder, prompts were included to promote the thinking and connections described in the covariation framework.

The fifth activity was designed to adhere to the six principles of model-eliciting activities (Lesh et al., 2000). This activity prompted students to provide a graph to represent the height as a function of volume for animating the filling of a container on a screen, and to produce a manual for a company to animate the filling of any shaped container. The reason for asking students to produce a manual was to place them in a situation where they needed to organize and verbalize their understandings with a level of clarity and detail that could be understood by others. We believe that this type of task is extremely valuable in promoting a robust and flexible understanding in students (Lesh et al., 2000).

A similar set of activities were developed for the concepts of limit and accumulation. While each activity had a primary conceptual focus on the calculus concept, whenever possible, prompts were included to encourage students to connect their newly developed calculus concept with their more informal knowledge that was acquired in the context of the covariational reasoning module. Students were provided tasks that required them to unpack their calculus understanding to explain such things as why a concave up graph conveys rate of change increasing or why an inflection point corresponds to the point where the rate changes from increasing to decreasing or decreasing to increasing. As appropriate, the problems used in the covariation module were revisited and extended in the limit and accumulation modules. The covariational reasoning abilities that emerged in these subjects over the course of one semester are reported. This is followed by two examples of how covariational reasoning was used when responding to specific limit and accumulation tasks.

Select Results

Results from a Covariation Task

Comparison of pre- and post-test data revealed positive shifts in these students' covariational reasoning abilities over the course of one semester. On a pretest assessment administered on the first day of class, students were asked to select the graph (from four possible graphs) that represented the height as a function of volume as water filled an empty spherical bottle at a constant rate. The four choices were an increasing straight line, a graph that was strictly concave up, a graph that was strictly concave down and the correct graph that was initially concave down; then concave up. Of the 24 students who completed both the pre- and post-test assessment, 13 selected the graph of the increasing straight line, 6 selected either the strictly concave up or strictly concave down graph and only 5 selected the correct graph.

On the post-test assessment (administered on the last day of class) students were asked to produce a graph of the height as a function of volume for a container with a spherical bottom, a cylindrical middle and a funnel at the top (Figure 1). They were also prompted to provide a clear description of the thinking they used to construct their graph. 23 of the 24 students provided a mostly correct response (i.e., a graph that was concave down, then concave up, followed by a linear segment, then another concave down construction). Only 8 of the 24 students had a minor error in the construction of the linear segment of their graph—the slope of the linear segment was not the same as the rate/slope of the tangent line at the point to the left of the linear portion).

Construct a graph of the height as a function of volume for the given container.

Explain the reasoning you used when constructing your graph.

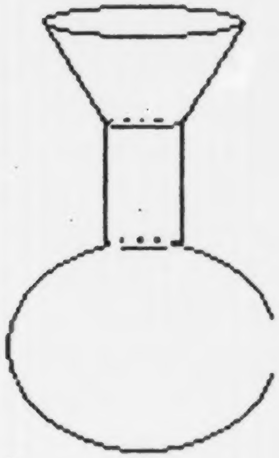


Figure 1. The Bottle Task.

The written justifications for their constructions varied in form, with seven different categories of reasoning emerging from their responses. The various response types (each presented in the words of a student) are followed by the numbers of students that provided that response and a classification of that response according to the covariation framework (e.g., MA 5).

a) For the bottom half of the bottle the height was increasing but at a decreasing rate; then after the middle, the height increased at an increasing rate (7 students) (MA 5).

b) At first there is an increasing amount of volume needed as the height raises; then there is an increasing height as the volume changes evenly (1 student) (MA5).

c) As the bottle gets wider, the rate it filled decreased; then as the bottle gets narrower the rate if filled increased (7 students) (MA5).

d) As the cross-sections get progressively wider the height increases at a slower rate because it takes more volume to fill the container evenly—as the cross-sections get smaller the height increases at a quicker and quicker rate (2 students) (MA5).

e) From the initial filling, the rate of the height increases more slowly than the volume, resulting in a concave down shape; then at the halfway point the rate of increase of the height increased more and more rapidly resulting in a concave up graph (3 students) (MA5).

f) Longer to fill results in a concave down construction and less time to fill results in a concave up construction (2 students) (not enough information to classify).

g) If the bottle is getting wider, it is concave down and if the bottle is getting narrower it is concave up (2 students) (not enough information to classify).

These results suggest that, at the completion of the course, most of the students in this study were reasoning at the MA5 level according to the covariation framework (i.e., they able to construct continuous images of increasing and decreasing rate while imagining the input variable changing over the entire domain). This result is particularly striking compared to previous results (Carlson, 1998) reporting that 75% of high performing second semester calculus students at the completion of the course were not able to create the correct graph for this situation. These students produced a straight line or a strictly concave up or concave down graph for this situation, while providing MA2 level reasoning for their justifications. Follow-up interviews with select students also revealed little or no ability of these students to attend to the covariant nature of this situation.

Follow Up Interviews

The follow-up interviews with eight students provided additional insights regarding the abilities of these students to access and apply covariational reasoning in a variety of situations. In addition to prompting students to provide a verbal explanation of the reasoning they used when completing the bottle task, the interview subjects were also asked to talk though the thinking they employed when responding to three additional dynamic tasks. As one example, students were asked to produce a graph of the distance (i.e., shortest distance from the starting line) as a function of time for a mile race that Tim ran at a steady pace on a quarter mile oval track. When completing this task, seven of the eight interview subjects produced an accurate construction of this situation, with their justifications again revealing diversity in their explanations, while their reasoning revealed a consistent patterns in their covariational reasoning abilities. Student responses on these tasks further confirmed that these subjects were able to effectively access and apply covariational reasoning when thinking about unfamiliar dynamic situations.

Results from A Limit Task

When asked to determine the $\lim_{x \to 2}$ for two different functions (e.g., an upside down parabola with a point removed at $x = 2$; a graph of a split function defined by

y=3 for $x <= 2$ and $y = 1$ for $x > 2$) most students were able to determine the limits as x approached 2 for these functions (only 2 of the 24 provided an incorrect response to the first item, and 3 of the 24 students provided an incorrect response when asked to evaluate the limit as x approaches 2 for the second function).

During the follow-up interview all eight students provided explanations that exhibited a consistent pattern of coordinating an image of the independent and dependent variables changing concurrently. Sue described her thinking to the first task by saying, "I thought about values of x that were approaching 2 from the left side and thought about what the y-value was getting close to, and this was 3. Then I thought about x-values getting closer and closer to 2 from the right and looked at what the y-value was getting close to while the x got closer and closer to 2. Since when x got close to 2 from both sides, y gets closer and closer to 3, this means that the limit is 3…it doesn't matter that it never really gets to 3. The limit is what it gets close to…I mean what the y-value gets close to as x gets close to some value from both sides". Even though one student did not provide the correct response, his difficulty appeared to result from the fact that he did not understand that y must approach the same value when x approaches 2 from both the left and the right. This finding is not surprising as it suggests that a robust and flexible covariational reasoning ability, although foundational for this task, does not assure that students will produce a correct response, as other difficulties/misconceptions may exist.

Results from an Accumulation Task

The following accumulation task (Figure 2) was administered on the post-test assessment given on the last day of class.

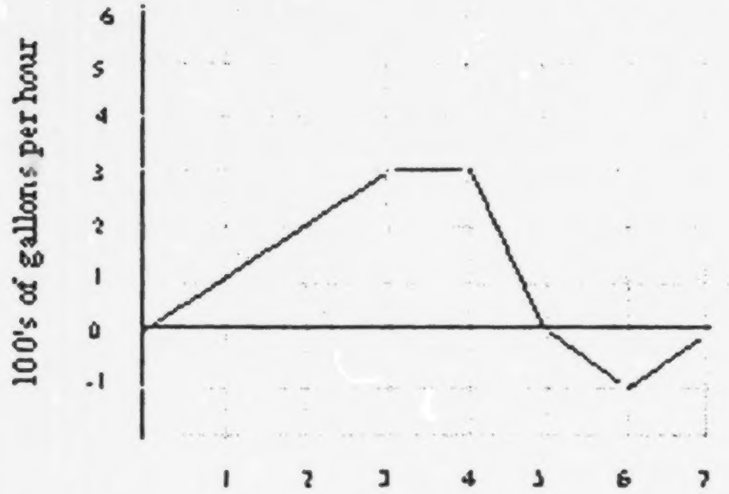


Figure 2. The Accumulation Task.

Let $f(t)$ represent the rate at which the amount of water in Tempe's water reservoir changed in (100's of gallons per hour) during a 7 hour period from 10 a.m. to 5p.m. last Saturday (Assume that the tank had 5700 gallons in the tank at 10 a.m. (t=0)). Given the graph of f,

a) Determine how much water was in the tank at noon.

b) Determine the time intervals when the water level was increasing.

c) Construct a graph of g, given that $g(x) = \int^{x} f(t)dt$.

d) Explain what $g(x)$ conveys about this situation.

e) Determine the value of $g(7)$.

On this item 21 students provided a correct response to parts b and c and 22 students provided a correct response to parts a, d, and e. The follow-up interviews suggested that these students were able to coordinate an image of the area accumulating while imagining the independent variable changing continuously. When prompted to explain the reasoning he used to construct the graph of g, Bill conveyed, "you just need to think about how much area has been added on up to that point, and then that amount gives you how much….I mean that value tells what the y-value for g is….so g gives you the amount of area that has been accumulated in the f function". Even more surprising was the fact that 21 of the 24 students in this study were able to accurately construct the graph of g. This type of task has been previously reported to cause difficulty for students (Thompson, 1994). Further analysis of the interview responses revealed that 7 of the 8 interview subjects appeared to determine the value of the output for g by continuously coordinating an image of the accumulating area under the graph of f while imagining the time changing continuously. This result was particularly encouraging and appears to provide support for the effectiveness of their covariational reasoning abilities, and in particular their ability to coordinate images of rate accumulating while concurrently thinking about the input variable changing.

When prompted to explain their reasoning for item e above, all eight interview subjects computed the areas for contiguous sections of the graph and proceeded to explain this situation. The interview subjects each conveyed that adding the areas corresponded to the accumulation of the rate, and that the accumulation of rate for the specified time interval corresponded to the change in the amount of water in the tank from t=0 to t=7.

Concluding Remarks

These results suggest that, at the completion of the course, most of the students had developed flexible and robust covariational reasoning abilities. These results are particularly encouraging as they convey that these research based curricular materials were effective in promoting covariational reasoning in this diverse collection of first semester calculus students.

In reviewing the results for the limit and accumulation problems, it appears that these first semester calculus students were able to effectively apply their covariational reasoning abilities in a variety of contexts; in particular when thinking about conceptual tasks for two major ideas of calculus. The images that emerged in these contexts suggest that covariational reasoning may be foundational (using the term of Tall—a cognitive root) for understanding and completing specific tasks for limit and accumulation

References

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education, 1* (7), (115-162). Providence, RI: American Mathematical Society.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema. *Journal of Mathematical Behavior, 15* (2), 167-192.

Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 153-166). Boston: Kluwer.

Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. Kelly & R. Lesh, (Eds.), *Handbook of research design in mathematics and science education* (pp. 591-646). Mahwah, NJ: Erlbaum.

Saldanha, L., & Thompson, P. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education North America*. Raleigh, NC.

Szydlik, J. (2000). Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for Research in Mathematics Education, 31,* 258-276.

Tall, D., & Schwarzenberger, R. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching, 82,* 44-49.

Thompson, P. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26,* 229-274.

Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education, 22,* 219-236.

THE ROLE OF CHILDREN'S SPONTANEOUS CONCEPTS ON THE DEVELOPMENT OF REPRESENTATIONS FOR MATHEMATICS LEARNED IN SCHOOL

Rochelle Goldberg Kaplan,
William Paterson University

The rationale for this study is based on the premise that the outcomes of children's learning are specifically shaped by the interaction of their individual cognitive structures with presentations of curricular content in the classroom. Therefore, since the interpretive tools that children apply to instructional content may vary substantially even within the same classroom, the representations they construct about curricular content are unlikely to have uniform meaning. Moreover, some of these representations may not necessarily be synchronous with the conceptual learning goals intended by curriculum developers. It falls to teachers, then, to be cognizant of these the potential pupil variations in order to adapt curriculum to take into account the ways in which the same seemingly objective content will be interpreted to mean different things to students with different preconceptions.

The intent of this study was to examine the ways in which first grade children's spontaneous concepts informed their understanding of school-taught ideas dealing with elements of algebraic reasoning. In particular, the study assessed children's responses to written activities in the school's developmentally sequenced spiraling curriculum as a function of differences in their previous knowledge of equivalence and related concepts. It was predicted that despite common instructional opportunities designed to "draw on the child's 'rich store of mathematical understanding and information'" (Fraivillig, Murphy, & Fuson, 1999, p.150), learning outcomes and modes of representing that curricular content would not be uniform. Rather it was predicted that they would vary with the quality of spontaneous concepts individual children brought to the learning situation.

Theoretical Framework

The broad theoretical framework guiding this research addresses questions about the extent to which first graders' spontaneous notions of equivalence and related concepts are characterized by mature or immature reasoning structures. This line of investigation emanates from Piaget's (1965) cognitive-structuralist stage theory and rests on the importance of the development of concrete operational thinking and the appreciation of conservation of quantity and the coordination of perspectives as the basis for understanding mathematical expressions of equivalence. It examines the extent to which underlying immature concepts might guide children's interpretations of school-taught conventional procedures so that they assimilate new information to old structures to build the foundation for inaccurate conceptual understanding. The study also relies on the distinctions that Vygotsky's (1986) socio-cultural theory makes about the initial separation of spontaneous and scientific thought in young children as part of the developmental process. In this view, successful learning experiences need to provide children with enough, but not too much new information, that can serve to transform spontaneous concepts into conventional culturally accepted concepts. A basic assumption of this study, therefore, was that some children's spontaneous concepts were not well matched with the starting points of instruction in the formal mathematics curriculum and that, therefore, effective scaffolding during instruction would not occur. As a consequence, assimilative distortions would take place and these children would develop idiosyncratic rather than conventional ideas about school-taught concepts and procedures.

Specifically, in the present study, it was expected that children would develop very different representations of first grade mathematics curricular content as a function of whether or not they had fully developed concepts about logical principles related to reversibility, compensation, equivalence, and appreciation of relative magnitude that underlie the development of concrete operational thinking. It was predicted that, very much like young children who deploy gestures while counting for a labeling rather than for a quantitative function (Kaplan, 1985) cognitively immature first graders would tend to attribute static, specific, non-quantitative linguistic meanings to relational concepts in the mathematics curriculum.

Methods of Inquiry

Previous investigations with somewhat older children on the effects of the curriculum used in this study, suggest that children who learn with it do not perform less well than students who learn with more traditional approaches and may even do better in some areas. These investigations, however, were based primarily on test results that considered the correctness of answers through which reasoning could be inferred (Fraivillig, et al.,1999). On a smaller scale, the present study attempted to probe the thinking processes of participating children directly in addition to attending to the answer outcomes of those processes. It did this through a series of individual interviews with children that examined both spontaneous notions of equivalence and non-equivalence and the ways in which the children translated these ideas into applications in the school curriculum.

The participants of this study were 12 first grade students who were interviewed at the mid-point and at the end of the school year. The children attended on class in a middle-class suburban community K-4 school composed of primarily Caucasian families. Participants included 8 boys and 4 girls ranging in age from 6-6 to 7-6 at the beginning of the study.. At the time of the interviews, the school district used the *Everyday Mathematics* curriculum (University of Chicago School Mathematics Project, 1995) in grades K - 4.

The children were seen individually in their classroom and were interviewed for about 2 hours spread out over 2-4 sessions. All interviews were videotaped. In the first set of interviews the children were asked to do a series of informal tasks including oral counting, counting of objects, some conservation tasks, combining small numbers mentally, and predicting how to make a scale balance. In the second interview, each child was asked to do some curricular worksheet problems involving fact families, figuring out number patterns, writing numbers in relation to place value conventions, predicting a number outcome based on applying an addition or subtraction rule, and constructing and representing word problems using number fact families.

The Data

The data for this presentation consisted of the videotaped interviews and written work of the children. Both formats were scored for accuracy of response and coded for strategies used to obtain answers. In addition, a case study examination was conducted in which patterns of responses were analyzed across tasks for individual children. One of these cases is presented here.

Quantitative Results

Quantitative scores were obtained for each informal and formal task. With the exception of seriation and class inclusion tasks, all tasks had between 4 and 20 parts to them with each part scored on a scale that ranged between 0 and 4. For the most part, a score of 4 indicated that the child was correct while a score of 0 indicated that the child's answer was incorrect. Scores in-between reflected different levels of distortions. Formal and informal task scores were correlated overall and for paired-task combinations using a Pearson Product Correlation.

As indicated in Table 1, there was a significant and substantial correlation between overall scores on the informal tasks and formal tasks (r (10) = 0.896, p < .001). As indicated in Table 2, we see that successful performance on six of the formal tasks was associated with more highly developed spontaneous concepts. However, performance on three other formal tasks showed no relationship to the children's development of spontaneous concepts with anticipated structural links.

Table 1. Overall Correlation Between Performance on Informal and Formal Curriculum Tasks

Informal Tasks (Mean Score)	Formal Tasks (Mean Score)	Correlation Coefficient
130.17	235.17	r(10) = 0.896*

*p < .001

Table 2. Correlations Between Specific Informal Tasks and Their Structurally Related Formal Tasks

Informal Tasks	Number Series	Place Value	Money	Fact Families	Number Names	Frames & Arrows	Function Machine	Equation Model	Create Stories
Conservation	-----	-----	0.561*	0.618*	0.632*	-----	0.605*	-----	0.280
Oral Counting	0.331	-----	-----	-----	-----	0.895**	------	-----	-----
Counting Objects	-----	-----	0.821**	-----	-----	0.895**	------	-----	-----
Mental Arith (Oral Facts)	-----	-----	-----	0.658*	0.536	-----	0.782**	0.585*	0.112
Class Inclusion	-----	0.319	0.128	-----	-----	-----	-----	-----	0.030
Seriation	0.385	-----	-----	-----	-----	0.702*	-----	-----	0.223
Number Stories	-----	-----	-----	-----	-----	-----	-----	0.914**	0.251
Balance Scale			0.499	0.696*	0.371		0.371	0.725**	0.363

The results of all these correlations, though, whether significant or not, do not address the important issue of how specific forms of representation on informal tasks inform the reasoning behind the answers obtained on the formal curricular tasks. Therefore, a more refined examination was made between performance on specific items within the informal tasks and their relationships to particular outcomes on specific formal curriculum tasks for individual students. This examination focused on particular kinds of mistakes and reasons that students gave. Specifically, it focused on their applications (or lack of applications) of the dynamic principles of reversibility, compensation, determination of equivalence, and relative magnitude as they were manifested first in the children's spontaneous concepts and then in relation to related formal concepts. This paper now focuses on one such case analysis.

The Case of Karen: Spontaneous Concepts

Karen was 6 ½ years of age when first interviewed. She was slow and careful in responding to questions and used counting on or counting down with her fingers as her primary problem solving strategies. Her performance on the informal tasks indicated that she had not yet fully formed structures for understanding the reversibility of operations, the idea that a change in one aspect accompanied a reciprocal change in a related aspect (i.e., compensation), or that equivalence was not determined by appearance but by underlying structural similarities of a quantitative nature.

Conservation

On the conservation tasks Karen was able to judge two initially equivalent arrays as remaining equivalent when one was spread out. Her justification, however, was simply that they were the same because "it was spread out." This response, which was actually typical of almost all children in the class, suggested that she probably was not concentrating on two properties at the same time. Such a one-sided approach to the task was even more apparent in Karen's responses to the conservation of substance and liquid tasks. On these tasks, her initial judgment of equivalence was easily challenged as Karen asserted that the amounts did not remain the same after the transformation occurred.

Class inclusion

When shown two groups of plastic circle chips, 8 blue and 4 white, and asked which was more, the plastic or the blue circles, Karen quickly judged the blue to be more. When questioned about how many plastic chips there were, she said "4" (actually the number of white chips). However, when asked specifically to count the plastic chips, she did count all the chips and obtained "12" as her answer. It was only at that point that Karen decided that there were more plastic chips than blue chips. Her initial response, however, demonstrated that on her own, she could not focus her attention on two dimensions at the same time and that she lost sight of the larger category (plastic) when she focused on a subcategory (color).

Balancing a scale

Karen was given a series of questions related to making a pan-scale balance by adjusting the weights on one side to match the weights on the other side. Each piece was of equivalent weight so that 6 pieces would balance 6 pieces and so the task was one about number. When given the chance to place additional pieces on the lighter side to balance the scale, the task posed no difficulty for Karen. By counting up from the smaller value until she reached the larger value, she was able to adjust the scale to balance 3 and 5, 5 and 7, and 3 and 6. However, when the task was complicated so that the balance had to be obtained within the set of weights that were presented (e.g., 8 and 4 - balance without adding any extra pieces, but just by moving them around from one side to the other), Karen's first try was to take 4 from the 8 and add it to the 4. Thus, she focused only on one side, the side she wanted to increase, by creating another 8 on the opposite side. After making this shift, however, she was surprised to see that she now had 8 on one side but only 4 on the other side. She then took off the 4 pieces that had tipped the balance, leaving 4 on each side and 4 in her hand. Then she split her 4 into two parts equal parts and then put 2 additional pieces back into each pan. She saw it balanced. When asked how many pieces were on each side, she then counted up the pieces on one side and said that she had 6 and 6. From her strategies, we can see that Karen remained focused on a one-sided effect rather than the relational aspect of considering the loss and gain simultaneously. Thus, she essentially used an adding-on trial and error approach until she got it right.

Counting objects

Karen was presented with 30 cubes and asked to count them. Her first count was by ones and accurate. When asked to count them by twos, she did take two cubes at a time and said the first few numbers from memory (2, 4, 6, 8, 10, 12). Then she proceeded to count numbers in her head essentially attempting to skip the appropriate number of names between each utterance. This worked until she got to number 20. After that she skipped over to 24 and then went on with 26, and 28. After that she jumped to 33 and then said 35. When asked how many cubes there were counting by twos, Karen said there were 35. Her skip counting by fives and tens was accurate and used appropriate number names. Nevertheless, she accepted that there were 30 cubes if counted by ones, fives, or tens, but that there were 35 if counted by twos.

Mental arithmetic for orally presented values

Karen was presented with two single digit numbers at a time and asked either "How much are # and #?" or "How much is # take away #?" The presentation of the numbers was ordered so that two doubles combinations preceded the combination between the doubles. The combination between the doubles was presented in AB and BA order. The doubles were expected to be memorized and so based on that knowledge, the question about the between values was intended to assess if the child could adjust or derive new quantitative information from the doubles fact, i.e., a form of compensation. After the addition combinations were completed, the sum of the combinations was presented in terms of a subtraction counterpart for that combination. The subtraction variant was used to assess whether the child understood the nature of inverse operations, i.e., reversibility. When Karen was asked "4 and 4" followed by "5 and 5," she immediately and correctly answered "8" and "10." I then reiterated that "4 and 4 was 8; 5 and 5 was 10; so how much is 4 and 5?" To this, Karen responded by counting on her fingers from 5, saying ;"6, 7, 8, 9. So 9." After this, I again reiterated the previous series of numbers combinations and their answers, adding, "so how much is 5 and 4?" Karen immediately responded with "9" and said she knew this because it was a turnaround fact. The combinations were repeated and next Karen was asked, "How much is 9 take away 5?" For this she began to count down from 9 using her fingers and saying "9, 8, 7, 6, 5, 4, 3. Three?" I did not correct her, but reiterated the combinations as she stated them and then asked her, "How much is 9 take away 4?" For this she again counted down using her fingers and saying, "9...8, 7, 6, 5. Five?" From her responses it was clear that the numbers were not seen as related to one another, but rather each combination was seen as an occasion to use counting up or down, usually accurately, but sometimes not. The logic of the quantitative relationships in terms of compensation and reversibility was not behind Karen's actions. Her mention of the turnaround fact was no more than a syntactical rule that really meant to her that it didn't matter which way you said it, but that you can count on from the larger of the two numbers regardless of which number was said first.

The Case of Karen: The Relationship of Spontaneous Concepts to Curricular Tasks

Karen's one-way and limited spontaneous responses on the informal tasks were next explored for evidence of how they might be informing her judgments and approaches to curricular tasks with similar underlying structures. It was noted that some of the curricular tasks did not rely on such structures and on these Karen performed consistently well. For example, she had no trouble place numerals in the correct place value position based on directions to, "Write a 3-digit number with 6 in the hundreds place, 7 in the tens place, and 8 in the ones place." For such items Karen knew which position each term referred to and was performed as a rote skill, much like reciting the fives and tens skip counting lists. She also understood that smaller numbers would require larger numerals to be positioned in the ones place and smaller numerals to be positioned in the hundreds place whereas larger numbers would follow the opposite pattern. Placement of digits, however, does not imply understanding of their relative quantitative values any more than recitation of number names infers understanding of cardinality.

Frames and arrows exercises

The frames and arrows curricular exercises required the child to identify and continue a pattern of addition or subtraction of a constant value in a series of numbers. To perform this task reasonably rather than as a rote exercise,, the child needs to have a sense of number as part of a series in a two-way relationship such that for any given number, the value before that place follows the rule in one direction and the number after that place follows the rule in the opposite direction. For example, in a simple pattern where "+2" is the rule and the list is given as, " 24, 26, 28, 30, ," the child should be able to recognize that 30 represents 2 more than 28, and 2 less than the number following it. However, in fact, to fill in this blank, the child needs only to count on 2 from 30 to get 32. A child can perform this level of the frames and arrows exercises without having to invoke any two-way logical principles. Karen, in fact, was able to do this type of format successfully. However, when the patterns required some element of appreciation of reversibility or compensation, Karen's performance retained its one-way structure. Thus when Karen had to use the rule "count up by 10s" in a series that began with the number 8, she could just count up to get the next number (i.e., 18). Then using her rote knowledge of place value, she was able to pick out a pattern in which the tens place increased by 1 for each continuing number. So Karen could errorlessly write, (8), 18, 28, 38, 48. However, when the rule just below this one was "count down by 10s," she could not apply this strategy. That was because in order to advance, she also had to move backwards at the same time. In this case the number given was 56 with four blanks following and arrows pointing forward. To get the correct answer, Karen needed to track the tens place digit backward while moving forward. This proved to be too much to juggle for Karen and so she just wrote, (56), 55, 54, 53, 52. Apparently going down by ones was rote enough for her to move forward and backward at the same time. The fact is, though, that she did not understand the nature of the operation that was required and so assumed that her approach was fine.

Expressing cardinal values by different number names

On another task Karen was asked to write several cardinal values in different ways as for example, list five different names for "10." A sample of 5 + 5 was given. Karen did not use the sample to guide her in any way, but instead came up with several + 1, -1, and +0 responses suggesting that she did not see how a cardinal value could be broken up into components and still maintain its value. This clearly related to her inability to understand that the amount of substance does not change because its shape changes. For Karen the wholeness of 10 does not represent several possible smaller values that represent components of "10ness," but rather that "10ness," does not really exist beyond the last number word said at the end of a count. The number does not represent the whole and a part simultaneously. This is also related to Karen's failure to understand the nesting of categories in the class inclusion task and in her failure to appreciate subtraction as an integrated and inverse process in dealing with orally presented small number combinations. In fact, one of her responses to the number names task further demonstrates that Karen uses the "turnaround" fact term not as a true exemplar of reversibility, but simply as a rote and probably overgeneralized idea. This shows up when she says that 8 + 2 is 10 and 8 - 2 is also 10. This is again demonstrated when Karen was asked to cross out any combinations in a given group of combinations that did not produce the value "12." For this task, Karen counted up and down to test every example including 10 - 2 and 4 +5 as well as 3 + 9 after having just counted 9 + 3. Thus, while she got the answers all correct, her procedure demonstrated that she did not appreciate the cardinal value simultaneously as the sum of its parts.

Fact family combinations

Fact family combinations are those single digit combinations using 3 numbers that exhaust all possible addition and subtraction combinations for those numbers. These combinations can be executed as a visual pattern or they can be executed using the logical relationships of reversibility and inverse operations. Karen was successful on some combinations and not on others. Those items on which she was successful do not reveal how she derived her answers. However, on the items in which she produced quantitatively impossible equations or failed to confine her responses to the three numbers in the family, we see that Karen used a rote and mechanical one-way operation. On the 4, 5, 9 combination presented with a domino containing 4 dots and 5 dots, Karen did not even consider the subtraction or inverse part of the relationship, but rather produced a series of four partially related addition combinations and even misused the operation sign that was provided. She wrote that 4 + 5 is 9; 5 + 4 is 9; 9 - 5 is 14, and 9 - 4 is 13, thus converting the subtraction pairs into addition and introducing two more numbers from outside of the system. Similarly, in recording the 7, 3, 10 family from the three numbers already given, Karen filled in blank spaces by not repeating any combinations, but also by not making sense of all combinations. She wrote that 7 + 3 = 10; that 10 = 7 + 3; that 10 - 7 = 3; but that 7 = 3 - 10. This confusion of reversibility with a visual pattern seems to be closely related to both Karen's failure to conserve substance and liquid in that the notion of equivalence in regard to two sides of an equation is lost. It also appears to be related to her count-on and count backwards strategies for dealing with orally presented number combinations in that the relationships among the numbers is lost to her.

Function machine exercises

On this task the addition or subtraction of a constant becomes the rule for operating the machine. The child first needs to figure out the rule by looking at the relationship between numbers already given in an "in column" and an "out column." Once the rule is recognized, it must then be applied to one side or the other of the "in" and "out" columns. Completing the "out column" when the "in column" value is given is fairly straightforward and requires only a uni-directional approach, i.e., count up or count down. However, when a combination of "in column" and "out column" values is missing, then the child needs to use inverse operations and reversibility knowledge in relation to number values. Still, though, if the numbers are small enough, the child can still use a counting up and down strategy, coupled with a rote procedural rule that goes something like: "if your 'in' column number is missing and the rule is subtract something, then add instead. Using this rote procedural rule and a consistent strategy of counting up and down by ones regardless of the rule and how much counting was required, Karen was generally successful in getting mostly correct answers on this task. However, if we examine those items on which Karen did not obtain correct answers, we see that her errors are not small counting mistakes, but broad conceptual mistakes.

Some of the misconceptions that Karen brought to this task were based on her incomplete understanding of reversibility and inverse relations as indicated in her performance on the conservation, oral number facts, and balance scale tasks. We see here that while Karen can generally apply any rule through counting by ones, her answers do not always make numerical sense. For example, when she had to count down from larger numbers, she could not continue to count by ones and so she tried to count by tens (e.g., to go from 130 to 80, she guesses that -30 is the rule. Then she fails to apply the rule accurately on all subsequent parts of the chart and even uses the same number [60] three times in the out column.) Also when the numbers are in place and the rule must be determined, she forgets about the relationship between the in-box and out-box numbers and instead just adds them together by counting on by ones. So she takes a -2 rule and changes it into a +8 rule by adding 5 *and* 3 instead of subtracting 3 *from* 5. This tendency to disregard the relationship between the numbers in favor of applying an arbitrary counting on by ones procedure is clearly related to Karen's uni-directional performance on the informal balance scale and conservation tasks. This seems to suggest that while a child like Karen can obtain many correct answers, the processes she uses to obtain these answers have little to do with the underlying curricular objectives of the task. To do this task properly a child needs to understand that the rule can work in either direction *because the operations are related to one another,* not because this is *what you are supposed to do.*

Equations and models of one step number stories

Karen's curriculum also provides opportunities for learning to write standard number sentences or equations. The form of these equations is usually mastered by all students, but the question is really what do these equations represent to the children who write them? To assess this meaning, Karen was given several short basic "story problems" depicting addition and subtraction relationships. She was asked to write down the answer to the question, then write an equation or number sentence that told what happened in the story, and then to show with some blocks what the equation was describing. Karen easily counted up or down to get her answers and had no trouble writing an equation to express the addition or subtraction action in the story. She also had no problem in modeling the addition stories. Where she did have difficulty was in showing with blocks what the equation and/or the story meant for subtraction. For example, for the story "Sarah had 9 pennies. She gave away 3 pennies. How many are left?" Karen first took 9 blocks. Then she took away 3 blocks. Then she put back 3 blocks. Then she took away 3 blocks. And finally, she put back 3 blocks. This depicted the equation: 9 - 3 + 3 - 3 + 3 = 9. This vacillation between addition and subtraction models of the equation suggests that Karen cannot hold both the question/action and the answer in her head at the same time. Unlike addition, subtraction destroys the set with which it started. It appears Karen she needs to keep the set intact in order to see where the subtraction took place. But if she does this, then the subtraction cannot take place. This dilemma is very much tied to her difficulty with class inclusion. The set of plastic circles disappears when her attention is focused on the color of the circles. In order for the plastic circles to maintain their integrity as a full set, they need to be disassociated from their blueness. In the same way, Karen cannot see what is taken away (3 pennies) and what is left (6 pennies) yet still keep in mind the 9 pennies with which she started. Thus, she continues to replace the part that was lost. The question then really is about the meaning of the written symbolic equation for a child who cannot see the forest and the trees at the same time. Like so much of the mathematics that Karen encountered in first grade, equations were just another procedural routine to be followed and replicated without consistent quantitative meaning.

Create a story problem

Another indicator that the numbers do not have a strong cardinal meaning for Karen and that they are not seen as being part of a balanced relationship comes from her responses to a task in which she was asked to make up a story problem for several sets of three numbers. The sets of numbers all constituted a number family relationship. In general, while she used all and only those numbers that were provided in each story, the stories did not depict any connection between the numbers. For example, for the set of numbers 6-3-9, she said, "I saw 6 dogs walking. Then I came back and I only saw 3. How many did I see altogether?" While the number 9 could answer the question of how many altogether, the fact that the story is framed in subtraction language, does not make sense. It appears that Karen responded to the order of the numbers, i.e., 6 is more than 3 so that means less, and then reversed the story line because the third number 9 was also more than 3. Her general sense of number magnitude for small numbers seems to be operating here, but it is operating in isolation because she has no logical framework for conserving the set as a related series of small numbers. As would be expected, she seems to be taking a syntactical rather than a semantic approach to these stories and the sets of numbers incorporated in them.

When larger numbers were used, even though they were "comfortable numbers," Karen's story became even less representative of connections between the numbers. For the numbers 100-200-300, for example, she said, "I saw 300 people then I saw 200 people. 100 were only left. How many people did I see altogether?" Here she seems to be connecting the 300 to the 200 in a subtraction direction, but does not carry through and instead of having 100 be the result, she includes it as another number to be counted and changes the story into neither an addition nor a subtraction model. In fact, this time she seems to be introducing the possibility of another number from outside the system.Overall, then, Karen's construction of simple word problem stories seems to reflect her inability to focus on more than one aspect of a quantitative situation at a time, her approach to numbers from a syntactical rather than a semantic framework, and her uni-directional understanding of quantity rather than an understanding of quantity through the principles of reversibility, compensation, and equivalence.

Discussion and Conclusions

Very much like young children who deploy gestures while counting for a labeling rather than a quantitative function (Graham, 1999; Kaplan 1985), this study suggests that first graders with immature spontaneous notions about number tend to attribute static, specific, non-quantitative linguistic meaning to relational concepts in the mathematics curriculum. Thus, they often get correct answers on content that does not call for relational reasoning, but fail to obtain correct answers on structurally more advanced items that demand the application of reversibility or compensation principles. Their paperwork assignments, then appear as "C" or even "B" work because of the unevenness of task demands within the curriculum and that based on their sometimes correct answers, it appears that they "get it" but are just "making some mistakes." In reality, children who are not using the abstract level of thinking that key elements of some tasks demand, confine themselves to one-way solutions, often employing counting on by ones, as the basis for these solutions. This strategy works for getting some answers, but it does not help build knowledge of higher level equivalence relationships, the precursors to algebraic thinking. As a consequence the children who are locked into the one-way syntactic solution process are likely to develop enduring misconceptions or only partial conceptions about the mathematics they are using in first grade.

My strong intuition is that just as lower level informal concepts create lower level approaches to dealing with more abstract equivalence concepts in the curriculum, so will lower level approaches to the first grade curriculum carry over to impair understanding of later curricular concepts that should be building on clear understanding and applications of reversibility and compensation principles encountered in the earlier curriculum. It is further suggested that these misconceptions interfere with subsequent school mathematics learning because children use them to inform their understanding of new material. As a result second grade children who functioned at a syntactic level on first grade curriculum are likely to continue to function at this level on second grade curriculum and beyond. Thus, we often see the rote application and parroting of rules with numbers rather than the development of a strong and flexible number sense as children move up in the grades. It would be expected that children would develop ritualized approaches using rule-based number manipulation techniques. They would overgeneralize a good thing in some cases such as using 10 as the basis for addition with 9s and then subtracting 1 until the technique becomes standard operating procedure and the child's justification for all answers even when this procedure does not make sense. My interest in the future is to develop some specific intervention strategies to prevent these misunderstood processes from carrying over from grade to grade.

References

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *Elementary School Journal, 97*, 3-20.

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics, 6* (4), 232-236.

Fraivilllig, J., Murphy, L., & Fuson, K. (1999). Advancing children's thinking in *Everyday Mathematics* classrooms. *Journal for Research in Mathematics Education, 30* (2), 148-170.

Fuson, K. C., Smith, S. T., & Lo Cicero, A. M. (1997). Supporting Latino first graders' ten-structured thinking in urban classrooms. *Journal for Research in Mathematics Education,28* (6), 738-766.

Graham, T. A. (1999). The role of gesture in children's learning to count. *Journal of Experimental Child Psychology, 74* (4), 333-55.

Kaplan, R. G. (1985). *The function of gestures in the counting activities of very young children.* Unpublished doctoral dissertation, City University of New York.

Piaget, J. (1965). *The child's conception of number*. New York: W.W. Norton.

University of Chicago School Mathematics Project. (1995). *Everyday mathematics.* Chicago: UCSMP.

Vygotsky, L. (1986). *Thought and language*. Cambridge, MA: MIT.

THE NOTION OF LIMIT AND LEARNING PROBLEMS

Fernando Hitt
Matemática Educativa Cinvestav-IPN,
México
fernando.hitt@matedu.cinvestav.mx

Rosa Páez
Universidad Francisco de Paula
Santander, Colombia
rpaez@mail.cinvestav.mx

Abstract: The purpose of this study is to analyze obstacles detected in students at first semester of University level while learning the concept of limit. The study explored how students construct their ideas of infinity connected to the potential infinity (*it expresses the idea of an inductive process*); and how this idea might conflict with the actual infinity (Kant, 1790, p. 112),*"regard this infinite, in the judgment of common reason, as completely given"*) involved in tasks given by their instructor. Questionnaires that included both concepts of function and on limit of functions were used as a means to collect data. At the end of the course, five students were interviewed to explore deeply their arguments and conceptualization around concepts that involve infinite processes.

Introduction and Theoretical Framework

Previous studies (Hitt, 1998; Hitt & Lara, 1999) on the same subject guided us to focus on understanding some strategies used by students and teachers (high school) when solving exercises that involved finding limits of real functions. Also, we studied the construction of the concept of potential and actual infinity from an historical point of view. We found that Zeno's paradoxes, for example Achilles and the Turtle, questioned the idea of potential infinity. The Greek philosophers did not understand fully those paradoxes and the germ of the actual infinity came from philosophy with Kant (1790, p. 112) who said: *"regard this infinity... as completely given"*. The idea of actual infinity and the limit-avoidance concept in mathematics appeared until XIX century when that problem was raised with the arithmetization of the analysis. From Grattan-Guinness' point of view (1970, p. 55), the quintessence of the analysis is: *the arithmetization of analysis by means of limit-avoiding*. He states that in the work of Bolzano (1817) the definition of continuous function provides the *"key"* for the development of the analysis by means limit-avoidance, that is: *$f(x)$ is continuous at $x = x_0$ if $[f(x_0 + \alpha) - f(x_0)]$ is small when α is small.*

In the present study we assumed that student's acquisition of a concept will be achieved when the student is capable of coordinating, free of contradictions, different representations of the mathematical object. In this qualitative research we tried to understand deeply which kind of conceptual structure about the idea of limit students have throughout questionnaires and interviews during a regular calculus course.

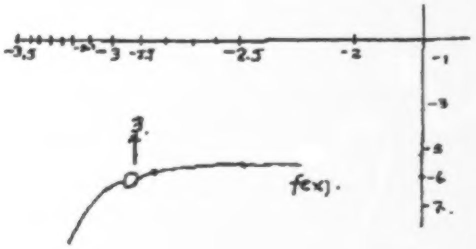
Then, from the theoretical point of view of representation (Duval, 1995), our qualitative research focused on documenting processes and meanings developed by students. In a first stage of the research, questionnaires (about functions and limits of real functions) were designed taking into account tasks within a semiotic system of representations and conversion tasks between different systems (for example, algebraic to graphic and conversely). Some preliminary results were consistent with those obtained by Sierpinska (1987, p. 371): *"Obstacles related to four notions seem to be the main sources of epistemological obstacles concerning limits: scientific knowledge, infinity, function, real number"*.

Because our theoretical approach is immersed in the theory of representations, student's acquisition of a concept will be achieved when the student is capable of coordinating, free of contradictions, different representations of the mathematical object. Related to the the concept of limit, Davis & Vinner (1986, p. 300) said: *"Thus will be stages in the student's development of mental representations where parts of the representations will be reasonably adequate and mature, but other parts will not be"*. From the previous studies we paid attention to particular results:

- Hitt (1998) reported that in a sample of 30 high school teachers, one third of them experienced problems with identification of Domain and Image Set in the graphic representation of functions.
- Hitt & Lara (1999) said that some teachers in high school teach wrong or incomplete ideas when introducing geometric approaches regarding the concept of limit. An specific example, a teacher used a graph to show a problem with this function: $f(x)=\frac{x^2-9}{x+3}$ *when "x" approaches -3. To find the limit by a numerical approach we need to tabulate the following data:*

x	-2.5	-2.9	-2.99	-2.999	-3.001	-3.01	-3.1	-3.5
f(x)	-5.5	-5.9	-5.99	-5.999	-6.001	-6.01	-6.1	-6.5

"O" means that in –3 the function is indeterminate, that is, discontinuous on that point, what means is that the function has a limit, that is a value towards the function can approach as much as possible without indetermination, this limit is on the right and left of the hole of the indetermination...



Methodology

Taken into account those previous studies we designed clinical interviews involving carefully constructed mathematical tasks in the sense of Goldin (2000). The interviews were designed in a semi-structured format, i.e. asking questions already formulated and including others in accordance with the answers given by the students. We asked students to give examples to analyze their spontaneous ideas. For example, if a student said that 1/n approaches more and more to zero when n increased, we asked them if 1/n approaches –1 when n increased, or when they were giving a formal definition of limit of a function $(\forall \varepsilon > 0, \exists \delta > 0, |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon)$ we asked them for a geometric interpretation of the definition and examples. Under the theory of representations and the importance of converting from one representation to another, we designed the tasks of conversions among, numeric, graphic and algebraic representations. We took special care about the design of the questions immersed in the theory of representations.

Results

We asked students to imagine a situation where they needed to explain the idea of limit, first to a student in secondary level, and second to his/her own classmate. Here we were interested in students' spontaneous ideas; What kind of resources does he/she utilize to explain this concept? Also, we asked them to provide a formal explanation about limits.

We knew from our previous studies that the use of different representations of a concept from teachers or books approaches produces different interpretations from the part of the students. For example, the graphic representation of the function (see problem 6 below) makes implicit that we need to imagine all the real numbers in the domain of the function and to extend the curve asymptotically below the line y = 1. Which kind of visual variables do we need to take into account when reading a graph? We were aware of some difficulties students face because of the lack of precision in the use of language in the process of teaching. For example, if I say: "find the limit of f as x approaches infinity" What do we mean by that? What do students understand by that?

In general, in their explanation about the idea of limit, the students mostly concentrated in the algebraic approach. Three of them gave explanations using sequences that they had recently studied in their calculus course, for example $\{1/n\}_{n \in N}$ and f(x) = 1/x, x ≠ 0. One of them gave some examples and a formal definition of $\lim_{x \to a} f(x) = L$.

Student EM explained the concept of limit using the function $f(x) = x^2$, a table and a graph. He said that the term "towards" or "approach" is intuitive but not formal. He added, a formal approach appears when the definition of $\lim_{x \to a} f(x)$ is given. Here he provided a formal definition and its graph representation.

Student EC: This sequence 1/n if we represent it as decimals there is a moment where is approaching to a value, a value where that value sometimes is going to be fixed, when this sequence goes to values at infinity.

It is interesting to see that her idea of limit is that the sequence indeed "reaches" the limit and she changed the notation *1/n* to a decimal notation to express that in certain point the elements of the sequence are going to be zero. The same student gave another interpretation when analyzing the limit of a function.

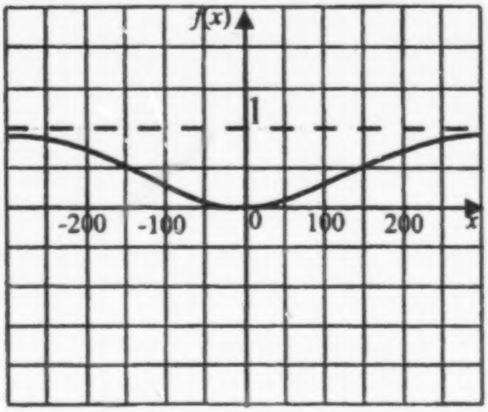
Student RH explained a general idea of approximation but didn't take into account his examples. He said: *the limit almost is reached... no exactly has to be the limit.*

We can say in general, that the idea of limit of these students is the one attached to the definition and the examples they were given.

Conversions among representations

Let's document some difficulties showed by the students when the task demanded conversion among representations.

Problem 6. Given the graphic representation of the function f(x) and the numeric table, answer the following questions:



x	0	1	2	3	10	100
$f(x)$	0	$\frac{1}{3}$	$\frac{4}{6}$	$\frac{9}{11}$	$\frac{100}{102}$	$\frac{10000}{10002}$

a) $\lim_{x\to\infty} f(x) = 0.999\ldots?$

b) $\lim_{x\to\infty} f(x) = 1?$

c) *What is the value of the difference 1-0.999...?*

e) *Is the function f(x) defined for x = 0? If yes, ¿What is the value f(x) when x = 0?*

In this problem, the students did not show difficulties answering the questions b), d), and e). The difficulties raised when answering question a) and c). Indeed, the results are similar to those discussed by Davis and Vinner (1986), Tall & Vinner (1981), and Sierpinska (1987). In our case the students could obtain from the graph of the function the information they needed.

Student ML: This 0.999… is increasing, little by little but increasing. The limit is one, because each time is increasing and approaching to one and we accept as it was one.

Student EC: If we consider the graph, if we continues giving values to x to obtain f(x), as these [x] are increasing, these [f(x)] also are increasing, then they are approaching one and there is one point where it touches one and will be the limit.

Student EC: By definition of limit, the limit of the function is the value, it means, the limit of a sequence is considered as the value where the sequence is approaching, but never reach the value.

Student EC: When this 0.999… is infinite, an infinite of nines, then this 0.999… you could say that is equal one. What happen is that 0.999… is not going to be one, … then, this 0.999… is different of one, even thought it be nearer, nearer, but different to one.

Student EC has displayed a contradictory idea of limit, sometimes she thinks that from one point, the elements of the sequence are going to be the limit as she explained with the sequence {1/n}. And as she said in the last two paragraphs, the sequence is approaching, but never reaches the value.

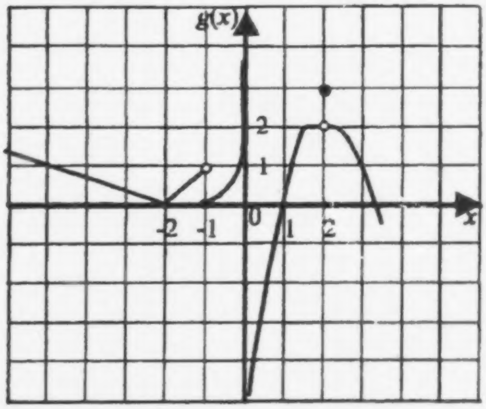
Student RH: The limit is one, but the function never will be the limit, it means that there is not a x that f(x) gives one.

Student EM: This 0.999… is a number with infinite decimals and it is in accordance with the geometric idea because the curve never will touch in this case the horizontal line y = 1, never is going to touch it, however, is approaching more and more, there are going to appear more and more nines in this decimal representation.

The student EM found immediately the algebraic expression and calculate the limit $\underset{x \to \infty}{Lim} f(x) = 1$. When asked for the comparison between 0.999… and 1, he said: *"we must add 0.000…1 in the infinite to obtain 1, something is missing a little gap"*. He proved algebraically that 0.999… = 1, but he said, *"I accept that the algebraic process gives 0.999… = 1 as a result, something is missing, a little bit, I accept the result but I do not feel satisfied"*.

These results agree with Fischbein's quote (1987, p. 92): *"When dealing with actual infinity –namely with infinite sets- we are facing situations which may appear intuitively unacceptable"*.

Problem 8. Analyzing the following graph of a function, answer the following questions.



a) $\underset{x \to 1}{Lim}\, g(x)$

b) $\underset{x \to -2}{Lim}\, g(x)$

c) $\underset{x \to 0^+}{Lim}\, g(x)$

d) $\underset{x \to 0^-}{Lim}\, g(x)$

e) $\underset{x \to -1}{Lim}\, g(x)$

f) $\underset{x \to 0}{Lim}\, g(x)$

g) $\underset{x \to \_\_}{Lim}\, g(x) = 1$

h) $\underset{x \to 2}{Lim}\, g(x)$

i) $g(2) =$

j) $\underset{x \to -1^+}{Lim}\, g(x)$

All kinds of cognitive problems came into surface with this question. For almost all the students, if the function is defined at one point, then the limit must exist and must be the point (x, f(x)), it doesn't matter if such function involves some kind of discontinuity at that point. For example, if the curve have a "hole or a jump", three students mentioned that the limit of the function is the isolated point (x, f(x)).

Indeed, one of them drew an open interval around the points (-2, 0), (-1, 0) and (1, 3)! Another one analyzed the behavior of the function in a global view (from -∞ to +∞) and not in a neighborhood of the point under question. Then because the behavior of the function was irregular she was claiming that the limit couldn't exist. All of them analyze the curve to deduce and claim that if the limit exist it must be on the curve. They did not consider the y-axe to analyze the images of the x variable.

It is clear students' performances in this problem indicated that they do not have a coherent idea of limit. That's is, it became difficult for them to deal with a graphic representation of a function and asked to find a limit.

Problem 11. Using a geometric, numeric or algebraic approach, find the following limits: (specify the method that you used). c) $\underset{x \to -2}{Lim} \sqrt{x+1}$.

In connection with algebraic processes we asked students to find some limits of real functions, and to provoke a conflict we asked to find $\underset{x \to -2}{Lim} \sqrt{x+1}$. In this case, they substituted *"x = -2"*. Two of them tried to find another algebraic method or procedure to avoid the problem (like in case a)), without considering if the function was or not defined at that point in connection with the continuity of the function in that point. Two of them said that the limit does not exist because they couldn't calculate $\sqrt{-2}$. One said that the limit, if exists, was a complex one, he said *"But I do not know how valid is an imaginary limit"*.

Student EC wrote the following on the blackboard and said: *The limit is that given that value where there is a cut* [vertical line from the point to the curve], *it means* *where there is an intersection, but the graph is not defined, there is not an intersection then we could say there is no limit. What I've observed, I would say that the limit of the function* $\sqrt{x+1}$ *when x tend towards –2, it is not defined in the real numbers, or that I do not find the right method.*

Student EC is the one that have got a contradictory idea of limit. She was not sure about her algebraic and graphic representations and to justify her inconsistency she added *"…or that I do not find the right method"*.

The following answer was given by EM:

Student EM: To this value [-2] the function is not defined. From there I say there is not a limit.

Discussion

Students when reading or interpreting a graph in terms of limits they showed incoherence, since they were concentrated on the curve and not in the domain of the function and images of the function. The students have an intuitive idea of limit related to the potential infinite. For example, they experienced difficulties in explaining informal approaches about limits and they often rely on the idea of substitution when calculating limits.

If we know that the mathematical idea of infinity entails an epistemological obstacle, then the instruction should emphasize the distinction between potential and actual infinity. From our point of view, a discussion about the limit avoidance taking into account Grattan-Guinness' interpretation in history must be considered in the classroom. Then, we need to design learning situations to provoke a conflict in our students to help them in the construction of the concept in question to overcome the epistemological obstacle related to the actual infinity.

It seems that under our theoretical approach of the construction of mathematical concepts that deals with the articulations among representations of the mathematical object, these students have not constructed a suitable concept because their inability to coordinate, free of contradictions, different representations of the mathematical object.

References

Fischbein E. (1987). *Intuition in science and mathematics. An educational approach.* D. Reidel.

Davis R., & Vinner S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior*, *5*, 281-303.

Duval R. (1995) *Sémiosis et pensée humaine: Registres sémiotiques et apprentissage intellectuels.* Suisse: Peter Lang.

Goldin G. (2000). A scientific perspective on structured, task-based interviews in Mathematics Education Research. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics education*. Mahwah, NJ: Erlbaum.

Grattan-Guinness, I. (1970) *The development of the foundations of mathematical analysis from Euler to Riemmann.* MIT Press.

Hitt F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior, 17* (1).

Hitt F., & Lara H. (1999). *Limits, continuity and discontinuity of functions from two points of view: That of the teacher and that of the student.* Lancaster, UK: British Society for Research into Learning Mathematics (pp. 49-54).

Kant, I. (1790). *The critique of judgement*. Translated by James Creed Meredith. Electronic Texts (EpistemeLinks.com).

Sierpinska A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics, 18*, 371-397.

Tall D., & Vinner S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

A Modeling Perspective on Metacognition in Everyday Problem-Solving Situations

Margret Hjalmarson
Purdue University
mhjalma@purdue.edu

Abstract: This paper describes a models and modeling framework that has been applied to various areas in teaching, learning and problem solving (Doerr & Lesh, in press). It examines the implications of that framework on metacognition and higher-order thinking during everyday problem-solving situations that required teams of students to produce complex solutions in approximately 1-2 hours. In a team of students, there may be multiple interpretations of a complex problem situation and communication of interpretations effects problem solving. This changes the role of metacognition.

Metacognition has been investigated often over the last 25 years and has remained an unclear, vague, and broad concept in part because of unclear definitions and a large number of other related problem-solving issues (Schoenfeld, 1992). Recent standards documents for science and mathematics list metacognition and higher-order thinking skills as important educational goals for students (National Council for Teachers of Mathematics, 2000; American Association for the Advancement of Science, 1993). However, the questions "What is meta? What is cognitive?" (Brown, 1987, p. 66). can be difficult to answer when considering everyday problems requiring complex products. The framework proposed here attempts to clarify some aspects of metacognition and its application during real-world problem solving. New questions include: what kinds of everyday problems provide opportunities for students to use metacognitive functions? How do new types of problems affect perspectives on problem solving? If problem solving is defined in a new way, then what changes for metacognition?

Model-Eliciting Activities

Model-eliciting activities represent one type of problem that provides opportunities for students to use metacognitive functions. The students are placed in small groups of about three and given a task from a real-world client with specific criteria for successful completion of the task. Their solution takes the form of a procedure, explanation, justification, plan or method for solving the client's problem. Their solution should serve not only the client's immediate needs, but also be generalizable and extendable to other problem situations. Part of this requirement often includes writing a letter to the client describing their solution. Their procedure should be described in enough detail that the client could apply it to a set of data and produce the same results as the students. Two examples of model-eliciting activities used as illustrations in this paper are the Paper Airplane Problem and the Amusement Park Problem. For the Paper Airplane Problem, the goal is to devise a set of rules that can be used to find the winners in four categories (Most Accurate, Best Floater, Best Boomerang and Best Overall) at a paper airplane contest. Each team throws their plane three times for three different paths (a straight path, a path including a right turn and a boomerang path so the plane comes back to the thrower). The data provided includes the time in air, length of throw, and distance from a target for each of attempt. The set of rules should work not only for the available data, but other data as well. The types of data analysis procedures the students use (e.g., averaging, ranking) are applicable to other situations where different types of data must be combined and analyzed. The Amusement Park Problem asks students to group a class going on a field trip to an amusement park into chaperone groups and devise a schedule for each group's day at the park. A table listing each student's five favorite rides, a line waiting time chart and a park map are included as data. In the letter to the client (in this case a teacher), the students have to outline their grouping and scheduling procedures so the teacher can use it for other classes going on field trips. The students have to coordinate different types of data to define their process. When they begin writing their schedule, they often use both the park map and the line waiting time chart. They may make a new table listing rides according to wait time (short, medium, long) and notes their location in the park. This information may then be combined with data from the ride preference chart in order to form groups.

The students must judge the usefulness of their solution by criteria explicitly defined by the client. These assessments happen throughout the solution process, but particularly at the end when they are writing their letter and reflecting about what they did. As they write their letter, they have to think carefully about their procedure and how to describe it so someone else could carry it out. Often, during the letter writing, the group may discover a detail they left out or a flaw in their procedure. In the Paper Airplane Problem, the rules have to be fair and realistic. Students will often use their experience with paper airplanes or other contests to evaluate their procedure. The students self-assess their work as they compare their results with the needs and specifications of the client. They also compare their solution with realistic constraints in the problem situation.

Placing the students in small groups means they have to discuss the problem with each other and monitor the work of their group members. Since different students have different interpretations of the problem or may focus on different pieces of information, they have to communicate effectively with each other. Sometimes they listen to each other and sometimes they don't. The small group aids self-assessment since they frequently ask each other for justifications and explanations. During the letter writing process, they have to come to agreement about their procedure. In some problems (e.g., Amusement Park), the group may have divided up tasks among members and they need to assemble their results. They have to communicate their way of thinking about the problem to their group thereby providing a window to the observer about the metacognitive functions they employ.

A Modeling View of Problem Solving

In order to describe the relationship between problem solving and metacognitive functions, there are some distinctions between model-eliciting activities and other more traditional problems that need to be made. In model-eliciting activities, the primary purpose of problem solving for students is to provide an explanation, justification, plan or method as a solution to a complex problem rather than a one word or one number answer. The solutions take the form of complex models (Lesh et al., 2000). The model may include equations, graphical and/or pictorial representations, systems and other mathematical constructs. The model can be generalized and applied to other mathematical situations. In model-eliciting activities, students are often asked to develop a process for solving a problem so "correctness" is often judged by the usefulness of the process. For example, the product of the Paper Airplane Problem is a method for finding a winner of the contest. The method is judged by its fairness and usefulness for the judges of the contest. "Process as product" is a different type of goal than a problem asking the students to find the winner of the contest using established procedures. Students have to clearly define their assumptions as part of describing their process. Defining assumptions helps the client understand why certain decisions were made as part of the procedure. For example, when awarding a prize for the most accurate plane, some students assume this means which plane came closest to the target most frequently. Others take "most accurate" to mean which plane on average came closest to the target. A third interpretation is that the "most accurate" plane came closest to the target at least once even though other planes may have been more consistently close to the target. These different interpretations result from different assumptions about the problem situation and different experiences students may have had with the problem context.

Many model-eliciting activities (and other real world problems) contain too much information or not enough information. So, sorting out the relevant information for solving the problem is important. Sometimes the information needs to be reorganized or reformatted (as in the Paper Airplane Problem or the Amusement Park Problem) in order to be useful. How the students sort the data can depend on their assumptions and interpretations about the situation. In the Amusement Park Problem, students coordinate pieces of data (e.g., the park map and the line waiting time chart) in different ways. They may reorganize information in the charts. They may give different weight in their process to different sources. Some groups start work by looking at the park map. Other groups ignore the park map until later stages of the session. In the end, most groups coordinate information from the different sources because ignoring sources can give an incomplete solution. For example, their schedule wouldn't be complete if they didn't consider the park map. In word problems found in textbooks, givens and goals are often clearly outlined and defined. The problem for the student is to find the correct path from the givens to goals. In model-eliciting problems, there is a greater need to define the givens, goals and relevant assumptions of the problem in order to develop a solution. The data provided often needs to be modified to be useful for a solution. (Zawojewski & Lesh, in press)

In addition, everyday problems are often solved in teams of people. This means that compromises must be made between interpretations and communication between team members is critical to successful problem solving. (Zawojewski & Lesh, in press; Developing a process may take a team 1 to 2 hours. This is in contrast to finding a one word or one number answer in only a few minutes. A 1-2 hour process requires more planning and monitoring. Tasks may be divided (e.g., data analysis, different parts of the problem) and students may reconvene at various points. This provides more opportunities for metacognitive functions such as planning, monitoring and reflecting. Planning a one to two-hour process for a group is different than planning a 2-minute process by an individual. Reflecting about what happened over a one to two hour problem solving session is different than reflecting about a 2-minute problem solving process. Metacognitive functions such as planning, monitoring, reflection, control, etc. play a significant role in the development and application of complex models by students working in teams in everyday problem solving situations.

Modeling Views of Metacognition

In general, a modeling definition of metacognitive behaviors includes that students use them whenever they move from thinking with a construct to thinking about the construct. Thinking about the model occurs while they are developing the model and in subsequent problems. For example, students frequently begin using a procedure to process data. Often, while using the procedure, they consider the uses of their results or steps they need to take to solve the problem after they have finished the present procedure. They may also monitor the accuracy of their results as they are developed. Group members may check their own results and the results of others. So, they are not only carrying out a procedure, but also simultaneously thinking about the procedure. For example, instead of planning and then executing a procedure, students may plan the next procedure as they are executing the first. They may evaluate the usefulness or the implications of a procedure as they are carrying it out. For example, the girls in the following excerpt are solving the Amusement Park Problem. At this point, they have just started dividing the class into chaperone groups. They decided they needed to know how many students wanted to ride each ride before they could group them. Three girls in the group are counting the number of students who want to ride each ride. The fourth girl (Emily) is writing down the names of the students who want to ride each ride. At this point (Figure 1), Rochelle has begun to notice patterns as she is counting. Rochelle makes a number of comments throughout the session about how they should make the groups when they are done counting, but is largely ignored until they are ready to start grouping. The two girls discuss their different interpretations

Rochelle: I think those guys should definitely be together. These three. Or ... like um. Cause there's only 3 rockets they should be together you know cause...
Emily: We could also divide it if we look up on here how long take it'll take each of 'em to get through the line if they're like right by each other.
Rochelle: These should definitely be together because there's not that many people and if they like... They both like ... Emily.
Emily: Oh wait. Yeah.
Rochelle: All of 'em both like... the three people who like the rockets also both like the snake so they ... {Emily starts to interrupt} let me finish

Figure 1. Amusement park excerpt.

here, but they have to compromise about what to do later. In this instance, Emily had already formulated a plan for what to do next, and she wasn't ready to hear alternatives. Later in the session, they use a combination of grouping by ride preference and using the line waiting time chart to form their chaperone groups.

There are three characteristics of metacognitive functions related to the shift from thinking with a construct to metacognitively thinking about a construct: (i) connection to lower-order functions, (ii) interactions with lower-order functions, (iii) cyclic development. These characteristics have been observed in students solving model-eliciting activities and examples from sessions will be used to illustrate these characteristics (Lesh, Lester & Hjalmarson, in press).

The first characteristic of metacognition functions is that they are connected in a context-dependent fashion not only to the logical and mathematical aspects of the problem but also to lower-order functions related to the problem context. The connection to lower-order functions results because modeling focuses on interpretations, and interpretations are not restricted by the mathematics of problem solving. Interpretations of a context can depend on prior experience with the context. For example, the girls solving Amusement Park reference their experience with rides at other amusement parks when grouping. They interpret a rides description from its name (e.g., the Monster ride is thrilling, the Boat ride is for little kids). The students' interpretations of the problem are affected not only by their mathematical knowledge, but also by interactions in the group, beliefs associated with the problem context, and other issues. These issues affect mathematical decisions made during everyday problem solving. For example, beliefs and previous experiences about a problem context affect assumptions students may make about the problem and decisions they make to solve the problem. Hence, students are able to examine their beliefs about a problem and the effects of those beliefs on their problem-solving process (Middleton, Lesh, & Heger, in press). The connection to lower-order functions is context-dependent because of students varying capabilities and experiences in different contexts. For example, in the Paper Airplane Problem, students who have had experience with paper airplane contests, flying paper airplanes or other contests will think differently about the problem than students with different types of experiences. Their experiences in the situation affect the types of assumptions they make about the context and procedures they use.

The second characteristic is that metacognitive skills occur in parallel to and interactively with lower-order skills and cognitive functions. The interactive and parallel connection is directly related to the context-dependent nature of metacognitive functions. If metacognitive functions are context-dependent, then the context affects their application. More experience in a particular context may provide more opportunity to use metacognitive functions. One implication of this is that if more thinking is required to carry out the procedure, less thinking will be used for metacognitively thinking about the procedure. When the girls solving the Amusement Park problem started counting how many students wanted to ride each ride, they were very focused on the process of counting. As this process progressed, they began to notice more patterns and consider what they would do once they were done counting. Also, different types of metacognitive procedures may be used when a student first learns about a procedure than later in their development. For example, checking that fractions have been added correctly may be more important or take more time when first learning to add fractions. Planning when to add fractions may be more important later. As procedures become more refined, there is more opportunity and need for students to think about them metacognitively. So, as students are developing mathematical models, they are also developing metacognitive functions that help them understand when, how and why to use those models effectively Figure 2 shows the difference in the relationships between metacognitive and cognitive thinking (Lesh, Lester & Hjalmarson, in press).

This represents an important shift from thinking of lower-order functions as prerequisite to higher-order functions or as a separate goal of instruction. When students solve model-eliciting activities, they develop lower-order skills at the same time as they are using higher-order skills to further the development of lower-order skills. Principally, since model-eliciting activities require more time and more complex solutions, there is more opportunity for students to use and need to use metacognitive functions in a meaningful way. They are also developing an understanding of when, how and why to use metacognitive functions. For example, a group of boys solving a model-eliciting activity called Paper Airplanes spends a significant amount of time working on the problem without monitoring what they are doing. At the beginning of the problem, they read the data table without regard to which numbers represent times and which are distances. Later, they carefully question each other about procedures to make sure they all understand what is happening. In the excerpt in Figure 3, Bob is explaining his averaging technique to Al.

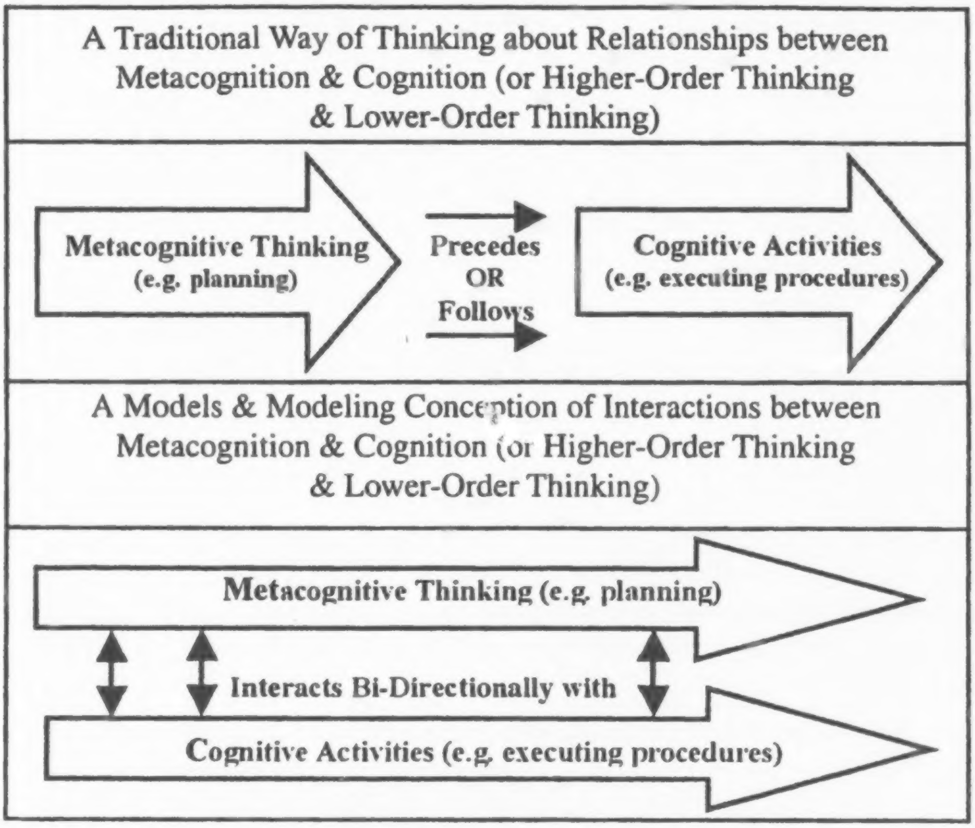

A Traditional Way of Thinking about Relationships between Metacognition & Cognition (or Higher-Order Thinking & Lower-Order Thinking)
Metacognitive Thinking (e.g. planning)
Precedes OR Follows
Cognitive Activities (e.g. executing procedures)
A Models & Modeling Conception of Interactions between Metacognition & Cognition (or Higher-Order Thinking & Lower-Order Thinking)
Metacognitive Thinking (e.g. planning)
Interacts Bi-Directionally with
Cognitive Activities (e.g. executing procedures)

Figure 2. Relationship between metacognition and cognition.

Al: Why we gotta find the average? How we gonna do that?...Let's get a calculator.

Bob: The average is, like, in baseball. ... It's the one in the middle. ... It's like you got three stacks and you gotta make 'em all the same. You move 'em up or down a little – if they're too high or low. ... Here, I'll show you. You gotta even 'em up. ... Take these. [Refers to 1.8, 8.7, and 4.5]. ... Now. Move... uh... move 7 from the big one [Refers to 8.7] to the little one [i.e. 1.8]... No, just move two.

Al: What are you doing? What are you doing? I don't get it.

Figure 3. Paper airplane excerpt.

Bob proceeds to explain the procedure until Al understands how he's finding the average. Carl (their third group member) then gets a calculator from their teacher and asks her how to calculate averages. They use averages in combination with other procedures to evaluate the rest of their data. The developmental view also means that students need the opportunity to develop metacognitive skills in the form of problems requiring such skills. The opportunity may also be provided by problems that require a group to solve. The complex nature of the problems in model-eliciting activities means that in order for students to successfully solve the problem they must plan, monitor, reflect, etc. and work effectively in a group. There is often too much data for one student to process alone in the time period they are allowed. Effective and useful solutions also require input from multiple sources and people.

The final characteristic of a models and modeling perspective is that models are developed in cycles. For example, the boys solving the Paper Airplane Problem spent some time initially examining the data table without much monitoring of each other's interpretations. They didn't pay attention to the different types of values in the table (some distances, some times). Later, they realize that how far a plane flies may be a different size number than how long it was in the air (e.g., 14.9 meters is a reasonable distance, but 14.9 seconds is not a reasonable time.) Later in the session, they carefully monitor each other's interpretations of the problem and procedures used to solve it. In one problem-solving session, there may be multiple cycles or phases of model development. From cycle to cycle, the models move along continuums from unstable to stable, incomplete to complete, and concrete to abstract. For the girls solving Amusement Park, at first they used only one piece of information (the ride preference table). This led them to find only the number of students for each ride. Later cycles of the solution process, they coordinated information between sources (e.g. the park map and the line waiting time chart) because the coordination led to better chaperone group formation. Just as students' models develop in cycles, metacognitive functions also occur and develop in cycles. As students improve their way of thinking about the problem situation, their solution and their model should improve. Hence, it is also important to note when metacognitive functions serve to help students change their way of thinking about the problem as they move from cycle to cycle. Often, at the beginning of the session, the students have not clearly defined the givens and goals and they may have an incomplete notion of the givens and goals in the problem (As Bob, Al, and Carl did for the paper airplane problem above). It is important for them to work through a few cycles of understanding so they can have a clearer, more complete understanding of the problem and how to solve it. One implication of this characteristic and the interactive nature of metacognition with cognitive activities is that it becomes clear that not all metacognitive functions are helpful all of the time in all situations. A metacognitive function that may be helpful at the beginning of the problem solving session may not be helpful at the end of the solution process. For example, reflection may be counter productive during initial brainstorming of ideas about the problem, but reflection is very helpful when finishing the solution to the problem in order to make sure the solution is appropriate and complete. Some metacognitive functions may be helpful when a model is still primitive and incomplete, and different metacognitive functions may be helpful once a model is more stable and complete.

The three characteristics discussed above (connection to lower-order functions, parallel interaction with lower-order functions, and cyclic development) have been accounted for in the development of model-eliciting activities that embody the types of everyday problem-solving activities requiring model development. For cyclic development to occur, it is necessary to have a problem situation requiring more than one cycle of interpretation. Outlining the client's specific criteria for a useful solution helps students self-assess their ways of thinking and move from cycle to cycle. This is aided by having problems where multiple interpretations of data and information are possible. It is also aided by problems asking for a process rather than a one number answer. A problem asking for a complex artifact means they have an opportunity to monitor and reflect on multiple interpretations.

Conclusions

The models and modeling framework as applied to metacognition attempts to clarify issues that have been unclear and examine metacognition from a perspective that is more easily observed and documented in students. Using model-eliciting problems, it examines new characteristics of metacognition for longer more complex problem solving situations asking for the development of a process. In particular, it addresses exactly what students are thinking about when they are thinking metacognitively. The model-eliciting activities provide problem-solving opportunities complex enough for students to have a need for metacognition and evaluate their interpretations of the problem. It also attempts to define how metacognitive functions develop and are employed in everyday problem-solving situations requiring the development of complex models and descriptions of systems. Metacognition remains a complex issue particularly when the interactions between metacognition and other aspects of problem solving are examined. However, this does not mean that it is any less important or that it cannot be examined and developed in a meaningful way.

References

American Association for the Advancement of Science. (1993). *Benchmarks for science literacy.* New York: Oxford University Press.

Brown, A. (1987). Metacognition, executive control, self-regulation, and other more mysterious mechanisms. In F. E. Weinert & R. H. Kluwe (Eds.), *Metacognition, motivation, and understanding* (pp. 65-116). Hillsdale, NJ: Erlbaum.

Doerr, H., & Lesh, R. (in press). *Beyond constructivism – A models & modeling perspective on mathematics problem solving, learning & teaching.* Mahwah, NJ: Lawrence Erlbaum.

English, L., & Lesh, R. (in press). Ends-in-view problems. In H. Doerr & R. Lesh (Eds.), *Beyond constructivism – A models & modeling perspective on mathematics problem solving, learning & teaching*. Mahwah, NJ: Lawrence Erlbaum.

Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for student and teachers. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* Mahwah, NJ: Lawrence Erlbaum.

Lesh, R., Lester, R., & Hjalmarson, M. (in press). Models and modeling perspective on metacognitive functions in everyday situations where mathematical constructs need to be developed. In H. Doerr & R. Lesh (Eds.), *Beyond constructivism – A models & modeling perspective on mathematics problem solving, learning & teaching*. Mahwah, NJ: Lawrence Erlbaum.

Middleton, J., Lesh, R., & Heger, M. (in press). Interest, identity, and social functioning: Central features of modeling activity. In H. Doerr & R. Lesh (Eds.), *Beyond constructivism – A models & modeling perspective on mathematics problem solving, learning & teaching*. Mahwah, NJ: Lawrence Erlbaum.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370), New York: Simon and Schuster, Macmillan.

Zawojewski, J., & Lesh, R. (in press). A models and modeling perspective on problem solving strategies. In H. Doerr & R. Lesh (Eds.), *Beyond constructivism – A models & modeling perspective on mathematics problem solving, learning & teaching*. Mahwah, NJ: Lawrence Erlbaum.

Zawojewski, J., Lesh, R., & English, L. (in press). A models and modeling perspective on the role of small group learning activities. In H. Doerr & R. Lesh (Eds.), *Beyond constructivism – A models & modeling perspective on mathematics problem solving, learning & teaching*. Mahwah, NJ: Lawrence Erlbaum.

LEARNING MULTI-DIGIT MULTIPLICATION BY MODELING RECTANGLES

Andrew Izsák
Northwestern University
izsak@northwestern.edu

Abstract: I report the results of interviews with 3 pairs of 5th-grade students during the course of a 2-digit multiplication unit implemented in a heterogeneous suburban classroom. The goal of the unit was for students to connect multiplication, arrays of unit squares, base-10 place-value[1], and the distributive property by modeling areas of rectangles. In the interviews, the students made such connections by (a) distinguishing between representational features for determining perimeters and areas and (b) refining contexts in which they applied criteria for using rectangular representations to solve multiplication problems.

Introduction

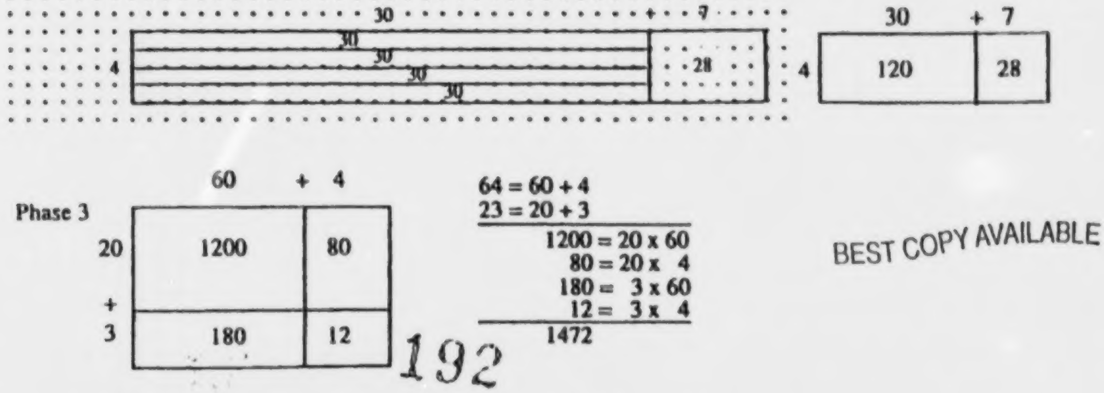
U.S. students often perform multi-digit multiplication poorly. In one international comparison, Stigler, Lee, and Stevenson (1990) reported that only 54% of U.S. 5th grade students in "traditional" courses could solve 45 x 26 correctly. The study reported here is a fine-grained analysis of three pairs of 5th grade students learning to solve similar two-digit multiplication problems with understanding through modeling activities. Modeling activities as contexts for learning are promoted by the National Council of Teachers of Mathematics (2000), and by others.

Existing research has established that learning to multiply involves recognizing classes of situations that can be modeled by multiplication (Greer, 1992), developing numerical strategies (Anghileri, 1989; Clark & Kamii, 1996; Kouba, 1989; Mulligan & Mitchelmore, 1997), coordinating psychological operations (Confrey, 1994; Confrey & Smith, 1995; Steffe, 1994), and restructuring whole number understandings when extending multiplication to rational numbers (Greer, 1992, 1994). This research has examined students' performance when using a range of external representations–including blocks, number lines, pictorial diagrams, tables, and rectangles–but has not examined in detail how students learn to use such representations for modeling and solving problems about multiplicative situations. Moreover, this research has not examined how students extend multiplication from single- to multi-digit numbers.

At least two mathematical issues arise when multiplication is extended from single- to multi-digit numbers and suggest that multi-digit multiplication is an important, though underrepresented, area of research. The first issue has to do with magnitudes of factors and products. Some single-digit products (e.g., 2 x 3 = 6) are of the same magnitude as both factors, but related products (e.g., 2 x 30 = 60 and 20 x 30 = 600) differ in magnitude from at least one factor. Thus students must coordinate understandings of place-value and single-digit multiplication to determine products. The second issue has to do with place-value and the distributive property. Efficient multiplication methods that generalize to arbitrary numbers of digits rely on multiplying each term in the expanded form for one factor by all terms in the expanded form for the second whether or not the expanded forms are made explicit. Thus students have also to coordinate place-value with the distributive property. Coordinating magnitudes of partial products, place-value, and the distributive property distinguishes multi- from single-digit multiplication and demonstrates that multiplication extends to multi-digit numbers in ways that differ from addition.

This study is part of the Children's Math Worlds project (CMW), an on-going project that develops instructional materials for elementary school mathematics and then conducts research on teaching and learning as teachers use those materials in their classrooms. Izsák and Fuson (2000) reported on an earlier design of the two-digit multiplication unit, implementation in one urban and one suburban 4th grade classroom, and strong whole-class results in both classrooms. This study extends Izsák and Fuson by examining in more detail how students can learn with the CMW materials. In particular, the study examines how 5th grade students refined their understandings of rectangular area representations and extended their understandings of single-digit multiplication. The students were in a second suburban classroom that was using a subsequent version of the unit. All classrooms were heterogeneous in terms of ability.

The version of the CMW two-digit multiplication unit used in this study relies on arrays of dots printed on whiteboards. Students use the whiteboards to draw rectangles and the unit squares inside. The goal is for students to coordinate magnitudes of partial products, place-value, and the distributive property by modeling areas of rectangles. Phase 1 lessons focus on single-digit multiplication problems, such as 4 x 7 (see Figure 1), and review connections among single-digit multiplication, repeated



BEST COPY AVAILABLE

Figure 1. The three phases in the CMW two-digit multiplication unit.

addition, arrays of unit squares, and areas of rectangles. Students draw rectangles on dot boards, focus on lengths connecting dots when thinking about perimeters, and focus on unit squares when thinking about areas. Phase 2 lessons focus on single-digit times two-digit multiplication problems. Students first examine the special case of a single-digit times a decade number (e.g., 4 x 30), and then the general case (e.g., 4 x 37). To prepare for work with larger numbers, students also sketch rectangles at the end of Phase 2. These rectangles are not to scale, and students are to understand that sketched rectangles can be used to multiply numbers of any size. Phase 3 lessons focus on two-digit by two-digit problems. For the example 64 times 23 shown in Figure 1, students sketch a rectangle; divide it into four parts that corresponded to 60 x 20, 60 x 3, 4 x 20, and 4 x 3; and add the partial products. Thus the CMW two-digit multiplication unit starts with connections among single-digit multiplication, repeated addition, arrays of unit squares, and areas of rectangles and develops methods that coordinate magnitudes of partial products, place-value, and the distributive property.

Theoretical Framework

This study is grounded in perspectives on learning found in (a) Smith, diSessa, and Roschelle (1993/1994), who described learning as the refinement and reorganization of prior knowledge that is useful in some contexts, (b) Izsák (2000) and Schoenfeld, Smith, and Arcavi (1993), who have emphasized the features of representations to which students attend as they construct knowledge structures through problem solving, and (c) diSessa, Hammer, Sherin, and Kolpakowski (1991), who coined the phrase meta-representational competence to describe students' understandings about the design and use of representations for solving problems. I focus on how students learn to model areas of rectangles with multiplication by refining and reorganizing the representational features to which they attend and the contexts in which they apply understandings about the design and use of representations.

Methods and Data

I interviewed three pairs of 5th grade students. In their teacher's experience, the students were articulate and worked well together. Naomi and Candis struggled with mathematics lessons, Jill and Eli were strong, and Sam and John were very strong. [2] I interviewed Naomi and Candis and Jill and Eli once a week for five weeks beginning in Phase 1 and continuing to the end of the unit. Due to scheduling constraints, I interviewed Sam and John at the middle and end of the unit. During the interviews I asked students to solve multiplication problems using dot paper that resembled the dot boards they were using in class. I used videotapes of whole-class lessons as a source of questions to pursue during the interviews and as a resource for understanding what students said.

I recorded the interviews using two video cameras, one to capture the students and one to capture what they wrote. I transcribed the interviews in their entirety and added notes indicating what students wrote and what hand gestures they used. I also kept all of the students' written work in case the videotapes did not capture important aspects clearly. I performed fine-grained analyses (Izsák, 1999; Schoenfeld et al., 1993) of line-by-line utterances, hand gestures, and evolving written work to determine how students used area representations to solve multiplication problems.

Analysis and Results

All of the students that I interviewed had used Everyday Mathematics (The University of Chicago School Mathematics Project, 1995) in the fourth and fifth grades. They could multiply two-digit numbers using the lattice method (see Figure 2), the method used in Everyday Mathematics, but could not explain the representation of place-value and the distributive property in the method. Thus their understanding was limited to the sequence of steps in the method. They had studied arrays using Everyday Mathematics activities that focused on counting dots and had limited experience with area, primarily in the third grade. Finally, they demonstrated connections between single-digit multiplication and repeated addition during the interviews when working on word problems such as the following: How many tomato plants are in Jim's garden if there are six plants in each of three rows? Analysis of the interviews led to three main results about how students can develop conceptual understanding of multi-digit multiplication by modeling areas of rectangles. I focus on Jill and Eli because data on these students led to all three results. Data on Naomi and Candis provided additional data for the first result, and data on Sam and John provided additional data for the third result.

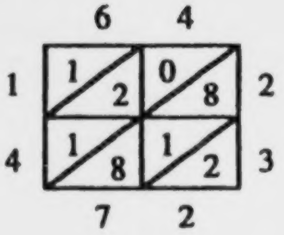


Figure 2. Lattice multiplication for 64 times 23.

1. Students distinguished representational features for determining perimeters from those for determining and areas.

When drawing and examining rectangles on dot paper, the students I interviewed focused at different times on dots, vertical and horizontal line spaces between adjacent dots, and square spaces between dots. The first interview with Jill and Eli occurred after initial CMW lessons. The students began with two methods for connecting multiplication to rectangles drawn on dot paper. Both methods used a single representational feature to answer questions about both perimeters and areas. The first method used dots, and the second used line segments that connected adjacent dots. Using the first method, the pair thought that the dimensions of a 5 by 7 rectangle drawn on dot paper were 6 and 8. They discussed the 6 rows of 8 dots and stated that the area was 48. Jill and Eli were explicit about their past experiences in which they had been told to always count dots when working with arrays.

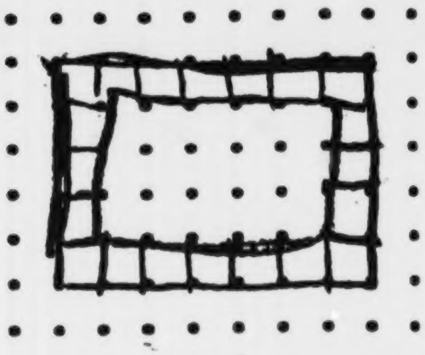


Figure 3. Jill and Eli used line segments to determine the area of a 5 by 7 rectangle.

Using the second method based on CMW lessons, Jill and Eli determined the correct dimensions of the same 5 by 7 rectangle and expected to count 35 line segments inside the rectangle as well. They drew horizontal line segments starting on the top edge and experienced conflict when they reached 35 before they were done (see Figure 3). The students stated that they had still to count the vertical segments. When I asked if they could count something else, Eli proposed unit squares and the students got their expected answer of 35. Learning to use line segments to determine dimensions, but unit squares to determine areas, was a significant representational accomplishment that allowed Jill and Eli to connect multiplication to areas of rectangles.

Naomi and Candis also had trouble distinguishing between representational features for determining perimeters and areas. Like Jill and Eli, they discussed two methods for determining perimeters and areas of rectangles drawn on dot paper, one based on dots and one based on line segments. Using the dot method, these students thought that the dimensions of a 5 by 7 rectangle were 6 and 8 and that the area was 48. Figure 4 shows how, when determining area using the second method, these students drew 20 line segments from the perimeter. The gestures on videotape made clear that the students counted line segments, not squares. They added the 24 line segments on the perimeter and got 44 as the area. In subsequent interviews, Candis continued having trouble selecting appropriate representational features when determining perimeters and areas. Sam and John had no trouble focusing on line segments when determining perimeter and unit squares when determining areas.

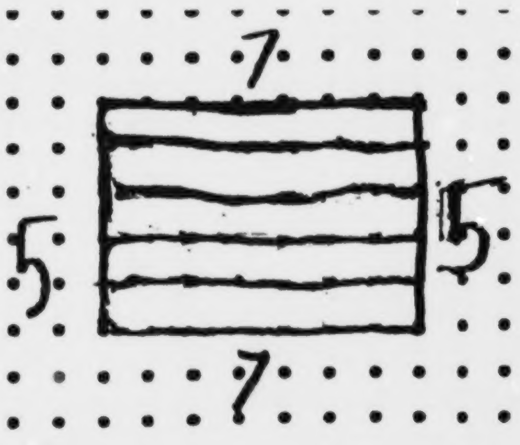


Figure 4. Naomi and Candis' method for counting line segments to determine area.

2. *Students refined the contexts in which they applied criteria for representations.*

Jill and Eli began the second interview, which occurred at the end of Phase 2, stating that rectangles, even when sketched on blank paper, had to be drawn to scale. The videotape showed the students drawing rectangles by stopping the movement of their pencils momentarily at the end of each imagined unit line segment. Eli eventually distinguished between multiplying "for fun," for example in an out-of-school game context, and multiplying to determine area, a school activity. She explained that rectangles had to be drawn to scale only when determining area.

In the third interview, Jill and Eli measured the dimensions of the rectangular interview room and drew representations on blank paper to determine the area. At first they drew rectangles to scale once more, but eventually focused on the array of unit rectangles formed when rectangles are not drawn to scale. Jill and Eli dropped their "scale" criterion for rectangular representations, a second significant representational accomplishment. I did not pursue issues of scale with Naomi and Candis because I continued to focus their interviews on representational features for determining perimeters and areas when rectangles were drawn on dot paper. Sam stated immediately in his second and last interview that rectangles did not have to be drawn to scale to determine the area of the interview room, and his comments evidenced understanding that the underlying array structure would give the same answer.

3. *Students demonstrated conceptual understanding of multi-digit multiplication.*

In the final interview, Jill and Eli developed their own correct method for multiplying two- by three-digit numbers that coordinated magnitudes of partial products, place-value, and the distributive property. They sketched rectangles, explained their method in terms of imagined arrays of unit rectangles, connected the rectangle and lattice methods, and explained how place-value is represented in the lattice method for the first time. Sam was also able to extend the rectangle method to two-digit times three-digit numbers and to connect the rectangle method to the lattice method. His explanations evidenced connections among magnitudes of partial products, place-value, and the distributive property.

Conclusions

This study extends research on multiplication by examining multi-digit multiplication and by applying fine-grained analytic techniques that have recently led to insights into students learning to use representations in other domains for solving modeling problems. The study examines how 5th grade students developed conceptual understanding of multi-digit multiplication through modeling areas of rectangles, and how they learned to model areas of rectangles by refining the representational features to which they attended and the contexts in which they applied criteria for rectangular representations. Both refinements were significant representational accomplishments. Such insights can inform future curricula that make conceptual understanding of multi-digit multiplication accessible to all students.

Note. The research reported in this paper was supported by the National Science Foundation under Grant No. REC 9806020. The opinions expressed in this paper are those of the author and do not necessarily reflect the views of NSF.

References

Anghileri, J. (1989). An investigation of young children's understanding of multiplication. *Educational Studies in Mathematics, 20*, 367-385.

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education, 27*(1), 41-51.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 291-330). Albany: State University of New York Press.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26* (1), 66-86.

diSessa, A., Hammer, D., Sherin, B., & Kolpakowski, T. (1991). Inventing graphing: Meta-representational expertise in children. *Journal of Mathematical Behavior, 10* (2), 117-160.

Greer, B. (1992). Multiplication and division as models of situations. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276-295). New York: Macmillan.

Greer, B. (1994). Extending the meaning of multiplication and division. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 61-85). Albany: State University of New York Press.

Izsák, A. (1999). *How Amy and Kate used criteria for algebraic expressions to develop knowledge structures for modeling the physical world.* Manuscript submitted for publication.

Izsák, A. (2000). Inscribing the winch: Mechanisms by which students develop knowledge structures for representing the physical world with algebra. *The Journal of the Learning Sciences, 9*, 31-74.

Izsák, A., & Fuson, K. (2000). Students' understanding and use of multiple representations while learning two-digit multiplication. In M. Fernández (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 715-721). Tucson, AZ.

Kouba, V. L. (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal for Research in Mathematics Education, 20* (2), 147-158.

Mulligan, J. T., & Mitchelmore, M. C. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education, 28* (3), 309-330.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA.

Schoenfeld, A. H., Smith, J., & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (pp. 55-175). Hillsdale, NJ: Erlbaum.

Smith, J. P., diSessa, A. A., & Roschelle, J. (1993/1994). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The Journal of the Learning Sciences, 3* (2), 115-163.

Steffe, L. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3-39). Albany: State University of New York Press.

Stigler, J. W., Lee, S,, & Stevenson, H. W. (1990). *Mathematical knowledge of Japanese, Chinese, and American elementary school children*. Reston, VA: National Council of Teachers of Mathematics.

The University of Chicago School Mathematics Project. (1995). *Everyday mathematics*. Evanston, IL: Everyday Learning Corporation.

Notes

[1] Heretofore I will use the term "place-value" to mean base-10 place-value.

[2] All names are pseudonyms.

CONCEPTUAL UNDERSTANDING OF FUNCTIONS: A TALE OF TWO SCHEMAS[1]

Mindy Kalchman
Northwestern University
m-kalchman@northwestern.edu

Karen C. Fuson
Northwestern University
fuson@northwestern.edu

Abstract: *Concepts-first* proponents propose that children are born with potential for conceptual knowledge in a domain and use this knowledge to generate and select procedures for solving problems in that domain. *Procedures-first* theorists propose that children first learn procedures for solving problems in a domain and then derive concepts from repeated experiences with solving those problems. A third view is that conceptual and procedural knowledge develop iteratively and influence each other. We propose a view that is based on Case's theory of intellectual development and that builds on and extends the third view. We suggest three ways in which conceptual and procedural knowledge relate to each other and support our view with empirical work on children's learning of mathematical functions.

Much has been written about the distinction between conceptual and procedural understanding in mathematics learning (e.g., Anderson, 1993; Greeno, et al., 1984; Hiebert, 1986). Considerable attention has also been given to whether conceptual or procedural knowledge emerges first (e.g., Gelman & Williams, 1998; Siegler, 1991; Sophian, 1997). *Concepts-first* proponents propose that children are born with potential for conceptual knowledge in a domain and use this knowledge to generate and select procedures for solving problems in that domain (e.g., Gelman & Meck, 1983; Halford, 1993). *Procedures-first* theorists propose that children first learn procedures for solving problems in a domain and then derive concepts from repeated experiences with solving those problems (e.g., Karmiloff-Smith, 1992; Siegler & Stern, 1998). A third view is that conceptual and procedural knowledge develop iteratively (e.g., Fuson, 1988; Rittle-Johnson et al., in press) and influence each other, with an increase in one type of knowledge leading to an increase in the other type, which stimulates an increase in the first, et cetera.

In this theoretical paper, we propose a view that builds on and extends the third view. Our view is based on Case's theory of intellectual development. It will be exemplified in this paper by empirical work on children's learning of mathematical functions. To emphasize knowledge in action, we will also use the terms "understanding" and "doing" for "conceptual knowledge" and "procedural knowledge," respectively. We suggest three ways in which conceptual knowledge (*understanding*) and procedural knowledge (*doing*) relate to each other: (1) *Understanding* and *doing* mathematics are always simultaneously present in any activity, but the proportion varies. (2)

Understanding and *doing* each arise initially within numeric/sequential and within spatial/analogic aspects of a domain, but then begin to relate across these aspects. (3) Developmentally, a mathematical activity that is viewed as a conceptual accomplishment, or *understanding*, at one age may be viewed primarily as *doing* at another age. The second and third ways of relating *understanding* and *doing* will be described after the theoretical framework is developed.

Relating *Understanding* and *Doing*

Most mathematical actions do not reflect only *understanding* or only *doing*. Rather, *understanding* and *doing* are always present in some proportion, with one or the other foregrounded. For example, people who understand why a particular algorithm works do not necessarily use those understandings while carrying out that algorithm, but they could shift into explaining mode and draw on these understandings. Similarly, it is not usually accurate to say that students who have learned an algorithm by rote understand nothing about it. Thus, it is more helpful always to consider that a mixture of *understanding* and *doing* are present or potentially present even when one is foregrounding *doing* or *understanding*.

The theoretical perspective we are taking for looking at how *understanding* and *doing* are related is that of Case's theory of intellectual development (e.g., Case, 1992, 1996). In Case's view, a deep conceptual understanding in any domain of learning is rooted in what he called a *central conceptual structure*. Central conceptual structures are central organizing features of children's domain-specific cognitive processing. They are constructed by integrating two primary mental schemas. The first of these is digital and sequential, and the second is spatial or analogic. In the first of four hypothesized phases of children's learning in a domain, each of these two primary schemas is elaborated in isolation. In the second phase they become integrated. The result is that a new psychological unit is constructed – the central conceptual structure – which constitutes deep conceptual understanding in that domain. During Phases 3 and 4, further learning and development (both in terms of biological maturation and learning experiences) build on students' understandings by elaborating further their numeric and spatial knowledge within the context of an already integrated understanding (a central conceptual structure).

Case's theory suggests the second and third aspect of the relationships between *understanding* and *doing*. As the second aspect, Case identified as conceptual understanding only those understandings that relate across numeric and spatial aspects of a domain. We also think that specifying conceptual understanding as related across numeric and spatial aspects is a useful point of view. However, we believe that a combination of *understandings* and *doings* are present in some proportion throughout each of these four stages of development. Therefore, we suggest the term "integrated understandings" or "integrated doings" for those *understandings* and *doings* that relate the numeric and spatial aspects.

For example, in Phase 1, each of the still-isolated primary numeric and spatial schemas includes both *understandings* and *doings*. In Phase 2, when the numeric and spatial schemas integrate to form a central conceptual structure the *understandings* and *doings* that were associated with each independent schema also integrate and become "integrated understandings" and "integrated doings". In Phase 3, variants of the Phase 2 central conceptual structure are developed as children apply their new structure to novel situations. These new Phase 3 structures involve further elaborations of either the embedded numeric or spatial structure. With these structural elaborations come sophistications in children's schema-relevant *understandings* and *doings* (i.e., spatial or numeric schema). Then, in Phase 4, numeric and spatial *understandings* and *doings* that are even further elaborated become the primary schemas that will integrate to form a central conceptual structure at the next level of development. That is, what is considered Phase 4 for one central conceptual structure is also Phase 1 for the next central conceptual structure.

The above model of development is best considered as an optimal learning sequence that should be supported and promoted through carefully designed instructional approaches. This view has already been proven effective in stimulating experimental learning units or curricula that result in powerful learning (e.g., Griffin & Case, 1996; Kalchman & Case, 1998, 1999; Moss & Case, 1999).

The third aspect of relationships between *understanding* and *doing* stems from Case's theory when one is looking across age levels of development. This aspect is that the same internal mental schema simultaneously indicates integrated conceptual *understandings* and *doings*, if viewed from one level, and not-yet-integrated conceptual *understandings* and *doings*, if viewed from the next level. This shifting across ages in viewing a mathematical accomplishment as *understanding* at one age but as *doing* at another age is fairly common in considering mathematical thinking. For example, learning that the last count word tells you "how many" (the count-cardinal principle) is a major conceptual accomplishment for 3 or 4 year-olds, but we take this understanding for granted for older children and consider their counting to be primarily *doing*, even though they automatically use this count-cardinal understanding.

Two Schemas-One Structure in the Domain of Functions

The topic of functions has been widely recognized as being central and foundational to mathematics in general. Literature indicates, however, that students of all ages have difficulty mastering the topic using traditional instruction approaches. The roles of numeric and spatial understanding in this domain are critical given that a concern among mathematics educators is that students have difficulty not only with moving among representations of a function (e.g., table, graph, equation, verbal description) (e.g., Goldenberg, 1995; Leinhardt et al., 1990; Markovitz et al., 1986), but also with understanding how and why the function concept is "representable" in tables, graphs, and equations (Thompson, 1994). Each of these representations embodies both spatial and numeric aspects of any function.

Two particularly relevant objectives have been among the major goals of our recent work. First, we have been working toward a comprehensive and coherent theoretical model for how students come to develop over time *understandings and doings* for functions, and how particular spatial and numeric characteristics of various functions influence development (Kalchman, 2001; Kalchman et al., 2001). Second, the first author has developed a curriculum intended to help students construct a central conceptual structure in a manner consistent with the proposed developmental sequence described in the model for learning (Kalchman, 2001; Kalchman & Case, 1998, 1999; Kalchman, Moss, & Case, 2001).

When students have a numeric understanding of functions, they can carry out calculations for making a table of values from an equation and plot the resultant coordinate points. This sort of understanding may be likened to what has been called a "process" or procedural understanding of functions (e.g., Kalchman & Case, 2000; Sfard, 1992). With this numeric understanding, students can use algorithms for finding, for example, the slope or y-intercept of a function. When students have a spatial understanding of functions, they can make qualitative judgements about the general shape of the graph of a function (e.g., straight or curved) or assess the magnitude of the slope of a function by comparing its steepness or direction (i.e., increasing or decreasing) to benchmark functions such as $y = x$ and $y = x^2$ (Confrey & Smith, 1994; Kalchman, 2001). This spatial understanding does not ensure that students have the computational skills for doing accurate quantitative comparisons.

When students have an integrated conceptual understanding, they can recognize and relate the spatial and numeric implications of the function concept in general and of each representation in particular. For example, a table of values is primarily numeric and sequential. However, there are spatial patterns that can be used for finding the slope, or even the overall shape, of a function: If the y-values in a table increase by 2 for every unit change in x, students can evaluate the pattern between successive y values, which is a constant increase of 2. From this pattern they can discern that the function must be linear with a relative steepness, or slope, of 2. This pattern can then be generalized into a symbolic expression of $y = 2x$.

Influences of Instruction on Students' Integrated Understanding of Functions

In a recent analysis of two modern secondary-level textbook units on functions (Kalchman, 2001), a strong emphasis was identified only on the development of numeric, sequential knowledge in the form of algorithms and mathematical notations. Spatial elements were barely addressed, and when they were, they were generally shown as isolated representations that *resulted* from numeric/algebraic procedures. That is, a graph of a function might be generated by carrying out calculations for finding coordinate pairs, but the graph itself was not an object of mathematical inquiry or thought. This sort of limited instruction cannot support students' developing of a well-constructed, balanced, and integrated conceptual framework for the domain. Without such a framework, students will have difficulty in simultaneously *doing* and *understanding* more advanced mathematics such as calculus.

Students' opportunities for constructing such an integrated conceptual framework are greatly increased when they experience instruction and curricula that focus on developing both numeric and spatial *understandings* and *doings* and ensure ample opportunity for integrating all of these. One such curriculum was developed by Kalchman and Case and has been shown experimentally to help students construct deeper and more flexible understandings of functions than do students who learn from textbooks (Kalchman & Case, 1998, 1999). In this integrative curriculum, the context of a walkathon is used to bridge students' spatial and numeric understandings and to help them foster a central conceptual structure for the domain (see Kalchman, 2001, for a description of this curriculum).

To illustrate differences in students' reasoning about functions following a textbook unit to that of the "walkathon" approach, we will exemplify differences between an integrated conceptual understanding of select functions problems and an understanding that favors the numeric, sequential aspect of the domain. We will use examples of how students in an advanced-level Grade 11 mathematics class ($n = 17$), who had at least three years of textbook-based instructions in functions, responded to two tasks. These examples will be compared to responses to those same items by students in a high-achieving Grade 6 sample ($n = 48$), who had experienced three weeks of the "walkathon" curriculum.

Methods

Forty-eight Grade 6 students and 17 Grade 11 students were involved in the present study. All students attended the same independent school north of a major urban center in Canada. The Grade 6 students comprised two intact classes at the school. Each of these sixth-grade classes had 12 classes of experimental instruction at about 50 minutes each for a total of 600 minutes of instructional time. The first-author of this paper taught these students. The Grade 11 students were an intact class at the same school. This group had 15 classes of a standard textbook unit of functions, with each class being 80 minutes long for a total of 1200 minutes of instructional time. The regular classroom teacher taught these students. Each class of students was given the same set of problems to complete both before and after the respective instructional units. Only posttest responses will be presented and discussed here because pretest results were so low for both groups and the effects of instructional type are most interesting.

Tasks and Results

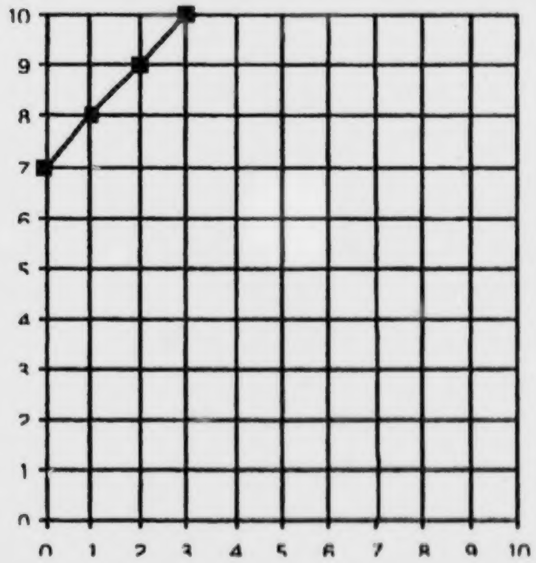
The tasks shown here are from a twelve-item functions test developed for a larger study (the complete test may be found in Kalchman, 2001). Both of the tasks below represent items designed to test for students' first integrated understanding of functions, i.e., their baseline integrated, conceptual understanding of functions.

In the first task (see Figure 1), students were asked to give an equation for a function that would cross through $y = x + 7$, which was shown graphically in the upper-right quadrant of a grid with a unitized scale from 0 - 10 on each axis. Students were told that their function had to pass through the given function within the observable space. We decided to have students work within the observable space in order to test their abilities to generate specific (albeit partial) representations of a more general function (Schwartz & Dreyfus, 1995). The solution space for this item is infinite. As a result of instructional experiences at both grade levels, students were likely to generate one of three particular types of functions: increasing functions with slopes steep enough to pass through the given function (with y-intercepts ranging from $0 \leq b \geq 7$); decreasing functions with y-intercepts ≥ 7; or increasing or decreasing curving functions again with a number of different y-intercepts possible.

Fifty-three percent of the older students and 73% of the younger students gave a correct solution for this item. A qualitative analysis of the approaches taken to the problem by each group overall also indicates how most of the younger students were

---

Can you think of a function that would cross the function seen in the graph below? What is the equation of the function you thought of?



Explain why the equation you chose is a good one.

Figure 1. Item 1.

using an integrated conceptual framework that involved an interactive dynamic of *understandings* and *doings* and how the older students were using primarily just a numeric approach.

A common error among the eleventh graders was first to use algorithms to find the slope and y-intercept of the graphed function and then to give the slope/intercept form of the equation. Students then substituted the negative reciprocal of the slope into the equation to get . Although this equation may be considered a correct solution, many students then proceeded to draw a line perpendicular to the one given to represent this new function, but with y-intercepts greater than 7. These students explained that "the line is perpendicular so it works." At first glance the approach seems sophisticated and suggests conceptual understanding. However, many of these students relied heavily on algorithms and did not recognize, or at least acknowledge, the algebraic implications of moving the y-intercept on the graph.

The younger students seemed to approach the problem from a more conceptually integrated point of view. Many of these students first drew a line that passed through the original one, and then derived the equation from the information on the graph. For example, one student drew a line from (0, 10) to (5, 0), made a table of values, wrote the equation $y = x*-2 +10$ and explained "Because it goes down by –2 [sic], so if it starts at 10 it will pass through the line." This sort of solution suggests an integrated conceptual approach to the task in that she is using a numeric approach for deriving the equation from the number pattern found on the graph while connecting it to the spatial entailments of the problem. Operating numerically and spatially suggests simultaneous procedural and conceptual activitations in the student's reasoning that most of the older students did not demonstrate.

In the second item, students were asked to make a table of values for the graph of an increasing linear function with a negative y-intercept (see Figure 2). The integrated conceptual understanding required here is in students' ability to use numeric and spatial *understandings* and *doings* to show that both the graph and the table for an

Make a table of values that would produce the function seen below.

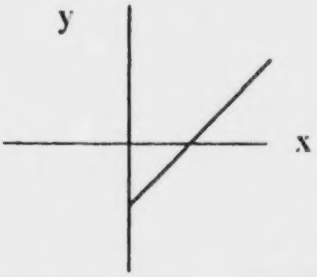


Figure 2. Item 2.

increasing linear function have a constant slope. This is done by generating a table that has a constant increase in y for every unit change in x. It was generally expected that students would produce a correct response by estimating the value of the y-intercept and constructing a table of covarying quantities that increase from there in a linear fashion (e.g., Confrey & Smith, 1995).

Only 18% of the eleventh-grade students gave correct solutions to this item compared to 54% of the sixth graders. There were two common errors for the older students. The first was to estimate coordinate points that would be on the line and simply record those points in a table. This strategy was used without regard for the idea that the x and y values must covary in a certain way -- the y-values must increase at a constant rate for every unit change in x. The second common error resulted from students' difficulty with identifying coordinate pairs on a line. Many students (29% of them) still erroneously determined coordinates by taking the y-value from an estimated y-intercept and the x-value from an estimated x-intercept and calling that a coordinate. The Grade 11 students' reliance on a numeric approach was a tenuous strategy at best, even when an algorithm was known or mastered. When *doings* were flawed, however, such reliance was a major impediment.

The sixth graders, on the other hand, showed a clear understanding that the table needed to have a constant increasing linear pattern, which they manifest with constant covariation between x and y. The limiting factor for these younger students was their difficulty with proficiently computing with negative numbers.

This second task was especially revealing with respect to showing how the textbook-taught students were not attending simultaneously either to the spatial and numeric aspects of a function or to the intrinsic *understandings* and *doings* required for the problem. Rather, they were relying primarily on numeric strategies and with an emphasis on *doing*. On the other hand, the younger students did show an integrated approach to this problem, which suggested again how numeric and spatial schemas and how *understanding* and *doing* are inter-active in sophisticated mathematical reasoning.

Summary and Implications

We used Case's theory of intellectual development and related empirical work on the teaching and learning of functions as a guiding framework to show how conceptual and procedural knowledge relate to each other as children construct an integrated conceptual cognitive structure for understanding in the domain. We argued that *understandings* and *doings* (procedural and conceptual knowledge) are present in some proportion when students are reasoning about sophisticated mathematical ideas such as those found in functions. We also presented the case for how numeric and spatial features of functions must be co-active when creating a central conceptual structure for understanding in a domain such as functions, which includes multiple ways of representing a common concept. Such co-activation may be promoted and facilitated through appropriate curricular and instructional design such as the "walkathon" curriculum presented here, with its meaningful bridge to students' previous *understandings* and *doings*.

The two tasks discussed above are particularly interesting because they show how young students who learned with the experimental curriculum were reasoning in an integrated conceptual fashion using both *understandings* and *doings* and were much more successful than older, more experienced students. The older textbook learners, on the other hand, demonstrated a mostly numeric and *doings* approach to the problems and struggled with these problems.

The implications for this work are many, especially with respect to practice, including curriculum design and classroom teaching. For example, sixth graders were found to be capable of integrating their primary understandings to form an integrated conceptual structure for functions. Thus, meaningful instruction on functions might begin earlier in students' school learning experiences. We would suggest that particular attention be paid to designing and implementing the sort of curricula and instruction used here, which are based in a cognitive theory that promotes the enrichment and integration both of the spatial and numeric aspects of function and of relevant *understandings* and *doings* in order to construct a deep integrated conceptual understanding in the domain. Such an integrated conceptual foundation might help prevent the difficulties found at present among older students' learning of functions.

We have longer-term goals in this endeavor. First, we must try to develop language that will adequately convey the complexities of these issues. We seek to bridge and eventually to blend the procedural-conceptual divide because we find these to be continually and inextricably intertwined. Furthermore, this divide stimulates political debates about learning goals and about pedagogy that, in our view, do not advance the public interest in all children learning mathematics in comprehensible ways. Second, in order to investigate and potentially measure the impact of procedural automaticity on conceptual gains and not-yet-conceptual structures for understanding functions (and other mathematical domains), further cross-sectional and longitudinal work needs to be carried out.

References

Anderson, J. R. (1993). *Rules of the mind.* Hillsdale, NJ: Erlbaum.

Case, R. (1992). *The mind's staircase: Exploring the conceptual underpinnings of children's thought and knowledge.* Hillsdale, NJ: Erlbaum.

Case, R. (1996). Introduction: Reconceptualizing the nature of children's conceptual structures and their development in middle childhood. In R. Case & Y. Okamoto (Eds.), *The role of central conceptual structures in the development of children's thought. Monographs of the Society for Research in Child Development, 61* (1-2, (246), 1-26). Chicago: University of Chicago Press.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics*, *26*, 135-164.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, 26 (1), 66-86.

Fuson, K. C. (1988). *Children's counting and concept of number*. New York: Springer-Verlag.

Gelman, R., & Meck, E, (1983). Preschooler's counting: Principles before skill. *Cognition*, *13* (3), 343-359.

Gelman, R., & Williams, E.M. (1998). Enabling constraints for cognitive development and learning: Domain specificity and epigenesis. In D. Kuhn & R. S. Siegler (Eds.), *Handbook of child psychology: Cognition, perception, and language* (5th ed., Vol. 2, pp. 575-630). New York: John Wiley & Sons.

Goldenberg, E. P. (1995). Multiple representations: A vehicle for understanding understanding. In D. N. Perkins, J. L. Schwartz, M. Maxwell West & M. Stone Wiske (Eds.), *Software goes to school* (pp. 155-171). Oxford University Press.

Greeno, J. G., Riley, M. S., & Gelman, R. (1984). Conceptual competence and children's counting. *Cognitive Psychology*, *16*, 94-143.

Griffin, S., & Case, R. (1996). Evaluating the breadth an depth of training effects when central conceptual structures are taught. In R. Case & Y. Okamoto (Eds.), *The role of central conceptual structures in the development of children's thought. Monographs of the Society for Research in Child Development*, *61* (1-2, (246), 83-102). Chicago: University of Chicago Press.

Halford, G. S. (1993). *Children's understanding: The development of mental models*. Hillsdale, NJ: Erlbaum.

Hiebert, J. (1992). Mathematical, cognitive, and instructional analyses of decimal fractions. In G. Leinhardt, R. Putnam, & R. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 283-322). Hillsdale, NJ: Erlbaum.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. New York: Macmillan.

Kalchman, M. (2001). *Learning and teaching mathematical functions: A neo-Piagetian framework for instruction and analysis*. Unpublished doctoral dissertation. University of Toronto.

Kalchman, M., & Case, R. (1998). Teaching mathematical functions in primary and middle school: An approach based on neo-Piagetian theory. *Scientia Paedagogica Experimentalis*, *35* (1), 7-54.

Kalchman, M., & Case, R. (1999). Diversifying the curriculum in a mathematics classroom streamed for high-ability learners: A necessity unassumed. *School Science and Mathematics*, *99* (6), 320-329.

Kalchman, M., & Case, R. (2000). Modeling the development of the concept of function. In Fernández, M. (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 241-246). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Kalchman, M., Moss, J., & Case, R. (2001). Psychological models for the development of mathematical understanding: Rational numbers and functions. In S. Carver and D. Klahr (Eds.), *Cognition and instruction: 25 years of progress* (pp. 1-38). Mahweh, NJ: Erlbaum.

Karmiloff-Smith, A. (1992). *Beyond modularity: A developmental perspective on cognitive science*. Cambridge, MA: MIT Press.

Leinhardt, G., Zaslavsky, O., & Stein, M. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60* (1), 1-64.

Markovitz, Z., Eylon, B., & Bruckheimer, M. (1986). Functions today and yesterday. *For the Learning of Mathematics*, *6* (2), 18-28.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30* (2), 122-147.

Rittle-Johnson, B., Siegler, R.S., & Alibali, M. (in press). Developing conceptual and procedural understanding and procedural skill in mathematics: An iterative process. To appear in *Journal of Educational Psychology*.

Schwartz, B., & Dreyfus, T. (1995). New actions upon old objects: A new ontological perspective on functions. *Educational Studies in Mathematics, 29* (3), 259-291.

Siegler, R.S. (1991). In young children's counting, procedures precede principles. *Educational Psychology Review, 3* (2), 127-135.

Siegler, R.S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: A microgenetic analysis. *Journal of Experimental Psychology: General*, 127, 377-397.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification: The case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epsitemology and pedagogy* (pp. 59-84). West LaFayette, IN: Mathematical Association of America.

Sophian, C. (1997). Beyond competence: The significance of performance for conceptual development. *Cognitive Development, 12*, 281-303.

Thompson, P. W. (1994). Students' functions and the undergraduate curriculum. In E. Dubinsky, A. Schoenfeld & J. Kaput (Eds.), *Issues in mathematics education, Vol. 4: Research in collegiate mathematics education. I* (pp. 21-44). Providence, RI: American Mathematical Society.

Note

[1] This work was made possible through the generous support of the James S. McDonnell Foundation.

YOUNG CHILDREN'S REPRESENTATION OF GROUPS OF OBJECTS: THE RELATIONSHIP BETWEEN ABSTRACTION AND REPRESENTATION

Constance Kamii
University of Alabama at Birmingham
ckamii@uab.edu

Abstract: Sixty Japanese children ranging in age from 3 years 4 months to 7 years 5 months were individually interviewed with three Piagetian tasks. Children's levels of representation were assessed by asking for a graphic representation of 4 dishes, 6 pencils, 8 small blocks, etc. A conservation-of-number task was then given to assess children's level of abstraction. It was found (a) that there is a close relationship between children's levels of abstraction and of representation, and (b) that children can represent at or below their level of abstraction but not above this level.

There is a common belief in mathematics education that children progress from the "concrete" to the "semiconcrete" level of pictures and then to the "abstract" level of numerals and mathematical symbols. However, Piaget and his collaborators (Furth, 1981; Greco, 1962; Piaget & Inhelder, 1948/1956) made a distinction between abstraction and representation and showed that pictures can be used at a high or low level of abstraction, and that numerals, too, can be used at a high or low level of abstraction. The purpose of this study was to investigate the relationship between abstraction and representation by building on Sinclair, Siegrist, and Sinclair's research (1983).

Method

The subjects were 60 Japanese children ranging in age from 3 years 4 months to 7 years 5 months. The 60 consisted of 15 each of the following four groups: 3 years 4 months to 4 years 5 months; 4 years 6 months to 5 years 5 months; 5 years 6 months to 6 years 5 months; and 6 years 6 months to 7 years 5 months. The children were randomly selected from class lists of two private day-care centers and two public elementary schools in Fukuyama and Okayama (without any consideration of gender). All came from middle-class families.

Three tasks were administered in individual interviews. The children's responses were categorized by three researchers using the criteria for each task as described below. The reliability coefficient was found to be .86 for the first task, .95 for the second, and 1.00 for the third. The researchers discussed disagreements until consensus was reached.

Representation-of-groups-of-objects task

The child was given a sheet of paper and a black marker. Four small dishes were then aligned, and the interviewer asked the child to "draw/write what's here so that your mother will be able to tell what I showed you." (In spoken Japanese, the words for "draw" and "write" sound exactly the same.) The interviewer was careful not to use words like "number" and "how many" which could have suggested quantities. When a child asked whether to draw or to write, the response was "*You* decide which way you like." The same procedure was followed with (a) 3 spoons, (b) 6 pencils, and (c) 8 small blocks, with a new sheet of paper each time.

The children's responses were categorized using three of the six types of representation conceptualized by Sinclair, Siegrist, and Sinclair (1983).

Level 1: Global representation of numerical quantity (absence of one-to-one correspondence). For example, the child drew 8 lines to represent 6 pencils or 12 shapes to represent 8 blocks.

Level 2: Representation of all the sets with one-to-one correspondence. Examples are: 6 lines for 6 pencils and "1234" for 4 dishes.

Level 3: Representation with one numeral indicating the total quantity. For example, writing "4" to represent 4 dishes.

Conservation-of-number task

This task was given to assess the child's level of abstraction. (According to Piaget, children construct number through *constructive* abstraction, which he also called *reflective* or *reflecting* abstraction.) With 20 each of red and blue counters, the child was asked (a) to make a row that had the same number (as the eight that had been aligned) and (b) (after the one-to-one correspondence had been destroyed) whether there were as many in one row as in the other, or more in one row or more in the other, and "How do you know that?"

The responses were categorized according to the following levels:

Level 1: Absence of one-to-one correspondence

Level 2: One-to-one correspondence without conservation

Level 3: One-to-one correspondence with conservation

To be categorized at Level 3, the child had to give one of the following logical explanations:

a. You didn't add or take away anything (identity).

b. I can put them back to the way they were before, and you'll see that it's still the same amount (reversibility).

c. This line is longer, but it has lots of space in between (compensation).

Writing-of-numerals task

This task was given to find out if, in the representation-of-groups-of-objects task, children used the numerals they knew how to write. Without the presence of any objects, the interviewer asked, in random order, "Can you write a *three* (then *four*, *eight*, *six*, and so on)?"

The responses were categorized into the following three levels:

Level 1: No knowledge of any numeral.

Level 2: Some knowledge of some numerals.

Level 3: Knowledge of all the numerals.

Results

Ten of the 60 children were excluded from the analysis because their representation of the objects in Task 1 did not include a quantitative aspect. For example, they drew only one dish for 4 dishes, representing only the qualitative aspect. As can be seen in Table 1, these children were found at the lowest level of abstraction as well as at the highest level.

Relationship between Abstraction and Representation

Table 1 shows the relationship between children's levels of *representation* revealed by the representation-of-groups-of-objects task and their levels of *abstraction* demonstrated in the conservation task. It can be seen in this table that most of the children (36/50, or 72%) showed a perfect relationship between the two variables in the diagonal cells. Eight (16%) were found to be at Level I on both tasks, 14 (28%) at Level II, and 14 (28%) at Level III on both tasks. In other words, 8 (16%) could not make a one-to-one correspondene in the conservation task (Level I) and drew an incorrect number of circles or sticks in the representation task (Level I). Fourteen (28%) made a one-to-one correspondence without conservation (Level II) and drew the correct number of pictures in the representation task (Level II). Another 14 (28%) conserved number (Level III) and wrote one numeral for the total number in the set (Level III). Below the diagonal in Table 1 are 14 children (4 + 10) who represented at a lower level than their level of abstraction. Four of them were at Level II in abstraction but at Level I

Table 1. Relationship Between Levels of Abstraction and of Representation

Levels of Abstraction	Levels of Representation			Representation Only of Qualitative Aspect
	I	II	III	
I	8	0	0	2
II	4	14	0	0
III	0	10	14	7

in representation. Ten were at Level III in abstraction but at Level II in representation. However, *no one was found to be at a higher level of representation than of abstraction* (above the diagonal in Table 1). These were significant findings both statistically and theoretically and will be discussed further below.

Relationship between knowledge of numerals and their use

As can be seen in Table 2, 33 (1 + 18 + 14) of the 50 children knew how to write all the numerals and were categorized at Level III of the writing-of-numerals task. However, only 14 of them used this knowledge in the representation-of-groups-of-objects task. The majority of those who knew how to write numerals used pictures and tally marks (Level II of the representation task). The significance of this phenomenon, too, will be discussed below.

Table 2. Relationship Between Knowledge of Numerals and Levels of Representation

Knowledge of numerals	Levels of representation		
	I	II	III
I	10	4	0
II	1	2	0
III	1	18	14

Discussion

Piaget (1977) pointed out that when children represent reality, they do not represent reality itself. They represent what they *think* about reality. If they do not yet have number in their heads and look at a set of objects, they cannot think about the objects with numerical precision. They therefore represent the set they are looking at at Level I, with a vaguely quantitative idea like "a bunch" or "more than one."

As they construct number (through constructive abstraction), children become able to see each set with more numerical precision. They then become able to represent each set at Level II, with numerical accuracy.

Level-II representations indicate that children are still thinking about individual objects. By contrast, Level-III representations show that children are now thinking about the total quantity as a higher-order unit. For these Level-III children, "4" seems better suited to represent 4 dishes than "oooo" or "1234."

Children's writing "1234" for 4 dishes is especially instructive because it shows that children use numerals, too, at their respective levels of abstraction. If they are still thinking about individual objects, they use numerals, too, in a way that allows them to represent their thinking.

It is also significant to note in Table 2 that most of the children who knew how to write numerals did not use this knowledge. They preferred to use pictures or tally marks that permitted them to represent each object in the set. This finding, too, supports Piaget's view that when children represent reality, they represent what they *think* about reality.

It was pointed out in connection with Table 1 that children sometimes represented at a level *lower* than their level of abstraction, but never at a level *higher* than their level of abstraction. This finding supports Piaget's view, elaborated clearly by Furth (1981), that children can represent their knowledge *at* or *below* their level of abstraction but not *above* this level. At Level I of the conservation task, for example, children cannot represent number concepts that are not yet in their heads. At Level II of the conservation task, they cannot represent a total quantity that is not yet one solid unit.

Mathematics educators often make statements about counting such as the following: "Children should learn that the last number named represents the last object as well as the total number of objects in the collection (NCTM, 2000, p. 79)." This statement reflects an erroneous assumption that numerals represent number. Representation is what human beings do. Neither numerals nor pictures represent. Therefore, children should not be taught that the last number named represents the total number of objects in the collection. Without any teaching, by the time they are at Level III of the conservation task, children become able to use numerals to represent the total number of objects.

This study shows the inadequacy of the belief that children progress from the "concrete" to the "semiconcrete" level of pictures and then to the "abstract" level of numerals and mathematical symbols. In the conservation-of-number task, we saw that children can think about concrete objects at a high or low level of abstraction. (At Level II of abstraction, they cannot conserve, but at Level III, they conserve the equality of the two sets.) Pictures and tally marks, too, can be drawn at a higher or lower level of abstraction. (At Level I of the representation task, children revealed their prenumerical thinking by drawing an inaccurate number of objects. The number became correct at Level II of representation, reflecting a higher level of abstraction.) Spoken and written numerals, too, can be used at a higher or lower level of abstraction. (The children who wrote "1234" were still thinking about individual objects. Those at a higher level of abstraction wrote "4" because they thought about the set of objects as a unit.)

The representation task in this study asked for children's *productions*. The relationship between abstraction and representation is more transparent in their productions than in their *reading* of numerals and mathematical symbols. For example, when first graders see "4 + _ = 6," some write "2" in the blank, but many write "10." This is because when children see mathematical writing, they represent to themselves the relationship that they are capable of making (abstraction). Those who can make a part-whole relationship represent this relationship to themselves and know that 4 is part of 6. Those who cannot make a part-whole relationship merely represent "4," "+," and "6" to themselves.

Older children may know symbols such as "0.321," "0.2," and "6/8 x 2/3" at some level. However, different children at different levels of abstraction give different meanings to these symbols in various contexts. The implication of this study for older children is that educators need to focus their efforts more sharply on the *mental* relationships children are making (abstraction) rather than simply believing that children understand or do not understand *the meaning* of "0.321," "0.2," and "6/8 x 2/3." The meaning is in the child's head, not in the symbols.

References

Furth, H. G. (1981). *Piaget and knowledge, 2nd ed.* Chicago: University of Chicago Press.

Greco, P. (1962). Quantite et quotite. In P. Greco & A. Morf, *Structures numeriques elementaires* (pp. 1-70). Paris: Presses Universitaires de France.

National Council of Teachers of Mathematics (2000*). Principles and standards for school mathematics*. Reston, VA: Author.

Piaget, J., & Inhelder, B. (1956). *The child's conception of space* (F. J. Langdon & J. L. Lunzer, Trans.). London: Routledge & Kegan Paul. (Original work published 1948)

Sinclair, A., Siegrist, F., & Sinclair, H. (1983). Young children's ideas about the written number system. In D. Rogers & J. A. Sloboda (Eds.), *The acquisition of symbolic skills*, (535-541). New York: Plenum.

Piaget, J. (1977). *Piaget on Piaget* (videotape). Yale University Media Design Studio.

"I WOULDN'T WANT TO GET MY HEAD WHOMPED OFF BY A BUTCHER'S KNIFE!" UNDERSTANDING CHILDREN'S PERFORMANCE ON STATE ASSESSMENTS

Elham Kazemi
University of Washington
ekazemi@u.washington.edu

Many states and school districts are increasingly trusting standardized and performance-based assessments to gauge student achievement. The rhetoric of accountability and standards has put teachers under pressure to prepare their students to pass these end-of-the-year tests (McNeil, 2000). We know very little about how students interpret items and why they choose the answers they do, yet we rely heavily on test scores to make policy, promotion, and instructional decisions. This study aimed to investigate what meaning students make as they solve items that appear on these assessments. Using interview data with 90 fourth-graders, this exploratory study examines why particular kinds of items might be more or less difficult for students. It raises questions about what we learn about students' skills and understanding from their performance on multiple-choice questions.

This study was motivated in part by my work with teachers. Like other mathematics educators, my work centers on supporting teachers to understand and make use of children's mathematical thinking in making pedagogical and curricular decisions. It has been striking to work with a group of teachers over the course of the year, helping them to learn to ask better questions of their students' and to elicit their thinking and then observe the same teachers transform their mathematics instruction as the end-of-the-year assessment approach. Classroom time shifts to practicing an array of mathematical procedures teachers' anticipate will be covered on the tests. Months later when the test results return, some teachers are dismayed with their students' performance. After living through the frustrations with teachers, I decided to explore children's performance on state assessments with the same focus on children's thinking that I use to guide my work with teachers in classrooms (Franke & Kazemi, 2001).

Theoretical Framework

A situated perspective on learning guides the interpretive stance of this paper (e.g., Greeno & MMAP, 1998; Lave & Wenger, 1991; Putnam & Borko, 1997). I take the position that the testing situation, like others that children encounter, is a particular activity setting with its own norms and rules (Miller-Jones, 1989). The assumption I make is that children bring a history of experiences that influences their mathematical skills but that also influences how they makes sense of a problem situation. I ask, in this paper, how children interpret the tasks they are given and what guides their choices. This study aimed to reveal not only the mathematics that children "know" but also other knowledge that children invoke in their problem solving efforts. It follows in the tradition of much of mathematics education research focused on documenting how children approach and understand mathematical problems (e.g., Carpenter et al., 1999; Lampert, 1991). It reveals not only the mathematics that children "know" but also how children's interpretation of a problem solving task influences their choices (e.g., Tate, 1994).

Methods

The study draws from interview data involving 90 fourth-graders (48 girls, 42 boys). Data was collected in April 2000, several weeks before the administration of the state assessment in Washington State. Children were selected from 5 urban schools, across 12 classrooms. The schools were selected based on their interest in participating in the study. They were "typical" of urban schools in that they had diverse student bodies, varying levels of curriculum innovation, and relatively poor performance on the state assessments. The sample was ethnically diverse (14% African American, 26% Asian, 12% Latino, 4% Native American, 43% White). Approximately 60% of the students at four of the five schools were on free or reduced lunch at 4 of the 5 schools. Less than 25% of the students at those four schools met or exceeded the standard for passing the mathematics portion of the state assessment the year prior to the study. At the fifth school, about a third of the students at the last school were on free or reduced lunch and 50% of the students met or exceeded the standard for passing the mathematics portion of the state assessment the year prior to the study.

I examined the multiple-choice mathematics portion of the fourth-grade state assessment, drawing from sample items that were widely distributed to teachers for use in helping prepare their students. I selected items that routinely appear on standardized tests, purportedly because they measure students' problem solving abilities (e.g., NAEP, ITBS, Stanford Nine). Based on my experience working with young children, I expected these problems to elicit a diverse range of student interpretation. (Because of space constraints in this report, I summarize findings from 3 of the 6 items.) For two of the three multiple-choice items reported here, a similar open-ended problem was posed to determine student responses in the absence of a fixed set of choices. The items were counterbalanced on two different forms of the interview. Half of the students were given Form A and thus solved the multiple-choice items before the open-ended items. The other half was given Form B and thus solved the open-ended items before the multiple-choice. When students were given the multiple choice item, they were asked to explain why they chose a particular answer. They were also asked to explain why they did not pick other choices. On open-ended items, they were asked to explain their solution. Transcripts were analyzed to document the explanations students gave.

Results

Finding the Appropriate Number Sentence

The first problem involved asking students to read a word problem and then select a number sentence that would help them find the solution. My conjecture for this problem was that young students would have difficulty with this task because they would not actually solve the problem first and would choose incorrect answers at a higher rate because they would not have fully thought through the problem.

> *Multiple choice item:* Juan and Bill worked together to unload bags of food from a van. On each trip, Juan carried 6 bags and Bill carried 4 bags. They each made a total of 3 trips. Which number sentence would you use to find how many bags they unloaded in all?
>
> a) 3 x 6 x 4 =
> b) (3 x 6) + 4 =
> c) (3x6) + (3x4)=
>
> *Open-ended item:* Thomas and Joelle were moving books to a new shelf. Thomas carried 3 books at a time. Joelle carried 5 books at a time. They each made 4 trips to the bookshelf. *Write a number sentence to find how many books they moved in all.*

In the multiple choice format, I expected students to read the problem and select a number sentence without necessarily solving the problem itself. The results shown in Table 1 confirmed that conjecture. Of the students who received Form A of the interview, meaning they had solved the multiple-choice question first, and then the open-ended one, 17 out of 45 or 38% of them chose the correct response. On Form B, when students had solved an open-ended question first, 31 out of 45 or 69% of the students chose the correct response. A rather trivial experimental manipulation (posing a similar open-ended problem first) almost doubled the number of correct responses.

Table 1. Number of Students Choosing Each Response on the Number Sentence Item

	Form A (multiple choice first)	Form B (open-ended first)
A. 3 x 6 x 4 = □	16	9
B. (3 x 6) + 4 = □	11	4
C. (3 x 6) + (3 x 4) =□	17	31
No answer	1	1

Further analyses indicated that students who chose the incorrect number sentences made decisions based on rules they held for equations. These included their beliefs that: (a) the answer to the number sentence (3x6) + (3x4) would be too big, (b) (3x6) + (3x4) has too many numbers in it, (c) the number 3 shouldn't appear twice in the number sentence, (d) the word problem had three numbers in it and those numbers appeared in choices A and B, (e) a number sentence cannot have more than one operation in it, and (f) the problem seemed to be about adding and multiplication is a faster way to add, so choice A must be correct. Because the question had multiple choices, the students reasoning revealed how they compared the number sentence. The first choice, 3 x 6 x 4 was chosen by 25 out of 90 or 28% of the students. They linked the idea of making *three trips* to multiplication, and eliminated the other two choices based on their understanding of what number sentences should look like. Likewise, the students who selected (3 x 6) + 4 made the connection between making 3 trips with 6 bags and saw the 4 as needing to be part of the equation – it made sense to them. They too eliminated other number sentences based on their understanding of what number sentences should look like. They thought the answer to (3x6) + 4 would produce an answer that seemed reasonable and both 3 x 6 x 4 and (3x6) + (3x4) would produce answers that would be too big. The number sentence, (3x6) + (3x4) violated several rules for students: the number 3 appeared twice, more than one operation was used. It was clear from the interviews that many students did not know what parentheses designate, yet students who correctly answered the question, reasoned that they divided up or separated the number sentence into two parts. Moreover, Form A students who picked the correct answer "C" thought through the context of the problem. Below is a typical response:

> I think it is C. (So can you tell me how you decided which answer to choose?) Well, they each made 3 trips and J carried 6 bags in each trip and that would be 3x6 - that part there. And B carried 4 bags each trip, so that would be 3x4 and then you need to add them to find the total. (OK, so can you tell me why you didn't pick A?) You don't want to do 6x4 because the number of bags they carried each doesn't have anything to do with each other. B could have done it by himself, but he would have had to do 6 trips. (And do you know what the parentheses mean?) It means you have to do that problem first. So you would say 18+12.

Form B students who selected the correct answer did recognize that the problem mirrored the one they had just solved. In the following examples, notice that students still may have expressed some puzzlement about the way the equations were written. The parentheses caused some confusion. At first, students leaned toward choice "A" because they saw both 3 x 6 and 6 x 4 in the equation 3 x 6 x 4. The following two explanations are representative of the way students explained their answer:

> I did mainly the same thing I did the first time, but I did 3x6 so 3x6 equals the number and then 3x4.... wait a minute. I don't want to do that. (You don't want to do A? How come?) Because I just figured it out. 3x6x4 - I just figured that out... at the start I was confused, so now I know it is C. (Tell me why you think it is C?) Well, 3x6 is the first one plus 3x4 equals both of their numbers. (Have you seen these marks - the parentheses - before?) Yeah. (What do they mean?) They mean that problem put together. It is like a problem that's not there. They put it there to make the sentence a little longer and also so that it makes it a little easier for you to tell. And also they do it for multiplication, they times it and then you know what it adds up to and then you plus the other one. That's how I usually think of it. (So now you are saying it is not A, it is C?) Yeah. (Because you figured it out? What do you mean by that?) Because first I thought it was 3x6, 3x4 and at first I didn't know that. Then I knew it had to be 3x6 PLUS 3x4.

> Like on the last one, we have to multiply. So Juan carried 6 bags and Bill carried 4, so I just multiplied that and so it's probably the same as this one here. (You first thought it was the same as A?) Because I think you have to multiply first then multiply these two together, but it doesn't have the little quotes to show you to do that first, like we did this one. (Hmm. The little quotes, you mean the parentheses - the half circles?) Oh - it's this one. (Now you changed your mind. Tell me a little bit about how you decided C was the one you wanted.) Because if we didn't do it that way, like 3x6 would be 18 times 4 again wouldn't equal whatever it equaled. It would be a lot more. But right here it is easier because 6x3= the answer right there, but it is telling you to do that first and then add the two answers together first. Exactly like what I did on the other problem. (Oh, so you would multiply 6x3 first and then 3x4 ...) and then add the answers. (OK, so why wouldn't it be say, B?) Because you don't just add 4 books. He didn't carry them one time, he carried them around three times, so you would have to multiply.

Analyses on the open-ended problem revealed that students were able to solve the problem. Children came up with 5 different correct equations that mirrored the way they worked through the problem:

$12 + 20 = 32$

$8 \times 4 = 32$

$5 + 5 + 5 + 5 = 20$, $3 + 3 + 3 + 3 = 12$, $20 + 12 = 32$

$3 + 5 = 8$; $8 \times 4 = 32$

$(3x4) + (5x4) = 32$

Only six students came up with an equation that mirrored the correct one on the multiple-choice item [(3x4) + (5x4) = 32]. All six of those students received Form A and thus had seen the model first. Students' own number sentences explain why some students were confused by the expression (3 x 6) + (3 x 4). They wrote rather straightforward equations. They used one operation at a time and separated out the steps they used to solve the problem.

It is interesting to note that 9 students solved the open-ended problem by first combining the books carried in one trip and then multiplying that number by 4 trips thus ending up with the equation 8 x 4 = 32. 7 of those 9 students took Form B of the interview and thus solved the open-ended problem first. All but two of these students, when they came to the multiple-choice question solved the problem first and matched the answer to the number sentence. The other 2 students chose B "(3x6) + 4" because they did not like the way C was written – the number three appeared twice or it seemed like the answer would be too big.

The students who took Form A produced 14 incorrect number sentences, the most common being 5 + 4 + 3 = 11 (a third of the responses). The next two most common incorrect number sentences were 5 + 3 = 8 and 3 x 4 x 5 = 60 used by 4 and 3 children respectively. They linked their problem solving skills to some part of the problem. 7 out of 11 of the students who took Form B who did not solve the problem correctly used a problem solving strategy of pulling the numbers out of the problem and deciding on an operation. Thus, they noted the numbers 3, 4, 5, and the words "in all" which signaled an addition problem. Typical explanations included, "Because it says 'how many they moved in all' and 'in all' means to plus so I added all the books they carried." The Form A number sentences reflect students' efforts to link the number sentence to part but not all of the problem. For example, students who wrote 5 + 3 = 8 said they wrote down how many books were carried in all, but the equation only refers to one trip. Students also recognized some of the problem as involving multiplication, which explains most of the other number sentences they created (e.g., 3 x 5 = 15; 5 x 4 = 20; 4 x 3 x 6 = 60; 3 x 5 = 15 + 4 = 19; 3 x 4 + 5 = 17)

Finding a Good First Step

For both of the following problems, students were shown a graphic of two different chocolate chip bags. One bag is labeled "Brand X" which holds 12 ounces and costs $1.20. The other bag is labeled, "Brand Y," which holds 16 ounces and costs $1.75.

> *Multiple choice item:* Pat finds the two brands below in a store. He wants to buy 30 ounces of chocolate chips for the least amount of money. Which is a good first step he could use to solve this problem?
>
> a) Find the total price of the bags he decides to buy.
> b) Find how many of each brand he would need.
> c) Find the price-per-ounce for each bag.

Open-ended item: Pat finds the two brands below in a store. He wants to buy 30 ounces of chocolate chips for the least amount of money. What should he buy?

Similar to the number sentence problem, I expected students to be able to select a good first step only after they had solved the problem. Since the problem does not require students to find a solution, they do not actually work out the problem. 12 out of 45 or 27% of the students chose the correct response when they solved the multiple choice question first. 21 out of 45 or 47% of the children who had an opportunity to solve the problem itself first chose the correct response.

A typical solution to the problem was to figure out how many bags of each brand are needed to get at least 30 ounces. It takes 3 bags of Brand X and 2 bags of Brand Y. Children added 12 three times and 16 twice. Then, they added up the dollar amount. Three bags of Brand X cost $3.60 while two bags of Brand Y cost $3.50. Only one student calculated the price-per-ounce when she solved the problem on her own (yet she did not select price-per-ounce as the correct answer in the multiple choice problem; she chose option A). Their choice of price-per-ounce however differed based on whether they took Form A or Form B. 21 Form A students selected price-per-ounce because (a) that's what you do when you have to buy something, "Because sometimes when I go shopping, you look at the price to see which one you should buy. When you find the price per ounce, I think it means what is cheaper." (9 out of 21) or (c) both ounces and prices have to be considered to solve the problem (9 out of 21). "Well, you kinda need to know how much they weigh AND the price before you go get a whole bunch." 3 students were not sure why they picked price per ounce.

16 of the 18 Form B students, chose price-per-ounce not for its literal definition but because they recognized it as the only choice that had both price and ounce in it. They connected that to their own efforts of considering both the price of the bags and its contents at the same time. One girl explained, "…so he wants the least amount of money and 30 ounces. So he'd have to look at each brand and see the money and add it up so it's the least amount and he wants 30 ounces and see if it adds up to 30 or 32."

Table 2. Number of Students Choosing Each Response on the First Step Problem

	Form A (multiple choice first)	Form B (open-ended first)
A. Find the total price of the bags he decides to buy.	12	6
B. Find how many of each brand he would need.	12	21
C. Find the price-per-ounce for each bag.	21	18

When children solved the open-ended problem, they chose at an equal rate 2 bags of Brand Y or 3 bags of Brand X. The children who chose Brand X did so because they found it to be the best buy, not necessarily the one that would give Pat 30 ounces for the least amount of money. Notably, 4 children who picked Brand X said they would only buy 2 not 3 boxes. When reminded that the problem indicated Pat wanted to buy 30 ounces, one girl said, "I'd ask him, is it okay to get 24 ounces?" She went on to explain that $2.40 is much less than $3.50, which was how much he would have to spend for 2 bags of Brand Y. She thought that he could get by with 24 ounces. Another boy said that buying 24 ounces would save him a lot of money and would only recommend that he buy 2 bags of brand Y "if it was real emergency, and he really, really needed it."

The students in this study could reason through this problem, yet their responses to the multiple choice question alone would have created some doubt and indeed alarm about their problem solving skills.

The Best Chance

> *Multiple choice item*: Special cakes are baked for May Day in France. A small toy is dropped into the batter for each cake before baking. Whoever gets the piece of cake with the toy in it is "king" or "queen" for the day. Which cake below would give you the best chance of finding the toy in your piece?
>
> Children were shown the following cakes:
>
> a) rectangle cut into sixths
> b) circle cut into fourths
> c) circle cut into fifths
> d) circle cut into sixths

An open-ended problem was not posed in this case because I was interested in seeing how the context of eating cake would influence children's responses. 58% (52) of the students chose the correct response. 3% (3) and 9% (8) chose responses C and D respectively. 30% (27) of the students chose A. The children who selected the rectangle cut into sixths brought their experiences eating cake to think about this problem realistically rather than probabilistically. Their responses fell into 3 categories. Most of the children (12 girls and 4 boys) who selected the rectangular cake cut into sixths discussed how and whether the *toy would fit in a slice*. Children commented about the shape of a toy: "D (circle cut into sixths) is in triangle shapes and there aren't too many toys that could actually fit in that kind of shapes. Usually you see toy boxes that are in rectangles and squares." Others commented on whether the entire toy would fit into a slice; "I would not want to be that toy if I got my head whomped off by a butcher's knife."

Ten students (8 girls and 2 boys) commented on the *size of the pieces*. They picked A because it had bigger pieces. They seemed to be thinking that it would be easier to find a toy in a bigger piece. They did not explicitly mention whether the toy would fit as other children did. For example, one boy said, "Because you would only get one piece, and so you want the biggest amount of pieces so you can get more chances of getting to the toy."

Finally, one boy thought that more slices constituted more chance because of the chance of *whether or not you would get a slice*. He said, "B doesn't give you much of a chance. You could be the fifth person in line and the person in front of you would get it. You don't get much of a chance."

For this problem, students who chose D responded to the question by thinking realistically about the problem. They were concerned whether and how toys actually fit and whether or not they would get a piece of cake. This was the only problem in which there was any gender difference. 20 girls versus 7 boys chose the rectangular cake cut into sixths. Girls may have brought in more of their own experiences baking, but that's a fairly stereotypical response.

Children who selected the correct response gave an explanation similar to the following one. "Because (B's) pieces are bigger than C. And although they are smaller than A and D, only 4 people can eat it, so there's less people so there's more of a chance of you finding it. (So, the pieces in B are bigger than C but smaller than D and A but...) only 4 people can eat out of it and 6 out of A and 6 out of D. (OK. So why is that better?) Because if there is less people, you have a better chance of getting it." Another child explained it this way, "It would be B. 1 in 4 chances is better than 1 out of 6 chances, with less pieces in it, it would be easier. If it were 1 out of 2, it would be really easy, but since four is the lowest number, it's going to be the lowest number is the easiest. (Now why do you say that less pieces give you a better chance?) If it was 5 pieces, someone else might get it, it might be five people getting it. D or A there are 6 pieces so 6 people, 1 out of 6 people get it. But if there is only 4 pieces, it would kinda be easier since there are 4 pieces of cake and one of them has the toy."

Conclusion

The results of this study show that students apply their experiences with mathematics and the real world to problem solving situations. Their responses convey various conceptions they hold about mathematics, about problem solving and about test language. What may seem as straightforward problems to adults such as selecting an appropriate equation or first step to solving a problem may in fact present difficulties to children who do not think to solve the problem before selecting an answer out of multiple choices. As the cake problem demonstrated, some contexts are so "authentic" to children that they pursue solutions that make sense according to their own everyday experiences.

This study has implications for teachers, policymakers, and test developers. For teachers, this study speaks to the importance of listening to how children understand and interpret mathematical problems. Teachers can help prepare students for tests by surfacing the assumptions and understandings that students bring to the test-taking situation. It underscores the importance of analyzing and learning from students' work. Teachers should discuss with children how language used in test items may differ from their everyday problem solving. For policymakers and test developers, this study raises the important question of score validity. It raises a healthy level of skepticism about what students' achievement scores reveal about their mathematical understanding and achievement. The results show that we must investigate more deeply how children use the knowledge they have developed about mathematics, about problem solving, and about test-taking as they work on problems that are meant to measure their mathematical abilities.

References

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.

Franke, M. L., & Kazemi, E. (2001). Teaching as learning within a community of practice: Characterizing generative growth. In T. Wood, B. S. Nelson, & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 47-74). Mahwah, NJ: Erlbaum.

Greeno, J. G., & MMAP. (1998). The situativity of knowing, learning, and research. *American Psychologist, 23* (2), 167-180.

Lampert, M. (1991). Connecting mathematical teaching and learning. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 121-152). Albany: SUNY.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.

McNeil, L. M. (2000). *Contradictions of school reform: Educational costs of standardized testing*. New York: Routledge.

Miller-Jones, D. (1982). Culture and testing. *American Psychologist, 44*, 360-366.

Putnam, R. T., & Borko, H. (1997). Teacher learning: Implications of new views of cognition. In B. J. Biddle, T. L. Good, & I. Goodson (Eds.), *International handbook of teachers and teaching* (pp. 1223-1296). Dordrecht, Netherlands: Kluwer.

Tate, W. (1994). Race, retrenchment, and the reform of school mathematics. *Phi Delta Kappan, 75* (6), 477-485.

SET FORMING AND SET ORDERING: USEFUL SIMILES IN OBSERVING THE GROWTH OF MATHEMATICAL UNDERSTANDING?

Thomas Kieren
University of Alberta
tom.kieren@ualberta.ca

Theorists, curriculum developers and teachers have long been interested in the nature of mathematical understanding and how its growth is manifested in the knowing of students of all ages. There has been considerable research and writing devoted to this topic over the last 25 years with many useful concepts developed for use by researchers and teachers (e.g., contrast between instrumental and relational understanding from Skemp; the overcoming of epistemological obstacles from Sierpinska; the reification of one's knowing actions from process to object from Sfard; concept image and definition from Tall and Vinner; and the dynamical model including folding back from Pirie and Kieren to name only some of the recent ideas). Many of these concepts and models are concerned with growing understanding. The questions discussed here arise in the context of the Pirie/Kieren model, but related questions might arise in the other frameworks as well.

Theoretical and Research Background

We (Susan Pirie and myself as well as others working with us such as Brent Davis,Lyndon Martin, Elaine Simmt, Jennifer Thom and Jo Towers) have been developing and applying a model *for* the growth of mathematical understanding through a series of studies over the last dozen years (e.g., see Kieren, 1994, Pirie & Kieren, 1994, and 2001 for details of the model). In particular, we have tried to show growing understanding as a non-linear process related to particular mathematical topics and contingent upon the lived histories of the knower and occasioned by various interactions with the environment and others in it. For any person and any topic or problem, we have posited that a number of enfolded and unfolding modes of understanding might be observed. Non-linear growth involves forming one's own images of the mathematics and working with them which we have termed this *Image Having* (generally more informal and local in nature). Such growth also may involve using more general *Formalizing*; and *Observing* and then generally Structuring theorems into a mathematical system. We have used our model to observe and portray understanding in the mathematical actions of students, groups and classes from elementary school to university across a variety of mathematical topics.

In using this model, several questions have arisen which form a focus of this paper: How can one better observe a person's growth in understanding to the point of having(and using) an image? How might such actions be occasioned? How might one better observe and prompt growth from less formal understanding acts to *Formalizing*; and later, to more abstract structuring? Thus the focus of this paper is on the transition to and the elaboration of *Image Having* understanding and the transition to *Formalizing*.

Based on a recent review of observational data drawn from a number of studies done over the last 10 years, it is suggested that ideas from mathematics (in action), *set forming* and *set ordering*, are useful as similes in observing fundamental features of growing understanding with respect to the questions raised above. It is proposed that observing a person having their own image of an idea, a problem or a mathematical process and expressing that image (describe, explain) is like observing that a person is forming or has formed a set. When one is able to use and justify a general method which "works" "for all" cases in some sense - what we term as *Formalizing* - appears from this review to be supported by actions which are like one is ordering or has ordered a set in some way and works with that ordering. It is the purpose of this paper to elaborate on these similes and illustrate their use in observing growing understanding. Further, the paper will illustrate how those similes might be used in generating teaching actions which may occasion such growth of mathematical understanding in action. Of course it is important to note that such observational use of these similes only adds to other factors in the complex process of observing and fostering the growth of mathematical understanding which have been developed in the course of our work on growing understanding; or only add to means of observing changing understanding under other theoretical frameworks.

Observational and Interpretive Stance

As suggested above we have been developing and testing what we see as a dynamical theory *for* the growth of mathematical understanding. The particular observational use of these set-related similes will be situated in this more general perspective on growing understanding. As highlighted above, the dynamical theory is a theory "for" not a theory "of". That is we are not saying that we have privileged insight into how that growth really occurs, or that such growth in action can in some way be reduced to the concepts of our theory. Instead we hold with Maturana's (1988) idea of objectivity-in-parentheses in which we invite the listener to consider just how our ideas might work under the various lived contingencies of mathematics knowing in action and teaching related to that knowing. In transforming ideas about sets into a means for observing understanding in action we are taking the same stance.

This work is set in a more general view of mathematics knowing and understanding in an individual's action as the bringing forth of a world of significance - including mathematics - in interaction with others in a particular environment. Such knowing is taken to be a coemergent phenomenon: the individual's actions are determined by their structure or lived history but co-specified by the environment and the actions of selected others in it. (see Simmt & Kieren, 1999). Mathematics knowing in action is observed as occurring in languaging which involves at least the recursive coordination of coordinations of actions and the related further coordinations of or distinctions made in such languaging (Maturana, 1991). As will be discussed below, the similes of set forming and set ordering are useful in seeing growing understanding actions as involving such recursive coordinations and distinctions.

"Data Sources"

This paper is not a research report as such but is a form of research - in- process whose intent is to add to theoretical tools useful in observing growing understanding. But it is based on material drawn from a number of studies done for a variety of purposes over the last 10 years. In each case, whether the study was done in classrooms - as was the case for the fraction and polynomial studies; or in special purpose settings - such as the parent child research; or in pair observation/interviews as was the case for the university students; the research involved multiple sources of data including video and or audio taped sessions; transcripts; the restructuring and fictionalization of transcripts; various artifacts gathered from students and teachers; interview data, etc. The studies also involved multiple researchers with multiple perspectives viewing and re-viewing the data, as well as reviewing their conversations with one another. The research material for this paper comes from the corpus of those previous research activities - limited in this paper to illustrations from studies of fraction knowing and understanding of grade three students in a Canadian city - as well as from manuscripts written for publication about that research.

Illustrative Cases

When a person forms a set, one might say they have collected together a number of elements or instances and further that they have a means by which they can judge whether an element or instance belongs to the identified collection. To see how set forming as an act might help us observe *Image Having* understanding, consider these instances drawn from the work of eight year olds working with fractional numbers. They have been introduced to a "kit" which contains 2 units worth of ones, halves, fourths, eighths and sixteenths in the form of color paper rectangles. As we have illustrated in other work, these children have been observed to use various mechanisms which can be deemed as determined by their own lived histories or structures (for example: dividing up equally; combining fractional units; reconfiguring fractional units drawn from the work of Behr, Harel, Post and Lesh, or splitting mechanisms drawn from the work Confrey - see references in Kieren, 1999) After a few days of such work, they are asked to find out and say what they could about the fractional amount, 5/4. Nearly all of the children in each of three research classes were observed to be able to adequately respond to this task with at least five different responses. Most provided many more or made various other elaborations on those that they had developed and kept working on the task for up to an hour or more. In that sense the illustrations below, though necessarily unique, are within the range of the activities which were more generally observed in the research classes on this task. Given that kind of prolonged action which was neither predicted or suggested by the teacher, it is reasonable to observe the knowing in action as a "bringing forth of a world including mathematics" rather than as providing a response to a question or task or even as problem solving -although obviously both of those kinds of activities may be seen as part of that bringing forth.

One of the girls, Jody, and her partner, Tina, were observed to be lying on the floor under their work table working on the task. After a few minutes Jody exclaimed, "We've found nine things!". At this point as was part of the practice in this class, the teacher had Jody and Tina display some of their work for others. A little later Jody announced, "Now we've got 23". Some of their statements included "1 +1/4 = 5/4"; and "1/2 + 1/4 + 2/8 + 4/16 = 5/4". Some time later Jody came up and asked the teacher, " Can I say 5/4 = 5/4? I think that it is right." The teacher asked why Jody thought this was the case. Jody responded, "They make the same amount, don't they?"

How does the simile of set forming help us observe Jody's understanding? It is clear that the girls were not simply completing a school mathematics task, but had created something more for themselves and -through their comments and displayed work- for others as well . While it is true that at the beginning they were simply generating sentences which described arrangement of the kit pieces which they could show equaled the quantity represented by 5/4 (five one fourth pieces), by the time of their first announcement one could infer that they certainly were deliberately forming a collection. The two girls were sharing fractional combinations that they were making up "in their heads" and were only referencing the kit to show combinations to one another when the partner questioned their fractional sentence. That they at least implicitly had a means of generating elements which fit in their "set" is seen in the nature of their very long list of "correct" instances. Their creation of the long list as well as their means of doing so suggests that they had created a "thing", a set if you will of instances of combinations which made 5/4. It is important to notice, however, that this set was being brought into being in a local manner - that is, the girls were generating instances one by one. Jody's query about 5/4 equaling five fourths shows at least one distinction which goes beyond this local set forming process. It can be interpreted as asking whether 5/4 can be in the set. All of the girl's other instances had involved combining fractional units (some of which went beyond the kit (e.g., 32nds and 64ths)) to "make up" 5/4. Jody's "5/4 =5/4" does not "fit" the previous pattern for being in the set. That is, it was not a report about a combination of fractional amounts which "made" 5/4. Interpreting Sierpinska's (1994) idea of epistemological obstacle in a local active sense, Jody can be seen as facing a quandary with respect to her implicit rule for set membership. But in overcoming this quandary or obstacle by adapting her "rule" for set membership, she was experiencing or in the process of generating a much more mathematically powerful meaning for "=". Thus thinking of Having an Image in set forming terms helps us observe *Image Having* understanding - the girls generate and collect together instances and implicitly and explicitly have "a rule" to generate and test for membership. Their activity and especially Jody's provocative question about 5/4 equaling itself is illustrative of the kinds of distinctions they are able to make. Again using our simile in observation, it is as if Jody is questioning her collection in terms of the "rule" element. Such an action can be observed as a recursive coordination of her previous coordination - one which in this case shifted her focus from fractional quantities aspect of her work to a consideration of the meaning of "equals" itself in this local context.

Of course, one might explain the girls' actions by simply suggesting that they had structures or lived experiences that allowed them to act as they did. For example, one might invoke the powerful contrast of Les Steffe's (1999) between acting using a unifying composition and using a composite unity (this time for the number 5/4). An observer so informed might observe the girls as being on the cusp of reorganizing the former scheme into the latter scheme and feel that this transformation of the structural mechanism is sufficient to portray what the girls were doing. I am suggesting that the use of our concept of *Image Having* and the simile of set forming in observing it in action allows to make a different kind of observation. Of course enacting the particular structures or schemes enables the girls to act as they do. But they are not satisfied with just knowing that there are alternate ways of seeing 5/4 decomposed. Their understanding action entails continuing generation and collection as well as working with and expressing themselves (mathematically and socially) about the "set" that they have created and the conditions for its creation. Thus using the set forming simile takes our understanding of their understanding actions beyond the fact that they are exercising a particular capability in a particular situation.

In another grade three class, an 8 year old boy named Sandy is faced with the same task. He does this by forming a table (see Figure 1 - the figures in this paper are not in the form given by their creators, but developed to fit the form of this publication):

1	1/2	1/4	1/8	1/16
1	-	1	-	-
	2	1	-	-
-	-	-	10	-
-	-	-	-	20
-	-	3	4	-

Figure 1.

He is observed as looking into the air and then completing the next "row" in his table one by one. When asked what he was up to by the teacher Sandy replied " I want to find them all" [all possible additive combinations of the fraction types from the kit which could show the quantity 5/4]. To this end Sandy was using his chart as a memory aid and a way of checking whether a particular instance had already been included. Using our simile we can see that Sandy, in a more symbolically sophisticated manner than the examples above, too is forming a set. The nature of the chart and his explanation of it to other students shows that he has a "rule" for generating instances which "belong to his set". But he, like the girls above is observed to be generating such instances on a one-by -one basis. In addition, one might see that he has observed a potential property of his set - he thinks that one can find "them all". That is he to has made a coordination of his previous coordinations. Unlike the girls he can imagine the set without having all the elements and in his way "know" that there will be a knowable number of elements.

When he is interrupted by the teacher to explain what he is doing for others Sandy does so in a rapid fire manner clearly aimed at the teacher but not providing much help to his peers. The teacher then redoes this explanation for the others by actually considering Sandy's chart-in-progress which he has been making as a public display at the board. In so doing the teacher and the others actually "work" on lines in Sandy's chart. He is annoyed when they actually alter his table. He starts to work furiously and secretly on the task again, now writing in tiny illegible script on another part of the board (Figure 2a). Exactly what he was doing could not be seen from the tapes but a day later at the request of the teacher Sandy creates a similar table with respect to 3/4 in less than five minutes, part of which is shown below (Figure 2 b). Without any checking of individual items he brings this work to the teacher and declares: "There, that's all of them!"

1	1/2	1/4	1/8	1/16
				*
			*	*
			*	*
			*	*
			*	*
			.	
			.	
			.	
			*	
		*		*
		*		*

Figure 2a.

1	1/2	1/4	1/8	1/16
-	-	-	-	12
-	-	-	1	10
-	-	-	2	8
-	-	-	3	6
-	-	-	4	4
-	-	-	5	2
-	-	-	6	-
-	-	1	-	8
-	-	1	1	6
-	-	1	2	4
-	-	1	3	2
	.			
	.			
	.			
-	1	-	2	-
-	1	1	-	-

Figure 2b.

BEST COPY AVAILABLE

In adding to other interpretations of what he has done or was doing, one can now use the simile of set ordering. In the illustrations in Figure 2, Sandy is not recording instances or elements of his set on a one by one basis but is imposing an order on them. In talking to him about his first board effort (Figure 2a, which was only noticed on the video tape after class was out) Sandy would only say that he hadn't finished that chart yet. From looking at the form of the chart entries, one might notice that rather than make entries one-by-one as he thought of them, the interruption of his previous work has provoked Sandy into reconceiving his previous charting activity. One might infer that he has imposed an order on his set, albeit aided by the form in which he represented it and by his previously observed property that the set had a knowable number of elements. This interpretation is clearer in looking at Figure 2b. From this chart one can see just how Sandy was proceeding in a methodical way to generate an ordered set of additive combinations equal in amount to , in this case, 3/4. It is this ordering process or method rather than the instances themselves which he sees as guaranteeing "allness". He has developed a method which he can justify - and can be observed as *Formalizing*. It is the interpretation offered here that Sandy has recursively coordinated previous coordinations several times in the activity traced above. Using the similes developed here one might say that his growing understanding in action is first like generating a set and working on it; and then like imposing an order on the set and working not with the set but the ordering.

Local Conclusions

In interpreting the knowing actions of children working in a particular situation on a particular prompt dealing with fractions as combinable quantities, I have suggested that the pair of girls Jody and Tina could be observed as *Having an Image* or enacting an *Image Having* understanding that can be characterized as follows: Five fourths can be made up in many ways by putting together (adding) other fractional amounts. I have suggested that using set forming as a simile is useful in observing for such understanding action. using such a simile one would say that the girls actions were like deliberately forming a set - albeit in a local way -, not just generating unrelated instances which fit the regimen of a teacher given task. Further their actions could be interpreted as being like having at least an implicit criterion for judging the membership in that set. So, at least in this case, looking for behaviours which are like set forming is useful in observing understanding actions which suggest that the children had their own way of working on the prompt independent from direction from the teacher or curriculum and independent from acting on the materials in a prescribed manner. It is important to not that such set forming as a human activity rather than a mathematical concept is contingent on the lived mathematical history of the girls; the ways in which they chose to represent and talk about their collection; the context in which they acted and the mathematical practices (e.g., making public displays and talking about them with others); as well as the emotioning with which their more cognitive actions were interwound (e.g., one might ask in what ways was Jody's delight in announcing the growing size of their collection related to her sense that their generated examples formed a collection which could be extended at least piece-wise in a particular way?) These contingencies in children's actions can also be seen in contrasting the ways in which the girls and Sandy represented their set - a list of "equations" vs. a structured chart. But both the girls - at least Jody- and Sandy could reflect on and make distinctions about these sets that they had created. Both the observed formation of the sets, their observed ability to make further distinctions based on that activity and its results and their own observed sense or confidence in what they were enacting allow the observer to characterize their actions as a form of mathematical understanding even if that understanding has an informal local character.

What strikes one in looking at Sandy's later work is that he notices that he could not just generate elements and use the chart as a convenient means of checking for duplicates, but that he can generate elements in an ordered fashion. More importantly, it is our observation of his working on that ordering rather than just on the set generation that appears to allow him to change his sense of 'all': "I'm looking for all of them." (If I keep careful track I can actually find all of the elements in this set.) This more local sense of 'all' can be contrasted then with his distinction not just about the collection itself but about the nature of his ordering of his ordering: "There that's all of them" (I don't have to even check the set; my method or ordering guarantees it.) In both cases it is clear that Sandy is exhibiting mathematical understanding and in both cases he was confident in what he was doing. The Pirie/Kieren theory and its concepts of *Image Having* and *Formalizing* allows the observer to distinguish the nature of these understanding acts as well as to see just how Sandy's earlier more informal actions and those of Jody were similar in character even though they looked different. What is being suggested here is that using the similes of set forming and set ordering is a tool in making such distinctions.

Broader Implications

To draw broader implications requires moving beyond the context above to other kinds of students; other topics ; other cases. It also requires that one consider whether using such similes allows use to better observe and why students do not move to more sophisticated ways of mathematical knowing and understanding particularly making the move to connect more formal, general and abstract understandings. Unfortunately, it is beyond the scope of this paper to develop research with other topics, or arising from other kinds of research data. Although as indicated above such data exists, that kid of exposition awaits another writing.

But space allows for one brief extension of the ideas above[1]. Shortly after working in the class with Sandy above, I had the opportunity to work with a teacher of a class of 11-12 year olds. After listening to the story of Sandy, it was decided to use his chart strategy with her class. Several of the students in this setting were considered to have learning difficulties and had histories of poor achievement in mathematics. One of these was a boy named Jake. With his classmates, he had been doing work with a fraction kit which included two units worth of halves, thirds, fourths, sixths, eighths, twelfths and twenty-fourths. The class was given a brief introduction to the chart strategy and asked to find combinations of fractions with denominators from their kit set which made up one of a variety of target quantities. Jake chose to make up combinations which equaled 1. He was observed to make several entries which form the "top" of his chart - see Figure 3 one-by-one. He then stopped and looked at his chart. He then, writing as fast as he could, made the other entries in his chart. During this process, he repeated his previous entries. He then very proudly walked up to his teacher and said, " I know everything about one. " With a little probing from the teacher, he suggested that it was anything over that thing. He then demonstrated that knowledge on 72nds and 193rds.

Using our similes, one can observe Jake recursively engaged in both set forming and set ordering. He starts by simply acting in an instatial manner. His act of stopping and studying his early entries in the chart might be observed of observing his items as a set and obviously not only working on them but ordering them in both a psychological and a mathematical sense. In so doing he was one of the students in his class to make and be able to talk about and apply a "for all" statement (even though it might be said that the mathematical scope of that statement was not broad). But the forming of a set, the ordering and ordering his ordering of that set can be observed as supporting his growing understanding in this situation.

This brief vignette - along with many others from a wide variety of students working on a variety of mathematical tasks in a variety of circumstances - as well as the central examples in this paper bring up the question: What might be the roles of [set] ordering and the ordering of that ordering in moving from more local informal understandings of mathematics to *Formalizing*?

1	1/2	1/3	1/4	1/6	1/8	1/12	1/24
				6			
		3					
						12	
	2						
1							
	2						
		3					
			4				
				6			
					8		
						12	
							24

Figure 3.

This is an open research question. But even without answering that question, the ideas raised and illustrated in this paper have possible pedagogical implications. In many cases, students see the their mathematical activity as successfully doing a [long] sequence of tasks each of which has a pre-given answer. For many students, it would be hard either for them or an observer to see this activity as bringing forth a world of *significance*. The work in this paper raises the question of whether such activities really promote students' understanding, particularly understanding which they sense as "their own". This research is suggestive of the value of having students see their work and its represented results as a set. Using prompts such as illustrated in this paper or prompting students to observe their item by item work as inter-related may prompt them to exhibit understanding which we would distinguish as *Image Having*. Further, occasioning students to order the elements of their previous efforts may provide a basis for further mathematical actions (conversations or discussions or individual actions) which one can observe as *Formalizing*. In either case, students would be prompted to generate and work with mathematical objects and ideas. Students would be possibly occasioned to do recursive coordinations of mathematical coordinations they have already made.

But of course these speculations awaits exploration in theory, in research, and in pedagogy.

References

Kieren,T. E. (1994). The Bonuses of understanding mathematical understanding. In D. Wheeler, D. Robitaille, & C. Kieran (Eds.), *Selected papers on mathematics education* (pp. 211-228). Ste. Foy, Quebec: Les Presses de l'Universite Laval.

Kieren, T. E. (1999). Language use in embodied action and interaction in knowing fractions. In F. Hitt & M. Santos (Eds.), *Proceedings of the Twenty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 109-128). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Maturana, H. (1988). Reality: The search for objectivity or the quest for a compelling argument. *Irish Journal of Psychology*, *19*(1), 25-82.

Maturana, H. (1991). Science and daily life: The ontology of scientific explanations. In F. Steier (Ed.), *Research and reflexivity* (pp. 98-122). London: Sage.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics*, *26*, 165-190.

Pirie, S., & Kieren, T. (2001). *Growth in mathematical understanding: From theory to practice*. A paper given at Rutgers University. May 4.

Sierpinska, A. (1994). *Mathematical understanding*. London: Falmer.

Simmt, E., & Kieren, T. (1999). Expanding the cognitive domain: The role(s) and consequences of interaction in mathematics knowing. In F. Hitt & M. Santos (Eds.), *Proceedings of the Twenty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 298-305). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Steffe, L. (1999). *The reorganization hypothesis.* Part of a collection of unpublished research papers. Athens, GA: The Department of Mathematics Education, University of Georgia.

Note

[1] The inclusion of this vignette arose from discussion of the ideas in this paper with Ralph Mason from the University of Manitoba.

A OVER B: MEANINGS OVER ALGORITHMS

Susan J. Lamon
Marquette University

Teaching and learning part-whole fractions have been persistent problems in the United States. Not only are our children unable to reason with fractions, but their performance on fraction computation as measured by the National Assessment of Educational Progress (Post, 1981; Kouba, Carpenter, and Swafford, 1989), even after years of practice with algorithms, is disappointing. Furthermore, instruction that highlights the part-whole interpretation of the symbol a/b provides limited access to the broad base of meanings underlying the rational number system Kieren (1976, 1980). This study addressed the question of whether other interpretations of the symbol a/b might provide a better foundation for understanding rational numbers.

Framework and Methods

A longitudinal study focused on children's development of rational number meanings and operations. Five classes of students participated in the study for four years, in grades three through six. Each class began fraction instruction using a different interpretation of the symbol a/b: measure, operator, part/whole comparison with unitizing (Lamon, 1993, 1994, 1996), quotient, and ratio. Instructional materials were developed for each of the experimental classes, based on Kieren's sub-constructs of the rational numbers (1976, 1980) as elaborated by Lamon (1999a, 1999b). A sixth class who received traditional part-whole fraction instruction served as a control group.

This study provided a unique opportunity to compare unconventional approaches with traditional instruction and to document the sequencing and growth of ideas, and the breadth and depth of the understanding that developed. Children were given the time, freedom, and encouragement to express their thinking in whatever manner possible. The two teachers who taught the five classes facilitated learning through the kinds of activities they encouraged and through the problems they posed. None of the groups were taught any rules or operations. Mathematical activity consisted primarily of group problem solving, reporting, and then individually writing and revising solutions for homework.

Rational Number Interpretations

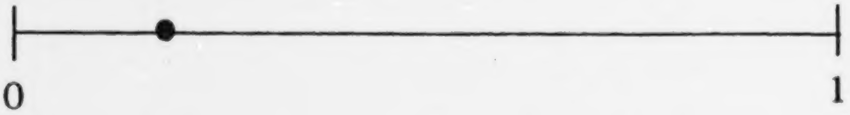
Table 1 summarizes the five rational number interpretations used in the study. Selected instructional activities that were used to highlight each of the interpretations are discussed below.

Table 1. Alternatives to Part–Whole Fraction Instruction.

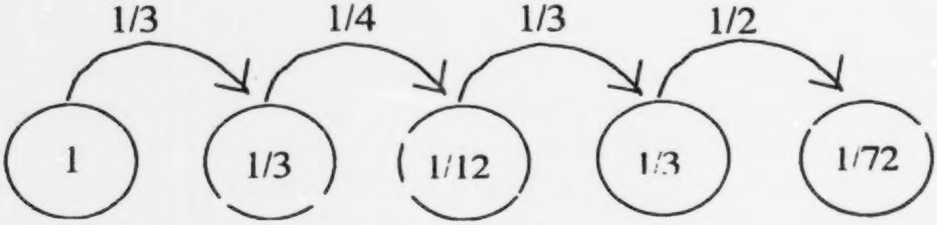
Rational Number Interpretations of 3/4	Meaning	Classroom Activity	First Concepts	Indications of Understanding
Measure "3(1/4-units)"	3/4 means a distance of 3 (1/4-units) from 0 on the number line or 3 (1/4-units) of a given area.	Successive partitioning	Ordering fraction numbers	Without partitioning, students employ their own methods to find two or more fractions between two given fractions.
Operator "3/4 of something"	3/4 gives a rule that tells how to operate on a unit (or on the result of a previous operation); multiply by 3 and divide your result by 4 or divide by 4 and multiply the result by 3.	Composition	Multiplication with identity and inverses	Students use multiple meanings for $\frac{3}{4}$: $3(\frac{1}{4}\ units), 1(\frac{3}{4}\ unit), and \frac{1}{4}(3-unit)$ and without drawings, can name a composition of operators with a single operator.
Part–Whole Comparisons with Unitizing "3 parts out of 4 equal parts"	3/4 means three parts out of four equal parts of the unit, with equivalent fractions found by thinking of the parts in terms of larger or smaller chunks. $\frac{3(whole\ pies)}{4(whole\ pies)} = \frac{12(quarter\ pies)}{16(quarter\ pies)} = \frac{1\frac{1}{2}(pair\ of\ pies)}{2(pair\ of\ pies)}$	Unitizing to produce equivalent fractions	Compensatory principle	Students discover common denominators and use them to compare fractions.
Quotient "3 divided by 4"	3/4 is the amount each person receives when 4 people share a 3-unit of something.	Partitioning	Equivalence	Without doing any work, students answer the questions 1) How much is in one share? 2) What part of the whole is each share?
Ratios	3:4 is a relationship in which there are 3 A's compared, in a multiplicative rather than an additive sense, to 4 B's.	Comparisons	Equivalence	Students perform ratio arithmetic (+, -, x, ÷).

Measure

The rational number interpreted as a measure says that ¾, for example, means a distance of 3 (1/4-subunits) from 0 on the number line. In this interpretation, the number a/b is strongly associated with the point whose distance from zero is a/b that we say the rational *is* a point on a number line. Students were given problems in which they were asked to use successive partitioning to name an indicated point on a number line.



Under this interpretation, children arbitrarily partitioned the unit again and again until one of their hash marks fell on the indicated spot and they are able to name the fraction that corresponds to the given distance from zero. Children used arrow notation to help keep track of the size of each sub-unit as the partitioning process when on.



Any unit of measurement can always be divided into finer and finer sub-units until you can take as accurate a reading as you like. Thus, this interpretation reinforced and extended concepts of measurement introduced in the early elementary years. This dynamic measuring process entails movement among an infinite number of stopping off places along the number line and it helped children to build a sense of the order and density of the rational numbers.

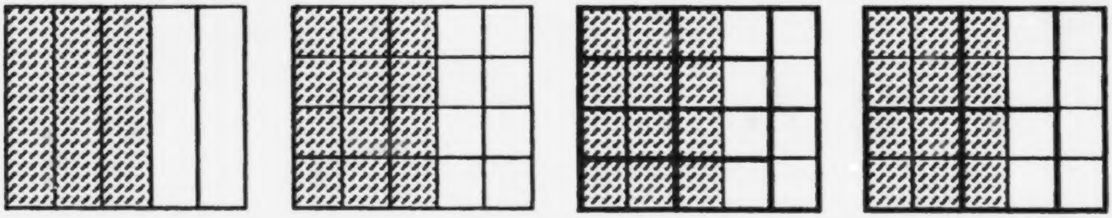
Operator

Under the operator interpretation of rational numbers, a fraction such as 3/4 is a rule for operating on a unit (or on the result of a previous operation): multiply by 3 and divide your result by 4, or divide by 4 and multiply the result by 3. Children used both discrete and continuous models to explore compositions: 2/3 of 3/4 of 8 cookies or 1/2 of 3/4 of a given unit of area. Other activities included shrinking and enlarging on a copy machine and paperfolding. Students first learned to multiply and divide fractions and later adapted their methods to deal with addition and subtraction.

Part-Whole with Unitizing

Unitizing is the cognitive process of constructing different-sized chunks in terms of which to think about a given commodity and is based on the compensatory relationship between a unit of measure and the number of copies of that unit in a particular quantity. Although the given unit in a fraction problem remains unchanged, it may be cognitively reconfigured in many different ways. For example, using an area model, 3/5 of the unit rectangle may be indicated as

$$\frac{3(columns)}{5(columns)} = \frac{12(small\ squares)}{20(small\ squares)} = \frac{6(rectangles)}{10(rectangles)} = \frac{3(larger\ squares)}{5(larger\ squares)}$$



Or, 8 eggs are what part of a dozen?

$$\frac{8(eggs)}{12(eggs)} = \frac{2(4-packs)}{3(4-packs)} = \frac{4pairs}{6pairs} = \frac{1\frac{2}{6}(half\ dozen)}{2(half\ dozen)} = \frac{1(8-pack)}{1\frac{1}{2}(8-pack)}.$$

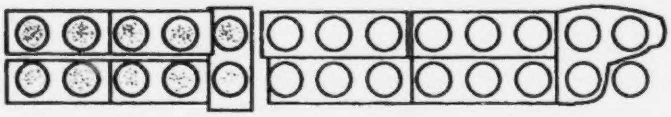
As opposed to current instructional techniques for generating equivalent fractions by multiplying 3/5 by 2/2, 3/3, 4/4, and so on, re-unitizing is a personal and creative process for generating members of an equivalence class by scaling up and down. Thus, proportional thinking was used from the start, and students learned the essential nature of a rational number, its emphasis on relative amounts, regardless of the size of chunks.

Quotient

A rational number is a quotient when it is the result of a division. For example, "If 6 people share 4 candy bars, how much candy does each person receive?" Algebraically, 6x = 4, and the answer to this question is x = 4/6 = 2/3 of a candy bar. Third grade children solved the problem by drawing 4 candy bars and dividing them into equal-sized pieces, and distributing the pieces fairly among the 6 people. While the process is not foreign to them (it builds on their early experiences of fair sharing with brothers and sisters), the more cutting they have done, the more difficult it is to name the amount of candy in a share (Lamon, 1996). Over time, the children's marking and cutting became more economical as they recognized equivalencies. Finally, they could answer two questions without engaging in the partitioning process: 1) How much candy does each person receive? and 2) What part of the candy (the total candy) is each person's share? (1/6)

Ratio

Children who studied the ratio interpretation of rational numbers began by making comparisons. For example, "Is it a better deal if 2 people get tickets for the school show for $3, or if 5 people get into the show for $8?" Their natural strategy was to divide one ratio out of the other (take 2 for 3 out of 5 for 8 as many times as possible). They adopted the idea of *cloning* for making copies of a given ratio that look like, act like, and can always be used in place of that ratio. For example, here we can see a 2-clone of 5 to 8. (Look at the columns to see the 5 to 8.) Clones were created and the 2 for 3 ratio was removed until one of the quantities (number of people) was used up.



All of the people got tickets, but there is still $1 remaining. Therefore, 5 for $8 is a better deal. Those who started with the 2 for 3 ratio and drew a 5-clone, discovered that they were short $1 to get two groups of 5 people into the show. Thus, the 2 for $3 price was more expensive. The children's reasoning with their dot drawings became very sophisticated and many interesting strategies emerged.

Examples of Children's Work

Under different rational number interpretations, children acquired meanings and processes in different sequences, to different depths of understanding, and at different rates. Even children who were taught the same initial interpretation of fractions showed different learning profiles. As can be seen in the following examples, the strategies children produced were far different from the fraction work produced in traditional fraction classes and clearly suggested the depth of their understanding. Other examples of students' work may be found in Lamon (1999a, 1999b, 2000).

Find three fractions between 1/8 and 1/9.

$\frac{1}{9} \quad \frac{8}{8} \div 9 \qquad \frac{1}{8} = \frac{9}{8} \div 9$

$1\frac{1}{8} \div 9$

$1\frac{1}{9} \quad 1\frac{1}{10} \quad 1\frac{1}{11}$

$\frac{10}{9} \quad \frac{11}{10} \quad \frac{12}{11}$

$\frac{1}{9} \quad \boxed{\frac{10}{81} \quad \frac{11}{90} \quad \frac{12}{99}} \qquad \frac{1}{8}$

Tell how you figure this out : 1 ÷ 2/3.

$\frac{2}{3}$ is twice as big as $\frac{1}{3}$ and I know $\frac{1}{3}$ goes into 1 three time. So $\frac{2}{3}$ can go in only half as many times.

Which is larger 1/2 or 3/5?

$$\frac{1\text{ pizza}}{2\text{ pizzas}} = \frac{5\left(\frac{1}{5}\text{-pizzas}\right)}{10\left(\frac{1}{5}\text{-pizzas}\right)}$$

$$\boxed{\frac{3\text{ pizzas}}{5\text{ pizzas}} = \frac{6\left(\frac{1}{2}\text{-pizzas}\right)}{10\left(\frac{1}{2}\text{-pizzas}\right)}}$$

BEST COPY AVAILABLE

Results

At the end of four years, three main criteria were used to compare achievement in the experimental classes and in the control group: the numbers of students able to reason proportionally in multiple situations, students' competence in fraction computation (at the 80% level), and the number of rational number interpretations students used meaningfully. Table 2 compares the six classes on these criteria. The time-honored learning principle of transferability was robust. Not only did children transfer their knowledge to unfamiliar circumstances, but to other interpretations that they were not directly taught. By the end of sixth grade, over 50% of the children had demonstrated the ability to apply their knowledge to at least two of the rational number subconstructs. The numbers of proportional reasoners far exceeded the number in the control group, and achievement was greater in computation in all five groups than it was in the control group.

Conclusions

It is likely that current part–whole fraction instruction could be improved merely by affording children the time and opportunity to build understanding and not directly presenting rules and algorithms. It took a long time for the children in this study to build meaning, and during the first 2-2 1/2 years, students in the experimental classes were no competition for other students who were already performing fraction computations. However, in the long run, all five classes of students surpassed the rote learners and their operations and explanations showed that they were performing meaningful operations.

Table 2. Achievement of the Four-Year Participants from Each Class on Three Criteria

	Numbers of Students Achieving Each Criterion						
			Number of Rational Number Interpretations				
Class	Proportional Reasoning	Computation[a]	1	2	3	4	5
Unitizing[b]	8	13	19	8	7	1	0
Measures[c]	8	12	17	11	9	3	0
Operators[d]	3	9	15	9	0	0	0
Quotients[e]	2	9	16	7	0	0	0
Ratio/Rate[f]	12	11	18	10	6	0	0
Control[g]	1	6	11	2	0	0	0

[a]80% accuracy or better; [b]n=19; [c]n=17; [d]n=19; [e]n= 18; [f]n=18; [g]n=20

Nevertheless, there is a case to be made for considering more dramatic changes in fraction instruction. It is apparent that when the goal of instruction is to provide as broad and deep a foundation as possible for rational number meanings and operations, not all of the subconstructs of rational numbers are equally good starting points. Part–whole instruction as it is currently delivered was the least valuable inroad to the system of rational numbers. The part–whole interpretation with unitizing and the measure interpretation are particularly strong. Both show exceptional promise for helping children to make the transition from whole numbers to rational numbers because they build on and extend principles of measurement with which the children have been familiar since early childhood: the inverse relationship between the size of the unit with which you are measuring and the number of times you can measure it out of a given quantity of stuff, and the successive partitioning of that unit of measure into finer and finer subunits until you can name the amount in a given quantity.

It is not yet clear how the five interpretations can or should be integrated in instruction. It seems clear that a cursory visit to each of them is unlikely to be of any real value because of the long time it took students to grow comfortable with the single interpretation with which they began fraction instruction. It is also likely that there is no "one-interpretation-fits-all" solution. For the children in this study, meaning preceded and suggested appropriate operations. Instruction designed to support children's thinking, rather that instruction in algorithmic processes, dramatically improves achievement in what has been the most challenging topic of the elementary and middle mathematics curricula. Although there are still many researchable issues, we have a much better notion than we had 20 years ago of what it means to teach fractions for meaning. We also have a much better understanding of the role that initial fraction instruction plays in enabling or disabling the development of rational number understanding.

References

Kieren, T. (1976). On the mathematical cognitive, and instructional foundations of rational numbers. In R. A. Lesh (Ed.), *Number and measurement* (pp. 101-144). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Kieren, T. (1980). *The rational number construct--Its elements and mechanisms.* In T. Kieren (Ed.), Recent research on number learning (pp. 125-149). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Kouba, V., Carpenter, T. P., & Swafford, J. (1989). Number and operations. In M. M. Lindquist (Ed.), *Results from the fourth mathematics assessment of the National Assessment of Educational Progress* (pp. 64-93). Reston, VA: National Council of Teachers of Mathematics.

Lamon, S. J. (1993). Ratio and proportion: Children's cognitive and metacognitive processes. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 131-156). Hillsdale, NJ: Erlbaum.

Lamon, S. J. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 89-121). New York: SUNY Press.

Lamon, S. J. (1996). The development of unitizing: It's role in children's partitioning strategies. *Journal for Research in Mathematics Education, 27* (2), 170-193.

Lamon, S. J. (1999a). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah, NJ: Erlbaum.

Lamon, S. J. (1999b). *More: In-depth discussion of the reasoning activities in 'teaching fractions and ratios for understanding.'* Mahwah, NJ: Erlbaum.

Lamon, S. J. (2001). Presenting and representing: From fractions to rational numbers. In A. Cuoco & F. Curcio (Eds.), *The roles of representation in school mathematics. 2001 Yearbook* (pp. 146-165). Reston, VA: National Council of Teachers of Mathematics.

Post, T. R. (1981). Fractions: Results and implications from national assessment. *Arithmetic Teacher, 28*, 26-31.

GROWING MATHEMATICAL UNDERSTANDING: TEACHING AND LEARNING AS LISTENING AND SHARING

Lyndon C. Martin
University of British Columbia

Abstract: This paper considers how mathematical understanding is observed to grow through the actions of the participants in an elementary classroom. The paper explores the growth of understanding both at the individual and the whole-class level, and seeks to elaborate the ways that the collective understanding of a class develops. Particular attention is paid to the role of the teacher in occasioning growing understanding, specifically through the ways in which she listens to and works with the ideas of the pupils as the class explore some initial concepts of probability. In developing an analysis of the classroom activities the paper draws on elements of the Pirie-Kieren theory for the growth of mathematical understanding, the enactivist principle of co-emergence, and the three 'modes of listening' proposed by Davis (1996).

The Study

The larger study, still ongoing, consists of a series of case studies of teachers, working in primary and elementary schools. The study began in Britain and was initially framed by the introduction of the National Numeracy Strategy in 1999. A key requirement of the National Strategy and the accompanying 'Framework for Teaching Mathematics' was that "teachers will teach the whole class together for a high proportion of the time, and oral and mental work will feature strongly in each lesson" (Department for Education and Employment, 1999, p. 2). However there was little mention in either document of the possible effect of this teaching style on the pupils' learning and understanding of mathematics. This study focuses on whole class interactive teaching in terms of the individual and collective understandings of the pupils.

This paper presents some initial findings and discussion drawing on a case study of a single British teacher. The teaching sequence lasted for five one-hour lessons. The series of lessons was observed and videotaped, and accompanying field notes were made, by the author. The teacher, Mary, was working with a Year Four (Grade Five) class of relatively high ability. She was introducing them to some basic ideas of probability, in particular the use of the language associated with probability to discuss events (for example, certain; likely; unlikely; etc.) Later she moved on to consider the numerical probability of an event occurring, leading to the theory of outcomes and to a consideration of how this compares with experimental results. Her teaching approach was based on that suggested by the British National Numeracy Strategy, and involved much direct, whole-class, interactive work

Analysis of the Data

In order to analyse the pupils' growing understanding use was made of elements of the Pirie-Kieren theory and model for the Dynamical Growth of Mathematical Understanding (Pirie & Kieren, 1994). This theory provides a way to analyse, describe, and account for the growing understandings of learners of mathematics, and, through the notion of 'interventions', provides a powerful mechanism for considering the actions and interactions of the teacher as teacher and student co-determine (Davis, 1996) this growing understanding. It is also a theory that can be used to describe the growing understandings of individuals or of groups or learners, and as such can be used to look at the growth of understandings of a whole class. This theory has been presented and discussed at a number of previous PME and PME-NA meetings 1 & 2. Use was also made of the work of Davis (1996), in particular the notions of 'hermeneutic' and 'interpretive' listening.

The Pirie-Kieren theory provides a way of considering understanding which recognises and emphasises the interdependence of all the participants in an environment. It shares and is intertwined in the enactivist view of learning and understanding as an interactive process, and rejects the boundaries between mind and body, self and other, individual and social, known and knower. This location of understanding in the "realm of interaction rather than subjective interpretation' and a recognition that "understandings are enacted in our moment-to-moment, setting-to-setting movement" (Davis, 1996, p.200) allows and requires the discussion of understanding not as a state to be achieved but as a dynamic and continuously unfolding phenomenon. Hence, it becomes appropriate not to talk about 'understanding' as such, but about the process of coming to understand, about the ways that mathematical understanding shifts, develops and grows as a learner moves within the world. Understandings are seen to be not merely dynamic but also "relationally, contextually, and temporally specific and so, while understandings might be shared during moments of interactive unity, they inevitably diverge as the participants come back to their selves." (Davis, 1996, p.200).

Although understanding is still the creation of the learner, the classroom, the curriculum, other students and the actions of the teacher occasion such understanding which can be seen as co-emergent with the space in which it was created (Varela, Thompson & Rosch, 1991) with an intersubjective character. Thus "the teacher through making space for the learning of mathematics can be seen as giving voice to the student and allowing for more constructive, less controlled or oppressed knowing." (Kieren, Davis, Mason & Pirie, 1993, p.2). Although the notion of 'personal understanding' is still a valid one within an enactivist view of understanding, this is seen as entwined with every other student's understanding. Collective knowledge and individual understanding co-emerge together. Davis (1995) suggests that "even when an individual is working independently and in apparent isolation on a mathematical task – the action is social, for it is framed in language and procedures that have arisen in social activity." (p.4).

A fundamental component of such co-emergence is listening, which Davis (1996) distinguishes from mere 'hearing' arguing that "important qualities of listening, then, are that it be active and participatory. Listening more involves a dissolution of static notions of the self, permitting a re-membering of intersubjective awarenesses - a "joining of minds." (p.38). Davis also goes on to offer three 'modes of listening' - 'evaluative'; 'interpretive' and 'hermeneutic'. Evaluative listening, Davis suggests is characterised in a classroom where student interactions with the teacher are limited and usually concerned with giving an answer that is judged as either right or wrong by the teacher. Contributions by the pupils have no effect on the planned sequence of the lesson, and thus only the pupils are engaged in 'listening'. Interpretive listening concerns a teacher attempting to get at what learners are thinking, to access the "subjective sense rather than to merely *assess* what has been learned." (Davis, 1996, p.53). Hermeneutic listening is defined as "a *participation* in the unfolding of possibilities *through collective action*." (p.53, original emphasis) and is characterised in the classroom by interactions which are "negotiated" and often "messy" - in contrast to planned and structured interactions designed to evaluate or to access the knowings of the pupils.

Listening and Sharing

Consider the following extract of transcript. This is taken from the second of the two lessons observed, and here Mary is engaged in working with the pupils to establish an image[1] for the theoretical and numerical probability of an event occurring. She has chosen to use a set of dice with the class. Each die has its faces coloured either green or blue, with each having a different number of green and blue faces. Mary is standing at the front of the class, the pupils are sitting in groups around tables, looking at her. It should be noted that is merely an illustrative example of Mary's style of teaching, and that some of the comments which follow this extract draw on the larger body of data recorded.

Teacher: Okay, who can predict for me, if I threw this dice with one green and five blues, **in theory** (*she emphasizes this phrase*), if I threw that dice six times, how many greens would I get? Have a think…in theory…if I threw that dice with one green and five blues – in theory – how many greens would I get?

(*Pupils are seen listening and thinking – the whole class is focused on the teacher*)

(Piece cut from transcript)

T: What do you think Gary?

Gary: Two

T: You think two greens out of six times?

G: Yeah…

T: Hands up if you agree with that

(*Some hands are raised*)

T: Okay, a few people agree..Chris what's your idea?

Chris: I think there'll be two greens.

T: You think, well I said do you agree with George, yeah..so if I've got a dice with six faces, I throw it six times, I've got one green and five blues you think the theory is I'm going to get two greens? Hands up if you disagree

(*Some hands are raised*)

T: Paul, what do you think?

Paul: One.

T: You think you'll get one green, why?

P: Because there's one green on this and then all the rest are blue.

T: So I've got five blue and one green, so out of six throws I should get one green. Out of six throws how many blues do people think I should get? If I threw it six times there are five blues on there. How many blues do you think I'll get, Nikki?

Nikki: Five

T: Five, how many people agree with that?

(*Some hands are raised*)

Pupil: Maybe.

T: It's tricky isn't it…who thinks it should be two greens and four blues?

(*Some hands are raised*)

T: Right, Chris, can you explain why you think it should be two greens and four blues?

C: Because when you said that in an ideal world if you had three blues and three greens and you would get the same amount I was thinking if you had four blues and two greens….erm five blues and one green I was thinking you would double it.

T: So how many blues do you think I would I get out of six throws? How many blues do you think I would get with five blues and one green?

C: Four

T: Is that logical? Which one do you think is more logical? Out of six throws I've got five blues and one green. Out of six throws I get one green and five blues, or out of six throws I get two greens and four blues? Hands up if you think getting one green and five blues is more logical?

(*Some hands are raised*)

T: Hands up if you think getting two greens and four blues is more logical.

(*Class are talking among themselves about the problem – some hands are raised*)

T: Okay, lets have a look at our theoretical result for a dice that's got no greens. Out of six throws how many greens am I going to get? Out of six throws how many greens am I going to get?

(*Many hands are raised*)

T: Barry?

Barry: None

T: None, so none out of six..starting to use a little fraction now *(she is writing on the board*), none out of six are going to be green. How many out of six are going to be blue, Jenny?

J: Six

T: Six out of six are going to be blue. Now we've got a query on this one (*meaning the dice with one green and five blue faces*). Some people think one out of six are going to be green when you've got one green and five out of six are going to be blue and some people think two out of six and four out of six. Okay. What if we took a dice with four green and two blue? If we threw that dice six times how many greens do you think we would get? I don't want practical, I want theoretical, we've done the practical…

(*The lesson continues in a similar manner*)

There are a number of interesting aspects to this transcript[2], and to the actions and words of the teacher.[3] Although Mary has in her own mind the image that she wishes her class to make for probability (i.e., that you can calculate the numerical probability of an event occurring, through looking at the number of ways the event occurs out of the total number of outcomes) she does not tell or directly 'teach' her pupils this. Instead she begins by asking a question, looking for what the pupils think the answer is, and why. In so doing she is allowing and encouraging her class to share with her (and each other) their understandings and indeed their misconceptions. Mary takes a

250

range of possible answers from her pupils, who continue to offer ideas throughout the interaction. At a number of points she asks 'Who agrees?' allowing her both to monitor and validate the kinds of understandings her class are constructing, to get a sense of how common these are to the group, and to see who might still have an alternative idea.

Interestingly, she does not say yes or no to either 'correct' or 'incorrect' answers. Instead she continues to probe what her students are thinking, through continuing to question them. Indeed the most powerful and striking aspect of the extract above is that after the pupil, Paul, articulates the image that she herself is trying to help the class to have and so gives a 'correct' answer, she then moves to work with the misconception – keen to explore why some pupils still think that the correct answer is two green and four blue outcomes. She seems to feel that the class as a whole does not yet share the understanding that Paul has articulated, is not ready to 'have the image' and so she is not prepared to tell the rest of the pupils that this is correct. Instead she wants the image to co-emerge from the interactions of the class (including herself), and for as many members as possible to have participated in this image making. The pupils too, seem to share in this desire, they do not engage in trying to 'guess what answer the teacher wants'. They are used to having to explain their answers, whether right of wrong, and are happy to be honest when they raise their hands.

Listening to each other is a fundamental and feature of this learning environment. Mary and her pupils do not merely listen for the correct answer, but instead criticise, comment and build on what others are saying. There is a sense of 'messy negotiation' occurring in the classroom and of a 'dynamic interdependence' of those participating (Davis, 1996). Although some of the actions of the teacher indicate she is listening in an interpretive way to the pupils, for example she does not ask particularly open-ended questions, the way that she reacts to the unexpected answers that she receives at times also suggest that she is listening hermeneutically. There is a fluidity to the whole lesson, which lasts an hour, that allows a wandering through the mathematics being considered. Notably, neither the teacher nor the pupils seem concerned by what might be seen as the slow pace of the lesson. There is no sense of a need to quickly reach a required rule or formula, or to 'know' a particular piece of mathematics, either on the part of the teacher or the pupils. This is not to say though that Mary is not 'teaching' - however she does not appear to see 'teaching' as "a matter of causing learners to acquire, master or construct particular understandings through some pre-established (and often learner-independent) instructional sequence." (Davis, 1996, p.114). Instead, teaching and learning is seen as a social process, and the teacher's role as one of participating and of interpreting. In a similar way the actions and interactions of the pupils are intertwined in the form the lesson takes - there is no way this lesson could have been pre-scripted or planned, and this allows the kinds of diversions occasioned by the pupil responses.

There is some element of tension apparent though in Mary's approach, and towards the end of the extract, we see her nudging the pupils towards the image she wants them to have, she uses the word 'logical' and attempts to get the pupils to again reason and to think about and around their ideas. She is shifting here to a more interpretive style of questioning and listening. However, she still resists the temptation to merely tell her class the answer, but neither does she leave them totally struggling with a misconception. Indeed, throughout the extract, and through the other lessons observed, there is a sense of Mary trying to find a different way for her class to understand the idea she is working with, and to allow a common understanding of some kind to co-emerge from the discussions. The developing understanding is being contingently occasioned here. What the pupils do is contingent on the actions of the teacher, but at the same time the actions of the teacher are contingent on those of the pupils, and genuine dialogue and "interactive unity" is thus created.

Conclusion

This study suggests, then, that within an interactive whole class environment, mathematical understanding can grow, both for individuals and for the class in a collective sense. Whilst Mary has a clear idea of the kinds of images she wants her class to make for probability, she does not merely tell her pupils a rule or formula. Instead, she works with them, allowing space for them to lag behind, to advance, confident in her teaching that she will at some point be able to gather them to a common image or kind of understanding - co-emergent and contingent within the class. It is her gentle coaxing that makes her interventions, constant as they are, effective in terms of achieving growth of mathematical understanding. She provides a 'thinking space'[4] for them to construct and modify their own images in response to her interventions. Even where she offers them particular images for the concept of probability, she does not intend her pupils merely to accept these. Instead she expects questions, challenges, discussion, and an articulation of different images that her pupils might have or be making, even where these involve misconceptions. She wants her pupils to have a real 'sense' of number, and not to merely be able to apply a relatively simple rule. The power her style of teaching is dependant on the kinds of listening engaged in during the lesson. Through a combination of hermeneutic and interpretive listening, the teacher and the pupils interact with each other in a meaningful way, to allow learning to occur through a social and shared process.

This collective growth of understanding can be interpreted as being *dependent on*, but not *determined by*, the actions of the teacher (Davis, 1996) and this is true of the mathematical actions within Mary's classroom. Although her interventions, actions and interactions do not determine the growing understanding of her pupils, each is contingent on the other, and pupils and teacher need to listen to each other to consider and co-determine where the discussion might next go. Such an analysis of classroom actions, leads to a complex yet powerful way of viewing growing understanding in the mathematics classroom as a co-emergent process.

References

Davis, B. (1995). Why teach mathematics? Mathematics education and enactivist theory. *For the Learning of Mathematics 15* (2), 2-8.

Davis, B. (1996). *Teaching mathematics: Toward a sound alternative*. New York: Garland.

Department for Education and Employment (DfEE) (1999). *The national numeracy strategy: Framework for teaching mathematics from reception to year 6*. London: DfEE.

Kieren, T., Mason, R., Davis, B., & Pirie, S. (1993). *Vignettes of the growth of mathematical understanding in a communicative classroom: An interactive and enacted review.* Paper presented at the Annual Meeting of the Canadian Society for the Study of Education.

Martin, L. C. (1999). *The nature of the folding back phenomenon within the Pirie-Kieren Theory for the growth of mathematical understanding and the associated implications for teachers and learners of mathematics.* Unpublished doctoral dissertation, Oxford University.

Martin, L. C., & Pirie, S. E. B. (1998) She says we've got to think back: Effective folding back for growth in mathematical understanding. In A. Olivier & K. Newstead (Eds.), *Proceedings of the Twenty-Second Annual Meeting of the International Group for the Psychology of Mathematics Education, 4* (pp. 255-262). Stellenbosch, South Africa.

Martin, L. C., Towers, J., & Pirie, S.E.B. (2000) Growing mathematical understanding: Layered observations. In M.L. Fernandez (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* (Vol. 1, pp. 225-230). Tucson, AZ.

Pirie, S. E. B. (1996) What are the data? An exploration of the use of video-recording as a data gathering tool in the mathematics classroom. In E. Jakubowski, D. Watkins, & H. Biske (Eds.), *Proceedings of the Sixteenth Annual Meeting of the International Group for the Psychology of Mathematics Education - North America, Vol. 2* (pp. 553-559). Florida State University, Panama City.

Pirie, S. E. B., & Kieren, T. E. (1992). The answer determines the question - interventions and the growth of mathematical understanding. In W. Geeslin & K. Graham (Eds.), *Proceedings of the Sixteenth Annual Meeting of the International Group for the Psychology of Mathematics Education,* (Vol. 2, pp. 1-8). New Hampshire.

Pirie, S. E. B., & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26* (2/3), 165-190.

Varela, F., Thompson, E., & Rosch, E. (1991) *The embodied mind: Cognitive science and human experience*. Cambridge, MA: MIT Press.

Notes

[1] The terms 'image'; 'image making', 'image having' are used as defined in the Pirie-Kieren theory. There is not space here to re-state these definitions. See Pirie and Kieren, 1992 for fuller definitions.

[2] It should also be noted that much detail of the life of this classroom is lost in the act of transcription. The analysis of the data always involved working with the videotape itself rather than with a transcript. See Pirie (1996) for a fuller discussion of this.

[3] There is also much that can be said about the way Mary approaches the idea of probability, and about the difficulties the pupils have with the concept, but it is beyond the brief of this paper to explore this.

[4] See Martin (1999) for an elaboration of the notion of a 'thinking space' in the growth of mathematical understanding.

THE ROLE OF TASK-ANALYSIS CYCLES IN SUPPORTING STUDENTS' MATHEMATICAL DEVELOPMENT

Kay McClain
Vanderbilt University
kay.mcclain@vanderbilt.edu

Introduction

The purpose of this paper is to describe an iterative process of task analysis and its role in supporting students' mathematical development. In doing so, I will present an episode taken from an eighth-grade classroom in which my colleagues and I conducted a twelve-week teaching experiment during the fall semester of 1998.[1] The goal of the teaching experiment was to support students' development of ways to reason about bivariate data as they developed statistical understandings related to exploratory data analysis. This teaching experiment was a follow-up to an earlier classroom teaching experiment conducted with some of the same students during the fall semester of the previous year. Over the course of the two teaching experiments our goal was to investigate ways to proactively support middle-school students' development of statistical reasoning. In particular, the teaching experiment conducted with the students as seventh graders focused on univariate data sets and had as its goal supporting students' understanding of the notion of distribution (for detailed analysis see Cobb, 1999; McClain, Cobb, & Gravemeijer, 2000). Our goal for the eighth-grade classroom was to build on this earlier work and extend it to bivariate data sets (for a detailed analysis see Cobb, McClain, & Gravemeijer, in press).

The particular focus of the analysis in this paper is on the role of whole-class discussions in helping to initiate shifts in students' ways of reasoning toward more efficient and sophisticated arguments, models and inscriptions (cf. Lehrer, Schauble, Carpenter, & Penner, 2000). These discussions comprise the third of three phases of the task-analysis process which are 1) posing the task, 2) analysis of data and development of arguments, and 3) a whole-class discussion and critique of the arguments. Within each of these three phases, there is an implied relationship between the negotiation of classroom social and sociomathematical norms and the students' mathematical development. In particular, the first phase is characterized by students actively involved in the data creation process during which they clarify for themselves (1) the question to be answered or dilemma to be resolved and (2) the ramifications of the data collection procedures on their analysis. In the second phase, the students work at computers in small groups on their analyses while the teacher carefully monitors their activity in order to make informed decisions about appropriate ways to orchestrate the subsequent whole-class discussion. The final phase is characterized by students explaining and justifying the results of their analysis in a whole-class discussion that is planfully orchestrated by the teacher. The teacher's role in this process is both to continually support the negotiation of norms for productive mathematical argumentation and to ensure that the discussions have mathematical validity with respect to the overarching goals for the sequence of instructional tasks. This is not to imply that discussions are funneled or scripted. The image that results is that of the teacher constantly judging the nature and quality of the discussion against the mathematical agenda in order to ensure that the mathematical issues under discussion offer means of supporting the students' development. This view of whole-class discussion stands in stark contrast to an open-ended session where all students are allowed to share their solutions without concern for potential mathematical contributions.

This description of the task-analysis process does not acknowledge the critical role of the iterative process of *cycles* of task analysis in supporting students' development. In particular, it is as the students engage in a *sequence* of tasks that they improve and refine their ways of analyzing data that results in cycle*s* of the task analysis process. For this reason, a better explanation of the task-analysis process would account for the student learning that occurred in the first iteration or cycle of this process as a second task was posed (and so forth).

In the following sections of this paper, I present an analysis of a classroom episode intended to clarify the different aspects of the task-analysis cycle. In doing so I highlight the significant issues that emerged and their contribution to subsequent phases of the cycle. I also point to the importance of the iterative nature of this process in supporting students' ways of reasoning statistically about data. It is important to clarify that my analysis is being conducted from the viewpoint of the teacher/researcher. Throughout the classroom teaching experiment, I took primary responsibility for the teaching, although on numerous occasions I was assisted by other members of the research team. I am therefore the teacher in the classroom episode in this paper.

Analysis of Classroom Episode

The final task posed to the students during the eighth-grade classroom teaching experiment involved their analyzing data from two speed-reading programs in order to make a judgment about which program was more effective. In *posing the task (phase one)*, the students and I engaged in a lengthy discussion about how such a decision could be made. The students delineated features of each program they would want to measure in order to make a comparison and discussed how these measures could be generated. In the process, the students created the design specification for conducting a study that would generate the data they deemed necessary to make an informed decision. This included discussion of issues such as sample size, testing procedures, accounting for variability within and across samples, and accuracy of timing. This discussion proved critical in grounding the students' investigation in the context of

the question at hand. Against this background, the students were given pre- and post-course reading scores on 138 people from one program (G1) and 156 people from another (G2) inscribed as scatter plots as shown in Figures 1 and 2.
The scatter plots were presented via a computer-based tool for analysis that was developed as part of the teaching experiment (for a detailed explanation of the tool and its features, see McClain, in press; Cobb, McClain, and Gravemeijer, in press). The task was then to analyze the data in order to determine which program was more effective. In doing so, students were asked to develop a written report and create an inscription that would substantiate their argument. The students spent the remainder of the class session working in pairs at the computers to *analyze data and develop arguments (phase 2)*. As they did so, I and other members of the research team circulated around the room to monitor their activity as we planned for the subsequent whole-class discussion.

The reader will recall that whole-class discussions comprise an integral aspect of the task-analysis process. As such, part of the teacher's role includes making decisions about which solutions and/or ways of reasoning should be highlighted. This requires that the teacher carefully monitor students' small-group activity with an eye toward planning for the subsequent discussion. As a result, while students worked on their analyses, I was focused on understanding their diverse strategies so I could build from them as I planned for the whole-class discussion.

As I and other members of the research team circulated among the groups, we noticed that all but one of them had structured the data into four quadrants using the "cross" feature of the computer tool as shown in Figures 1 and 2. We found this surprising in that few of the students had used the cross option in prior analyses. In addition, it had rarely been the focus in whole-class discussions. Nonetheless, they were using the cross to reason about the four quadrants, making comparisons across the two

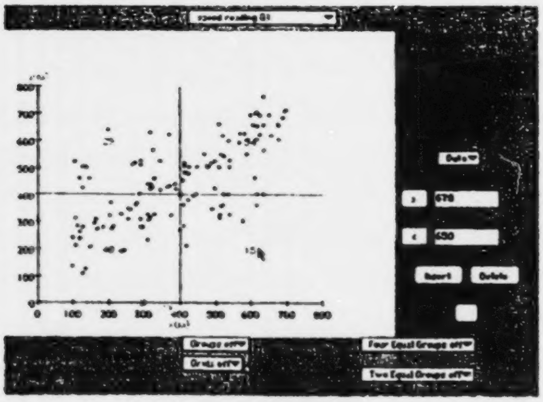

speed reading G1
0
100
200
300
400
500
600
700
800
Data
678
530
Insert
Delete
Groups off
Grids off
Four Equal Groups off
Two Equal Groups off

Figure 1. Speed reading data on program G1 with cross option.

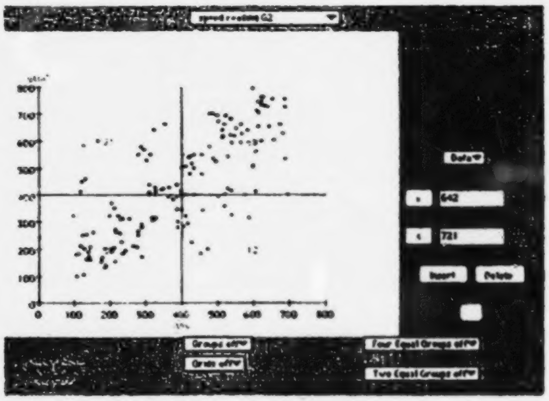

speed reading G2
0
100
200
300
400
500
600
700
800
Data
642
721
Insert
Delete
Groups off
Grids off
Four Equal Groups off
Two Equal Groups off

Figure 2. Speed reading data on program G2 with cross option.

BEST COPY AVAILABLE

data sets. For us, this was a less persuasive way to reason about the data as it left many questions unanswered. Further, we speculated that the students were not viewing the data sets as bivariate distributions, but simply reasoning about qualitative differences in the data in each quadrant. As a result of assessing their activity, we felt it was important to begin with the students' ways of organizing the data, but then attempt to initiate shifts in their ways of reasoning by posing questions that could not be answered using displays of the data generated by structuring the data with the cross.

The following day, the first pair of students chosen to share their analysis in *whole-class discussion and critique of student solutions (phase 3)* had partitioned the data into four quadrants and then reasoned about each quadrant in terms of whether it was a "good" quadrant or a "bad" quadrant as shown in Figure 3.

good	stayed the same
29	54
40	15
about the same	worse off

good	stayed the same
21	68
55	12
about the same	worse off

Figure 3. Data partitioned into four quadrants.

Dave and Susan were chosen as the first in the series of solutions to be highlighted because I judged that their argument was based on qualitative distinctions in the quadrants and did not appear to be viewing the data in terms of distributions. Further, making direct additive comparisons about the number of participants in each quadrant was problematic due to the fact that the two data sets contained unequal numbers of data points. From their inscription, the students did not appear to have created a way to reason about the data in multiplicative terms. My goal was therefore to raise issues and questions that could not be reconciled with this type of model in an attempt to shift the discussion towards ways of reasoning about the data in terms of bivariate distributions which were structured multiplicatively.

As Dave and Susan explained their inscription and argued that program G1 was better because it had more participants in the "good" quadrant, one of the other students in the class challenged their argument. Kyra noted that the two programs did not have the same number of participants so direct additive comparisons would not be a valid argument. Dave responded by noting that Program G2 had more participants, but Program G1 had more participants whose scores were in what he described as the "good" quadrant.

Kyra: One has more than the other.

Teacher(McClain): One group has more than the other so . . .

Kyra: So that's not a fair way of reasoning.

Dave: Well, this one, G2 has more and this one [points to G1] still has more in this [points to the good] quadrant.

I then asked Ryan to share his way, which involved using the cross option in a similar manner but also finding the percentage of participants in each program. After the percentages were recorded on the board, the students agreed that Dave and Susan's initial analysis had been confirmed by calculating the percentages since program G1 had 21% of it's participants in the "good" quadrant whereas program G2 had only 14% of its participants there. However, when questioned, the students were unable to use this model of the data to offer any additional information about what results, say, people who read over 400 words per minute could expect from either program.

At this point I asked Brad and Mike, two students who had structured the data into vertical slices, to share their way of thinking about the data. They had created vertical slices using the computer tool and made comparisons of slices across the data sets (i.e. people who read between 300 and 400 words before enrolling compared to their rates after the programs) as shown in Figures 4 and 5. I felt that this structuring of the data provided the means of supporting students coming to reason about the data in terms of a bivariate distribution by focusing on the change in the post-test scores and the pre-test scores increased. As an example, the students began by identifying what "slice" of the graph represented those people who entered each program reading 300 words per minute. They then reasoned about the slices across the horizontal axis by noting results on the vertical axis as facilitated by the structuring of the data. After some

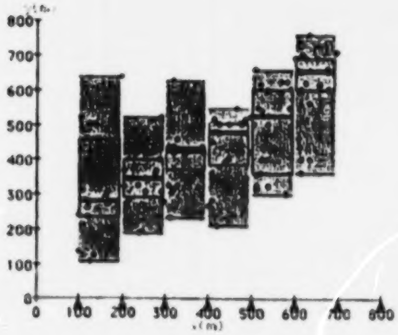


Figure 4. Speed reading data from program G1 partitioned into vertical slices.

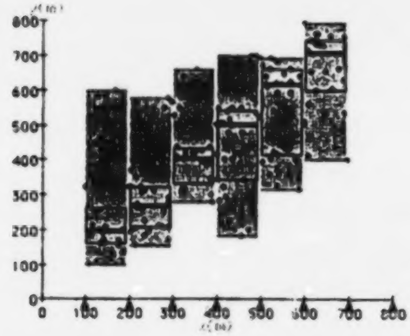


Figure 5. Speed reading data from program G2 partitioned into vertical slices.

discussion, the students agreed that by reasoning across the slices, they could make informed decisions about which program was more effective for certain groups of readers.

Conclusion

Throughout the episode, my decision-making process in the classroom was informed by the mathematical agenda but constantly being revised and modified in action based on the students' contributions. This is in keeping with earlier analyses that describe mathematical teaching as pursuing a potentially revisable agenda as informed by inferences about students' beliefs and understandings (cf. Ball, 1993; Carpenter & Fennema, 1991; Cobb, Yackel, & Wood, 1991; Maher, 1987; Simon & Schifter, 1991; Thompson, 1992). To this end, I worked to capitalize on the students' contributions to advance the mathematical agenda. In doing so, I first took their current understandings as starting points and then worked to support their ways of reasoning about the problem context in more sophisticated ways. In this way, I was able to start with the students' models and inscriptions and build toward the envisioned mathematical endpoint, that of reasoning about data in terms of bivariate distributions. The students' data models and ways of structuring data were critical in supporting this effort. The students' models of their analyses served as thinking devices (Wertsch & Toma, 1995) as they reasoned about the adequacy of both their arguments and their ways of structuring the data.

It is also important to highlight the importance of prior task-analysis cycles in supporting the mathematical agenda. During this episode, I was able to build on the students' prior task-analysis activity of investigating data sets and developing models and inscriptions in order to initiate a shift in their ways of reasoning about the speed reading data. Because the students had developed an understanding of data structured using vertical slices option on the computer tool during prior task-analysis activities, questions which could not be answered with more simple inscriptions were made accessible by building from students' earlier analyses. This highlights the importance of the cyclic nature of the task-analysis process located within the learning trajectory in supporting students' emerging understandings

Acknowledgement. The research reported in this paper was supported by the National Science Foundation under grant no. REC-9814898 and by the Office of Educational Research and Improvement through the National Center under grant no. R305A60007. The opinions do not necessarily reflect the views of either the Foundation or the Center.

Note

[1] The research team was composed of Paul Cobb, Kay McClain, Koeno Gravemiejer, Jose Cortina, Lynn Hodge, and Cliff Konold.

References

Ball, D. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal, 93* (4), 373-398.

Carpenter, T., & Fennema, E. (1991). Research and cognitively guided instruction. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 1-16). Albany, NY: SUNY Press.

Cobb, P. (1999). Individual and collective mathematical learning: The case of statistical data analysis. *Mathematical Thinking and Learning*, 5-44.

Cobb, P., McClain, K., & Gravemeijer, K. (in press). Learning about statistical covariation. *Cognition and Instruction.*

Cobb, P., Yackel, E., & Wood, T. (1991). Curriculum and teacher development: Psychological and anthropological perspectives. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 83-120). Albany, NY: SUNY Press.

Leher, R., Schauble, L., Carpenter, S., & Penner, D. (2000). The inter-related development of inscriptions and conceptual understanding. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms* (pp. 325-360). Mahwah, NJ: Erlbaum.

Maher, C.A. (1987). The teacher as designer, implementer, and evaluator of children's mathematical learning environments. *Journal of Mathematical Behavior, 6*, 295-303.

McClain, K. (in press). Task-analysis cycles as tools for supporting students' mathematical development. In D. Lesh & H. Doerr (Eds.), *Models and modeling.* Dordrecht, The Netherlands: Kluwer Academic.

McClain, K., Cobb, P., & Gravemeijer, K. (2000). Supporting students' ways of reasoning about data. In M. Burke (Ed.), *Learning mathematics for a new century* (2001 Yearbook of the National Council of Teachers of Mathematics). Reston, VA: NCTM.

Simon, M., & Schifter, D. (1991). Towards a constructivist perspective: An intervention study of mathematics teacher development. *Educational Studies in Mathematics, 22*, 309-331.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

Wertsch, J., & Toma, C. (1995). Social and cultural dimensions of knowledge. In L. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 175-184).

THE EMERGENCE OF QUOTIENT UNDERSTANDINGS IN A FIFTH-GRADE CLASSROOM: A CLASSROOM TEACHING EXPERIMENT

James A. Middleton
Arizona State University

Teruni de Silva
Vanderbilt University

Zulbiye Toluk
Abant Izzet Baysal Universitesi

Wendy Mitchell
Madison Meadows Middle School

Theoretical Orientation

Research on understanding children's rational number concepts has been of keen interest to mathematics educators in the last two decades. Children's difficulties with rational numbers can be attributed to the fact that rational numbers take different meanings across different contexts (Kieren, 1976; Behr, 1983). One of these meanings is the quotient interpretation (e.g., Kieran, 1993). Building phenomenologically from the contexts of fair sharing, under the quotient interpretation, rational numbers become quotients of whole numbers. More specifically, a rational number can be defined as the yield of a division situation. Considering this meaning, the relationship between division as an operation and fractions as quotients becomes a critical consideration for instruction.

Children first learn the concept of division partitioning whole numbers—e.g., fair sharing—and they express quotients as whole number partitions. Division situations with remainders are found to be difficult because instruction fails to engage students in the interpretation of resulting fractional quotients appropriately within the given problem context, instead putting off fractional quotients for later study. However, in research on division, there is a lack of study addressing the relationship between rational numbers and division of whole numbers and whether this traditional separation of whole number quotients-with-remainder and fractional quotients is pedagogically sound practice. Moreover, developmentally, there is little research that documents plausible trajectories by which children come to see the two as isomorphic conceptually (see Behr, Harel, Post, & Lesh, 1992, p. 308 which illustrates the limits of our current understanding in rational number research).

At any rate, current curricular offerings in the United States present whole number division and fractions as separate and distinct topics. Division is treated as an operation one performs on Whole numbers, and fractions are taught almost exclusively as Part-Whole concepts. Only in the late middle grades, when an understanding of quotient becomes necessary for dealing with algebraic entities such as rational expressions, are children expected to connect these two heretofore distinct topics. At no point in the preceding 7 or 8 years is explicit attention focused on teaching fractions as division, nor on teaching division as expressing a rational number: A quotient.

Toluk (1999; Toluk & Middleton, 2000), in a series of individual teaching experiments, studied how children can develop conceptual schemes for making sense of Quotient Situations by explicitly connecting their prior knowledge of whole number division with remainders and their knowledge of fair sharing situations resulting in fractions. Specifically, by pairing problems that generated fractional reasoning, with isomorphic problems that yielded reasoning about division, this work led children to the understanding that $a \div b = a/b$ and $a/b = a \div b$ for all (a,b), $b \neq 0$. As children progressed through the teaching episodes, they developed the following schemes in order: *Whole Number Quotient Scheme, Fractional Quotient Schemes, Division as Fraction Scheme* and *Fraction as Division Scheme* (see Figure 1).

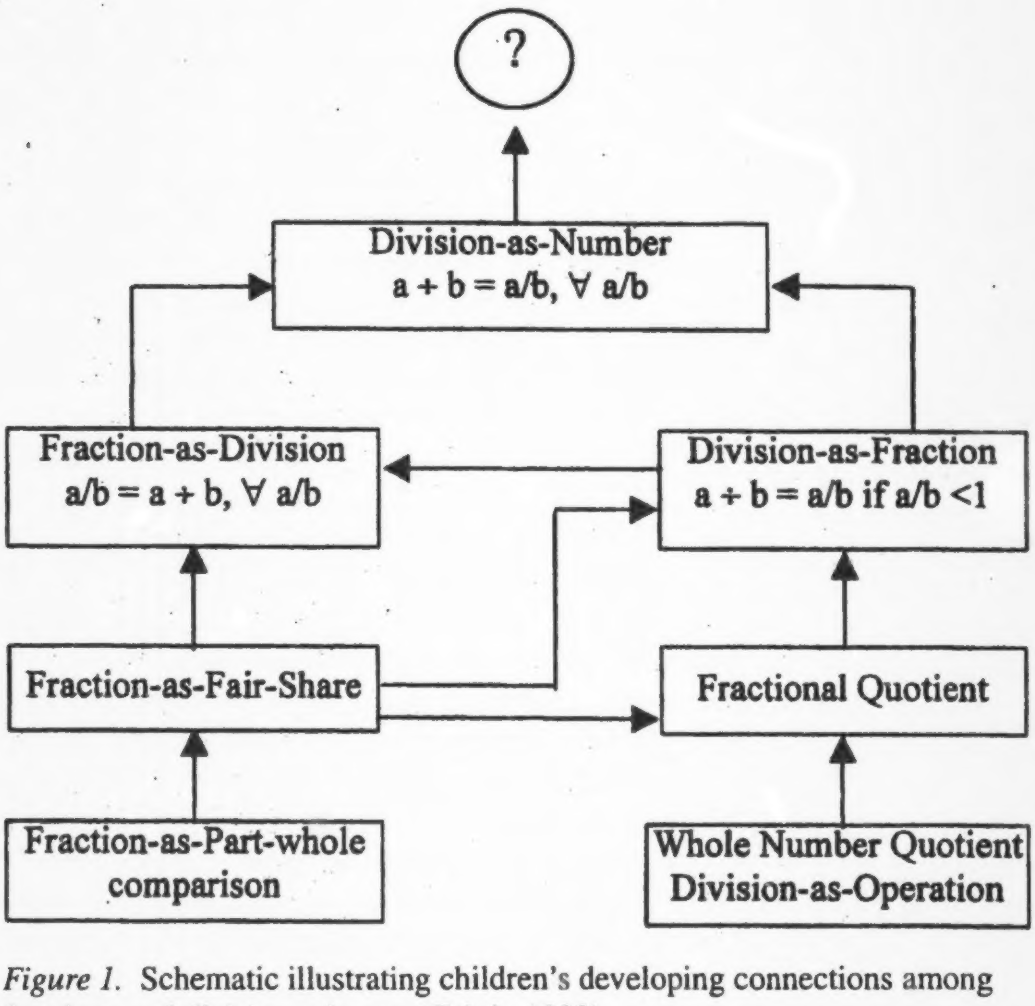

?
Division-as-Number
a ÷ b = a/b, ∀ a/b
Fraction-as-Division
a/b = a ÷ b, ∀ a/b
Division-as-Fraction
a ÷ b = a/b if a/b <1
Fraction-as-Fair-Share
Fractional Quotient
Fraction-as-Part-whole comparison
Whole Number Quotient
Division-as-Operation

Figure 1. Schematic illustrating children's developing connections among fractions and division schemes (Toluk, 1999).

In most prior work, including that of Toluk, inferences about plausible developmental of rational number concepts and skills have been made from an exclusively individual perspective. There are severe limitations to the kind of pragmatic conclusions that can validly be drawn from exclusively individual research. This study remedies these shortcomings by providing a test of recently proposed depictions of children's quotient understanding (Toluk, 1999, Toluk & Middleton, 2000), and by providing a framework that explains their development and interaction in a classroom teaching experiment from an emergent perspective (e.g., Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1996) that treats *psychological* (i.e., individual) development as reflexive with the development of *social* classroom practices and norms. From this theoretical frame, individual development is constrained by the norms and practices established in the collective, while the development of the collective is made possible by and does not occur apart from, the psychological activity of individual students. The current project targets our work at understanding both the development of individual children's understanding of the quotient and the classroom norms and practices that constrain or enable that understanding. To do this, we triangulate between the microgenetic study of a group of 4 children, with the sociological study of classroom practices within which the group is situated over the same instructional sequence. By understanding the nature of classroom discursive practices aimed at developing mathematical understanding, and the concomitant actions and understandings of children as they engage in those practices, we will be able to report not only how children's understandings develop, but why and under what conditions they develop. Moreover, we hope to be able to project ways in which consistent and coherent understandings of quotients can be fostered in the future.

Method

The case for this classroom teaching experiment centered around a fifth-grade classroom (20 students) in an urban school district in the Southwest United States. The teacher, Ms. Mitchell, just finished her Masters Degree and has been a teacher leader and actively involved in reflecting about mathematics teaching practices within the district. In particular, Ms. Mitchell was involved in a project designed to develop an understanding of children's mathematical thinking in teachers in the areas of Algebra, Geometry, and Statistics. Rational number, as a fundamental conceptual underpinning in each of these areas, was a key focus of the staff development.

The research team (which included Ms. Mitchell) developed a 5-week instructional sequence based on Toluk's (1999) developmental model. A pretest of students' knowledge of fractions and division was administered one week prior to instruction (the students had just completed a TERC unit on Fractions, Percents and Decimals), and again as a post test, one week following instruction. Four children (2 boys, 2 girls), who comprised a cooperative group, were selected as target subjects for the analysis of individual knowledge within the collective. These students were interviewed using the pretest as a protocol the week prior to instruction, and again following instruction. Two clinical interviews of Ms. Mitchell were conducted prior to instruction, as she reflected on the purpose of the sequence, particular strategies she would employ, and her assessment of how she expected students to perform on various tasks. During instruction, Ms. Mitchell was formally interviewed once each week, and informally interviewed following each lesson. Lessons lasted approximately 45 minutes per day. All interviews and lessons were captured on digital video. During instruction, one video camera was focused on Ms. Mitchell and the whole-class interaction, while a second camera was focused on the selected group of 4 students. Additional documentation includes copies of all children's individual work, group models drawn on large sheets of chart paper, the teacher's notes, and field notes by members of the research team. The general structure of the data collection reflects the cyclic process of design experiments and developmental research in particular (e.g., Cobb, 2000).

Results

The results of this study indicated that the practices of the classroom moved *more or less* along the hypothetical learning trajectory abstracted from the model proffered by Toluk (1999; Toluk & Middleton, 2000). That is, they progressed from treating fractions as exclusively Part-Whole to having at least two parallel conceptions: Part-Whole *and* Fair Share (Quotient), while at the same time began to conceive of the division operation as generalizable to any pair of whole numbers, and some children began to see the operation as generalizable to division by fractions Table 1 illustrates the general learning trajectory of the class in three more-or-less distinct instructional Phases (see Figure 2).

In Phase I, key norms were established in the classroom that included the explication of forms of representation in terms of their ability to be partitioned, the acceptance of nested sets of units (such as 2 packets of gum with 5 pieces of gum in each pack) and the need to select an appropriate unit with which to coordinate partitioning. This need was established when the teacher would introduce units at different levels in a nested structure to force students to communicate precisely how they conceived of the quantities under consideration. For Ms. Mitchell, a huge jump in conceptualizing the difficulties in coordinating both partitioning and unitizing occurred early on when students were unable to develop strategies other than repeated halving for "odd" denominators (e.g., Pothier & Sawada, 1990). For example, when Ms. Mitchell asked children to describe "other names" for the (Fair-sharing) fraction 2/3, they were readily able to provide 4/6, 8/12, and 16/24, but stated that 6/9 did not follow the pattern, and thus, was not a legitimate name for 2/3. Students were able to make sense of 2/3 = 6/9 in a Part-Whole context, but were not in one involving Fair-Sharing.

In Phase II, as children began to symbolize fractional quotient situations as answers to "division problems," they encountered a new kind of quotient: One that

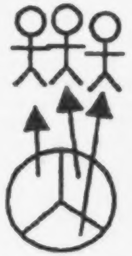
Phase		Focus of Classroom Activity	Forms of Reasoning	Key Transitional Strategies
I	**Representing Quotient as the Answer to a Fair Sharing Problem**	Partitioning: Transforming a whole into countable units. Understanding equivalency of shares. How many pieces make up a share. Representing answer as a fraction. Conceptualizing magnitude of fractional answer.	Primarily Pictorial: Partitioning and assigning. eg: 	Directing students' attention to the magnitude of the partitions, and the composition of shares—eg: away from "How Many?" to "How Much?" Presentation of problem contexts that could be represented by a "mixed number"—a quotient with a whole number portion and a fractional portion.
II	**Symbolizing Fair Sharing Problem Using Standard Divisor and Dividend Notation**	Determining the meanings ascribed to the divisor and dividend in a÷b notation. Mapping the meanings of partitions and whole from Phase I onto a÷b. Symbolically representing fair sharing as division, with a fractional answer.	"Stuff to split" became defined as the dividend (unit) The number of groups to distribute shares across became defined as the divisor (partitions) The bar in the fractional notation came to represent division e.g., "You divide the stuff to split by the number of groups."	Confront conception that "bigger number must divided by smaller" through examination of problem context. Consistently pairing a÷b notation with a/b notation, especially with the introduction of number sentences: $a \div b = a/b$ Using multiplicative relationship as a conceptual support e.g.: $a \times b = c \quad c \div b = a$ where a, b, or c could be < 1.

Figure 2, Part A. Description of classroom learning trajectory as played out in three instructional phases.

Phase		Focus of Classroom Activity	Forms of Reasoning	Key Transitional Strategies
III	**Flexible use of Fraction and Division Notation with Conceptual Understanding**	Examining the relative magnitude of divisor and dividend to anticipate quotient. Predicting the answer to a fair sharing problem without computation. e.g: $6 \div 9 = 6/9$ and $6 \div 9 < 1$ Switch to symbolizing division relationship first, then computing and symbolizing answer. Proving correctness of answers by using inverse operation	Anticipating the magnitude of the result of fair sharing as being either greater than or less than 1. Developing a rule for determining if quotient is greater than or less than one. Connecting the previously separate symbolizations for division and fractions conceptually—e.g., recognizing that they are the same. Prior to this Phase, students focused on figuring out how to symbolize the division relationship and fractional answer separately.	Now pictorial representations were used to justify reasoning following an attempt at solving fair sharing problems, as opposed to being used as tools to come up with solutions.

Figure 2, Part B. Description of classroom learning trajectory as played out in three instructional phases.

had a familiar whole number part, and an extra piece that could further subdivided. In particular, the provision of a continuous partitionable unit in the problem context (e.g., Brownies) enabled children to distribute fractional pieces of objects across groups and begin to conceive of answers to division problems as containing both a whole number part and a fraction part. Later, these fractional parts became the objects of discussion, and "division" problems were introduced that involved only fractional parts.

A watershed moment arose in Phase III when students realized that the notation $a \div b$ yielded a fraction a/b for every $a>b$. Previously, children tended to write given any improper fraction as a mixed number, and this tendency this hindered their ability to recognize the relationship. By providing successive problems that moved towards a numerator less than a denominator, the class was able to come to a taken-as-shared understanding that $a + b$ yields a fraction a/b for every $a<b$. Not all children reached this understanding individually, however. While in the context of classroom discussion, for example, children would work together to develop a communal solution to a problem utilizing this "Division as Number" scheme, but on the post-test, a number of children did not utilize this understanding for improper fractions.

Discussion

Previous research has shown that young children deal with quotient situations—fair sharing—easily (Lamon, 1996; Kieren, 1988). In a fair sharing situation, a typical behavior is to partition quantities and write the resulting fractions as the quotient of a given situation. From their partitioning behavior, it is concluded that children conceptualize those fractions as quotients. The results of the present teaching experiment, however, suggest that ability to partition quantities into its parts didn't necessarily indicate that children actually conceptualized the resulting fractions as quotients. This was evidenced by the general reluctance in symbolizing quotients less than one as division number sentences.

Children were reluctant to symbolize the situations in this manner for a variety of reasons. First, for them, "division" always yielded a whole number quotient with or without a remainder. It was not that they didn't recognize the partitionability of the remainder, it was the fraction, the result of the partitioning that they didn't conceive of as a part of the quotient. Typically, in their minds, "fractions" resulted from fair sharing situations which involved cutting or splitting of whole quantities into parts less than one, whereas "division" resulted from partitioning quantities into groups with number greater than one. Second, students' resistance to consider a fraction as a quotient was due to the the fact that children had such a strong understanding of the Part-Whole subconstruct which, from prior experience, was the only way they were used to thinking about a fraction.

Moreover the existence of an understanding of fractions representing fair shares and part-whole relationships and division as a whole number operation as separate entities does not necessarily imply that children will connect these understandings into a coherent quotient scheme without intervention. Rather, the results of the teaching experiment showed that some explicit connections have to be made between these concepts. By making the commonalities in context and notation between whole number division and fraction situations problematic, children need to be encouraged to reflect on the equivalency of fair sharing situations and whole number division. Using a common symbolization; children's conceptions of division can be challenged. In this teaching experiment, division number sentences were used to show that the result of a division operation is a fraction represented by the common divisor/dividend notation). These results suggest that, while the mathematics of quantity that connects rational number subconstructs to each other in an epistemic framework is well thought out, without mediation, children's proportional reasoning does not recapitulate this epistemic frame.

The findings of this study suggest that a fragmentary approach to teaching quotients, which arranges instruction into two distinct and separate conceptual dimensions, i.e., division as an operation and fractions (as primarily part-whole quantities, but also applying to simple fair sharing), may lead children to develop a fragmentary understanding of the quotient. Instead, the present cases suggest that providing common problems in both fractional and division forms, and confronting children with the basic equivalencies of these forms may be fruitful in developing a more powerful understanding of the quotient subconstruct.

References

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, & proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan.

Behr, E. A. (1983). Rational-number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 92-124). New York: Academic Press.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307-334). Mahwah, NJ: Erlbaum.

Cobb, P., Gravemeijer, K., Yackel, E., McClain, K., & Whitenack, J. (1996). Mathematizing and symbolizing: The emergence of chains of signification in one first-grade classroom. In D. Kirshner & J. A. Whitson (Eds.), *Situated cognition: Social, semiotic, and psychological perspectives* (pp. 151-233), Mahway, NJ: Erlbaum.

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and measurement* (pp. 101-144). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. J. Behr (Eds.), *Number concepts and operations in the middle grades* (pp.162-181). Hillsdale, NJ: Erlbaum.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 49-84). Hillsdale, NJ: Erlbaum.

Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education*, *27* (2), 170-193.

Pothier, Y., & Sawada, D. (1990). Partitioning: An approach to fractions. *Arithmetic Teacher*, *38* (4), 12-17.

Toluk, Z., & Middleton, J. A. (2000). *Fifth graders' conceptualizations of the quotient subconstruct of rational numbers.* Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, LA.

Toluk, Z. (1999). *The development of childrens' understanding of the quotient subconstruct of rational number.* Arizona State University. Unpublished doctoral dissertation.

ARITHMETIC THINKING STRATEGIES AND LOW ACHIEVING JUNIOR HIGH SCHOOL STUDENTS IN AUSTRALIA

Bob Perry
Peter Howard
University of Western Sydney
Australian Catholic University, Australia
b.perry@uws.edu.au

Abstract: This paper reports on a systemic mathematics initiative—*Counting On*—which targets low achieving students in government junior high schools in New South Wales (NSW), Australia. In NSW, many students enter secondary school with inefficient methods for calculation, often relying solely on schemes such as 'counting by ones'. The *Counting On* program aims to extend the range of arithmetic thinking strategies used by these students to include much more efficient 'composite' strategies, based on a thorough understanding of place value—a trip across what Gray, Pitta and Tall (1997) have dubbed the 'proceptual divide'. This paper reports on an extensive evaluation of the *Counting On* program, particularly in terms of changes in the students' arithmetical thinking strategies.

Introduction

In NSW, Australia, children start school between the ages of 4. 5 and 5.5 years and complete seven years of elementary school (grades Kindergarten to Year 6). (Junior) high school commences with Year 7, with the students usually aged between 11.5 and 12.5 years. Government schooling throughout NSW is run by the Department of Education and Training, which has responsibility for approximately 1650 elementary schools and 400 high schools.

Since 1992, work on the development of the mathematics skills and knowledge of low achieving elementary school students has been a systemic priority in NSW government schools. This work was originally undertaken by Wright and his team (see, e.g., Wright, Stanger, Cowper & Dyson, 1996) and was specifically aimed at students in the early years of elementary school. In the ensuing years, the program—*Count Me In Too*—has been extended throughout the elementary grades and adopted in other countries, including the USA and United Kingdom.

The *Counting On* program which is the subject of this paper, is an extension of *Count Me In Too* and is aimed at assisting mathematics teachers address the arithmetic needs of low achieving students in the first year of junior high school—Year 7. It was first introduced into NSW junior high schools in 1999, when a trial was undertaken in four schools. In 2000, the program was extended to 40 junior high schools across the state, involving more than 600 students, 120 teachers and 40 district mathematics consultants.

Background to *Counting On*

The *Counting On* program focuses on the professional development of teachers in identifying and addressing the students' learning needs and relies on the notion that improved teacher knowledge will result in improved student learning outcomes. It operates on a team approach involving, in each school, the Head Teacher, Mathematics, a Year 7 Mathematics Teacher, a Support Teacher Learning Difficulties and the District Mathematics Consultant.

The research base for the program is provided through the *Counting On* Numeracy Framework (Thomas, 1999) which is an extension of work by Cobb and Wheatley (1988) and Jones et al. (1996) and relates to the *Count Me In Too* Learning Framework in Number (Wright, 1998; Wright, Martland, & Stafford, 2000). The teaching program in *Counting On* concentrates on assisting students to move from unitary to composite-based mental strategies in the four operations on whole numbers. It is related to two sequences of development, with seven levels for place value development (including addition and subtraction) and six for multiplication and division (see Table 1).

Evaluation Methodology

The evaluation of *Counting On* in 2000 used a combination of quantitative and qualitative methods of inquiry. In this paper, we concentrate on the major data collection strategy consisting of the pre- and post-test implementation of a specifically designed 19 item instrument. The teaching teams had been instructed on its use during the program's professional development days. On each occasion, each student was individually interviewed using this instrument by a member of the school's *Counting On* team. All 'interviews' were videotaped and the *Counting On* team from each school, together, analysed the student's responses, not only in terms of the levels in the framework at which the student appeared to be performing but also in terms of the strategies which the student was using. The initial testing of students occurred

Table 1. Numeracy Framework Levels of Conceptual Development in Place Value and Multiplication and Division

Place value		Multiplication and division	
Level	**Descriptor**	**Level**	**Descriptor**
0	Ten as count	0	Unable to form equal groups
1	Ten as unit	1	Forming equal groups
2	Tens and ones	2	Perceptual multiples
3	Hundred as unit	3	Figurative units
4	Hundreds, tens, & ones	4	Repeated abstract composite units
5	Decimal place value	5	Multiplication and division as operations
6	System place value		

early in second term, 2000 (April/May), with post-testing at the end of third term (August/September). The first implementation of the instrument was completed by 671 students, with data for 544 students being received from the second implementation. (Data from one school were not available. As well, there was a number of students who refused to be involved in the second assessment or were absent when it was implemented.) The test instrument and its associated scoring rubric had been piloted in 1999 and was developed in conjunction with the Numeracy Framework. The *Counting On* teaching program was implemented in each school between the two 'interview' periods. For the most part, this program was the only arithmetic instruction received by the students during this time.

For each student on each of the 19 questions, the *Counting On* teams ascribed a level of student response based on the strategy used by the student. Following the analysis of each question for each student, an overall level derived from the learning framework was given by the team for place value, and multiplication and division. This was done through discussion and debate in each team, usually led by the Head Teacher, Mathematics and/or the District Mathematics Consultant. Hence, for the 544 students who completed both T1 and T2, this resulted in two levels for place value, and two for multiplication and division. These levels reflect the types of strategies used by students to solve problems in each of these areas. For further details of these framework levels see Mulligan and Mitchelmore (1996), NSW Department of Education and Training (2000) and Wright, et al. (2000).

Results

The spoken script for each of the 19 questions, along with the percentage of the cohort undertaking each test who obtained a correct answer (T1: initial interview; T2: second implementation), are given in Table 2. It should be noted that each question was usually accompanied with the possibility of the students using manipulatives or illustrations to arrive at an answer.

For each of T1 and T2, each student was assigned a place value level (from 0 to 4 only as only whole number place value was assessed) and a level for multiplication and division (from 0 to 5). These were derived from the Numeracy Framework. Tables 3 and 4 show the distribution of these levels for each of T1 and T2, while Figures 1 and 2 provide a graphical representation of these distributions.

Discussion

Table 3 and Figure 1 show clearly that there has been an overall increase in the place value levels with, for example, 31% of the cohort being at Level 0 in T1 and only 8% in T2, while only 9% of the cohort was at Level 3 in T1 and 22% in T2. Further analysis can track individual growth across these levels. Figure 3 shows the difference between the level ascribed for each student in T1 and the corresponding level in T2. Small percentages of students have fallen back in terms of their place value levels and many have remained at the same level. However, the overriding feature is the large

Table 2. Script for Each of the 19 Questions Plus Percentage Correct on T1 and T2

Question	Correct T1 (%)	Correct T2 (%)
1. Imagine that there are 7 counters under here and 5 counters under here. How many counters are there altogether?	89	93
2. Imagine that there are 17 counters under here. I have some more counters under here. Altogether there are 21 counters. How many counters are under here?	83	90
3. Can you tell me two numbers that add up to 19?	76	87
4. Can you tell me how many squares are on this card? (24) Can you circle the number of squares that this part of the number (2 in 24) stands for?	53	65
5. What does 37 and 25 add up to?	77	80
6. What does 53 take away 27 equal?	25	28
7. Put these numbers (4680, 8640, 6480, 6840) in order, from smallest to largest.	86	89
8. Show card with 2 grids (of 100 dots), 4 strips (of 10 dots) and a 5 strip of dots. How many dots are there altogether?	92	94
9. Using card of ten strips, uncover more and more dots, asking 'how many are there altogether now?'.	87	90
10. What is the number which is 10 more than this number (1593)?	50	58
11. I have 18 dots here and there are some more dots hidden over here. There are 64 dots altogether. How many dots are hidden?	31	35
12. If you had 621 counters, how many groups of 10 could you make?	25	33
13. Here are 7 rows of 5 dots. How many dots are there altogether?	89	94
14. Here is a row with 5 dots. There are 6 rows exactly the same as this on the page. Some dots are hidden by this card. How many dots are there altogether on this page?	85	89
15. Altogether there are five rows like this one on this page. Some rows are hidden. How many dots are there altogether on this page?	80	86
16. There are 12 dots altogether on this page. There are the same number of dots in each row. How many rows are there altogether on this page?	75	86
17. There are 28 cards placed in 4 equal rows. How many cards are in each row?	56	61
18. Under this cover there are 6 plates, and under each plate there are 3 counters. How many counters are there altogether?	84	87
19. These 12 students need to sit at tables. There must be 4 of these students at each table. How many tables will I need?	93	95

Table 3. Percentages of Students in Each Place Value Level—T1 and T2

Level 0		**Level 1**		**Level 2**		**Level 3**		**Level 4**	
T1	T2	T1	T2	T1	T2	T1	T2	T1	T2
31	8	36	36	24	30	9	22	1	4

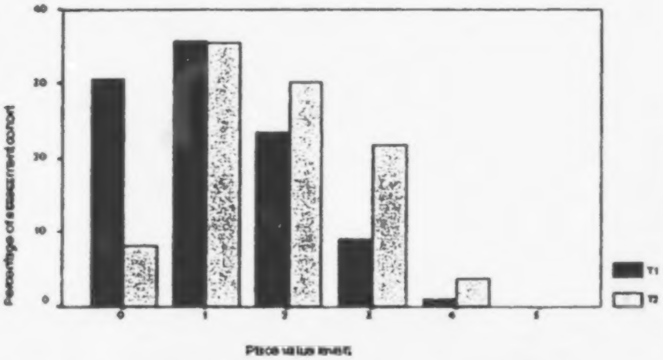


Figure 1. Percentage of students in each place value level—T1 and T2.

percentage increase of at least one level, with more than 15% increasing by at least 2 levels.

A paired sample t-test showed that the increases in level were highly significant for the overall cohort (t=13.96, p<0.001).

There was a general increase in multiplication/division levels from T1 to T2. This can be seen clearly in Table 4 and Figure 2 above. For example, 24% of the T1 cohort were at Level 2 while only 14% of the T2 cohort were at this level and 21% of the T1 cohort were at Level 5 but this was increased to 32% in T2. Figure 4 shows the results of tracking individual growth across these levels. Clearly, small percentages of students have fallen back in terms of their multiplication / division levels. However, there is a large percentage of students who have maintained their level and a substantial proportion who have lifted their performance by at least one level. A paired sample t-test showed that the increases in level were highly significant for the overall cohort (t=9.72, p<0.001).

Conclusion

It is clear from Figures 3 and 4 that the *Counting On* program—given that it was almost the only arithmetic instruction carried out with these students in the period between T1 and T2—has had a positive effect in terms of the growth in levels in both

Table 4. Percentages of Students in Each Multiplication/Division Level—T1 and T2

Level 0		Level 1		Level 2		Level 3		Level 4		Level 5	
T1	T2	T1	T2	T1	T2	T1	T2	T1	T2	T1	T2
2	0	11	5	24	14	26	28	16	21	21	32

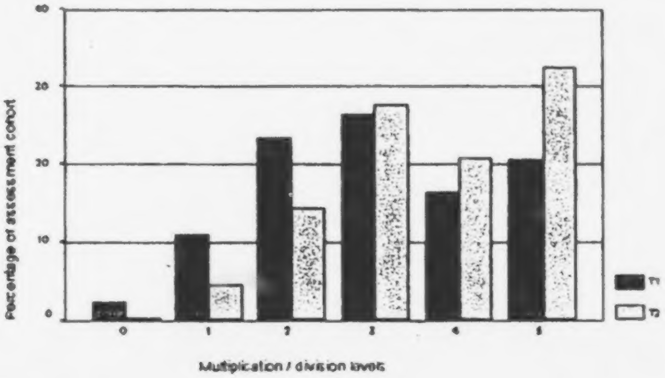


Figure 2. Percentage of students in each multiplication/division level—T1 and T2.

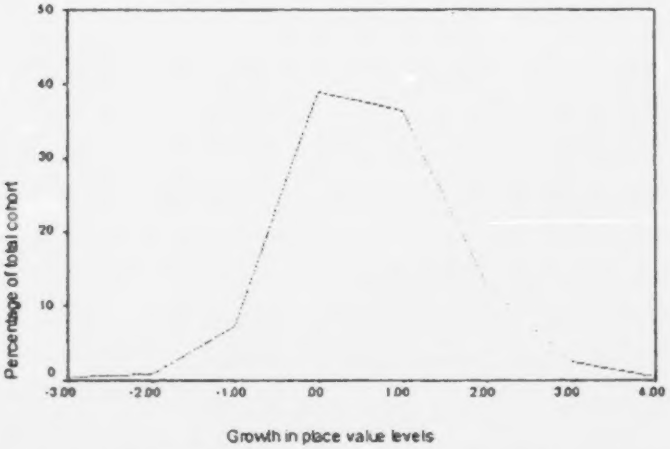


Figure 3. Difference graph showing growth in place value levels from T1 to T2.

place value and multiplication/division. There is some concern about the students who seem to have fallen back in terms of these levels. There are many possible reasons for this. However, the relatively small percentages involved suggest that it may simply be a natural regression caused by issues such as health or developing analysis skills of the teaching teams.

As important as getting the correct answer to a mathematical question might be—at least in the *Counting On* program—is the strategy which is used to get the answer. In particular, the activities introduced through *Counting On* were designed to facilitate a move by the students from the unitary counting strategies which are found among many lower performing students at the beginning of high school to the coordination of equal groups and the use of composite units (composite strategies). Figure 5 shows the change in use of composite strategies for all questions where this can be measured. Clearly, there has been an increase in such use for each of these questions.

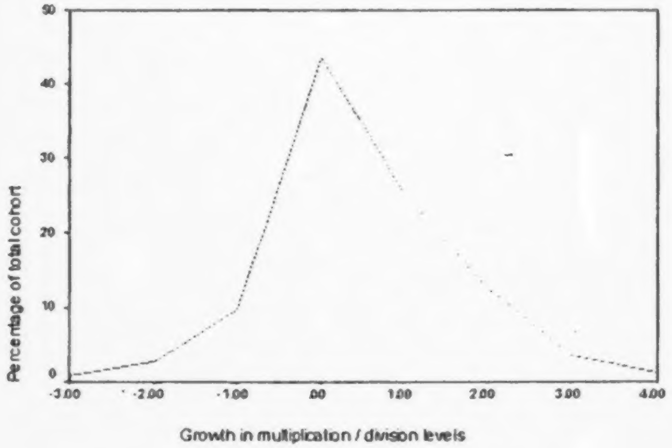


Figure 4. Difference graph showing growth in multiplication / division levels from T1 to T2.

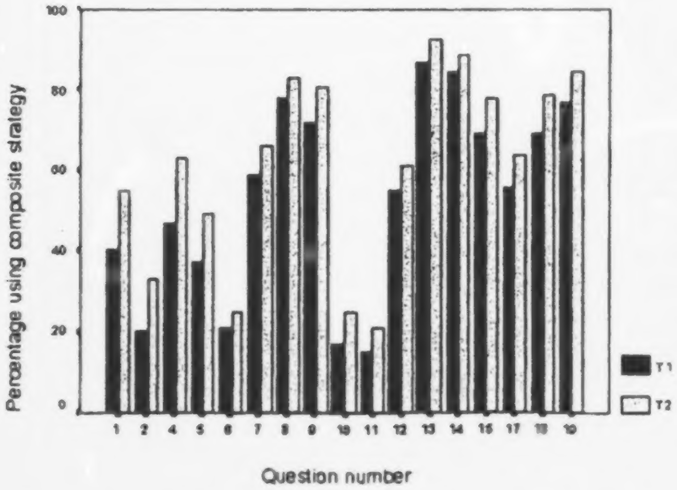


Figure 5. Increase from T1 to T2 in use of composite strategies in all questions where this can be measured.

One of the key aims of the *Counting On* program is "to assist the movement of students from unitary to composite-based mental strategies, specifically, on building the four operations and place value through grouping" (NSW Department of Education and Training, 2000, *The learning framework*, p. 1). Given that such distinctions between strategies are contained in the varying levels of the learning framework in place value and multiplication and division, with higher levels reflecting more composite-based strategies, it would appear that the *Counting On* program in 2000 has achieved this aim. With a relatively short period of intervention and focussed teaching activities, the students have generally advanced on the learning framework, as assessed in the pre- and post-tests. The increases in levels are statistically significant and pedagogically important. For the first time in many years, students were beginning to enjoy their mathematics as they found some level of success.

References

Cobb, P., & Wheatley, G. (1988). Children's initial understandings of ten. *Focus on Learning Problems in Mathematics, 10* (3), 1-28.

Gray, E., Pitta, D., & Tall, D. (1997). The nature of the object as an integral component of numerical processes. In E. Pehkonen (Ed.), *Proceedings of the Twenty-First Conference of the International Group for the Psychology of Mathematics Education, (Vol. 1,* pp. 115-130). Lahti, Finland: PME.

Jones, G. A., Thornton, C. A., Putt, I. J., Hill, K. M., Mogill, T. A., Rich, B. S., & van Zoest, L. R. (1996). Multidigit number sense: A framework for instruction and assessment. *Journal for Research in Mathematics Education, 27* (3), 310-336.

Mulligan, J. M., & Mitchelmore, M. C. (Eds.). (1996). *Children's number learning.* Adelaide: Australian Association of Mathematics Teachers.

NSW Department of Education and Training. (2000). *Counting on 2000*. Sydney: Author.

Thomas, N. (1999). Levels of conceptual development in place value. *The pilot Counting On numeracy project.* Sydney: NSW Department of Education and Training.

Wright, R. J. (1998). An overview of a research-based framework for assessing and teaching early number. In C. Kanes, M. Goos, & E. Warren (Eds.), *Teaching mathematics in new times* (pp. 701-708). Brisbane: Mathematics Education Research Group of Australasia.

Wright, R. J., Martland, J. R., & Stafford, A. (2000). *Early numeracy: Assessment for teaching and intervention.* London: Sage/Paul Chapman.

Wright, R., Stanger, G., Cowper, M., & Dyson, R. (1996). First-graders' progress in an experimental mathematics recovery program. In J. T. Mulligan & M. C. Mitchelmore (Eds.), *Children's number learning* (pp. 55-72). Adelaide, Australia: Australian Association of Mathematics Teachers.

CREATING THE SPACE OF SOLUTION FUNCTIONS TO DIFFERENTIAL EQUATIONS[1]

Chris Rasmussen
Purdue University Calumet
raz@calumet.purdue.edu

Michelle Stephan
Purdue University Calumet
stephan@calumet.purdue.edu

Abstract: The purpose of this report is to investigate a foundational aspect of students' understanding of solution functions to differential equations. In particular, we examine processes by which students create a structured space of solution functions rather than on their use of particular methods for finding solutions. Our research reflects and furthers theoretical advances regarding representations and current interests in social aspects of mathematical learning.

Introduction

Advances in technology and mathematicians' evolving interests in dynamical systems are currently prompting changes to the first course in differential equations. Traditional approaches in differential equations emphasize analytic techniques for finding closed form expressions for solution functions whereas current reform efforts are emphasizing graphical and numerical approaches for analyzing and understanding the behavior of solution functions. An emerging body of research on student learning in differential equations (e.g., Rasmussen, in press; Zandieh & McDonald, 1999) highlights several aspects of student reasoning surrounding the notion that solutions to differential equations are collections of functions.

The purpose of this research is to investigate a foundational aspect of understanding the space of solution functions to differential equations in a graphical setting. We view this research as foundational because our analysis focuses on the processes by which students *create* a structured collection of solution functions rather than on students' *use* of particular graphical or numerical methods for finding solution functions (c.f., Habre, 2000). Creating a structured space of solution functions is mathematically significant since such a structure is a basis for interpreting the impact that varying parameters might have on the evolution of a quantity of interest.

Our analysis is significant in two respects. First, we treat slope fields and graphs of solution functions as inscriptions that potentially emerge out of students' reasoning. This perspective builds on recent theoretical advances in symbolizing (Cobb, Yackel, & McClain, 2000) and differs from typical reform-oriented approaches in differential equations. For example, Smith (1998) posits that after drawing a slope field, "the meaning of a solution is then clear: We seek a function whose graph 'fits' the slope field" (p. 12). This stance assumes a degree of transparency (Meira, 1995) of the slope field, a degree of linearity in conceptual development—slope fields then solution functions, and it assumes that students unproblematically conceptualize solutions to differential equations as functions. Our theoretical orientation holds these assumptions in question. Second, in this report we add to a growing interest in documenting the collective aspects of learning mathematics (Wenger, 1998). In particular, we expand a methodology for determining the emergence of classroom mathematical practices (Cobb, Stephan, McClain, Gravemeijer, 2001).

Methodology

We conducted a 15 week classroom teaching experiment (Cobb, 2000) in an introductory course in differential equations primarily for engineers at a mid-sized university in the United States. Data sources include videorecordings of every class session, videorecorded interviews with students, copies of all written work, instructor's journal, researcher fieldnotes, and audiorecordings of project meetings and debriefing sessions. In terms of data analysis, we build on Stephan, Bower, Cobb, and Gravemeijer's (in review) and Yackel's (1997) methodological approach of analyzing argumentation in order to develop criteria for claiming that a community of learners' ways of reasoning (i.e., classroom mathematical practices) become taken as shared. Both Stephan et. al. and Yackel modified Toulmin's model of argumentation (1969) to develop criteria for determining when a practice, from the researcher's perspective, has become established within the classroom community. Figure 1 illustrates that for Toulmin, an argu-

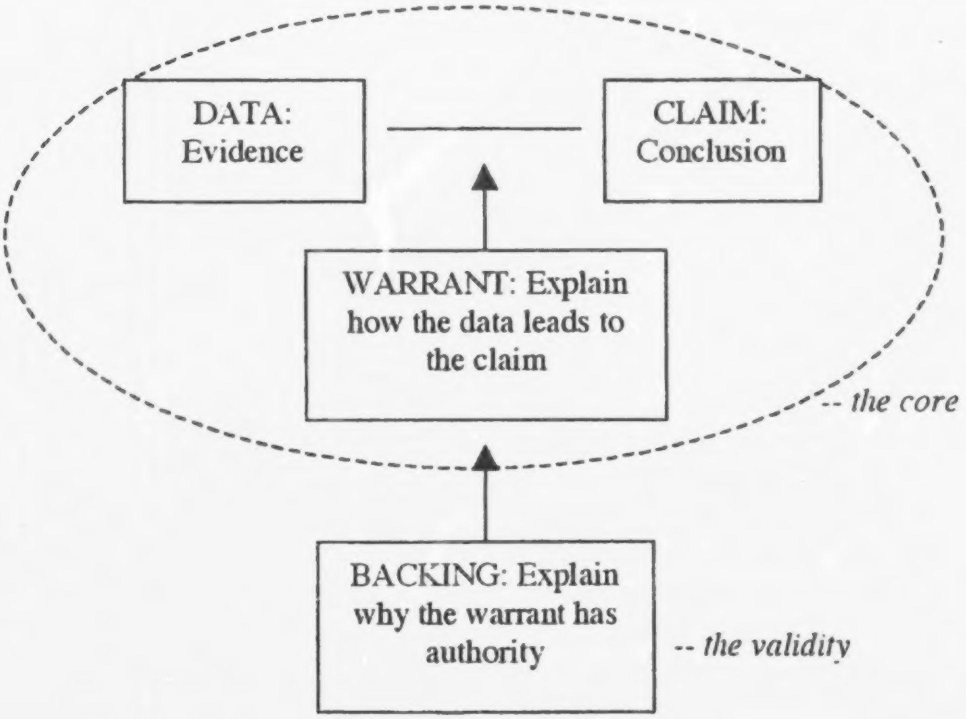

DATA: Evidence
CLAIM: Conclusion
WARRANT: Explain how the data leads to the claim
-- the core
BACKING: Explain why the warrant has authority
-- the validity

Figure 1. Toulmin's model of argumentation,

ment consists of at least three parts, called the core of an argument: the data, claim, and warrant.

In any argumentation the speaker makes a claim and if challenged, can present evidence or data to support that claim. The data typically consist of facts that lead to the conclusion that is made. Even so, a listener may not understand what the particular data presented has to do with the conclusion that was drawn. In fact, she may challenge the presenter to clarify why the data lead to the conclusion. When this type of challenge is made and a presenter clarifies the role of the data in making her claim, the presenter is providing a warrant. Another type of challenge can be made to an argument. Perhaps the listener understands why the data supports the conclusion but does not agree with the content of the warrant used. In other words, the authority of the warrant can be challenged and the presenter must provide a backing to justify why the warrant, and therefore the core of the argument, is valid.

We contend that mathematical practices become taken-as-shared when either 1) the backings for an argumentation no longer appear in students' explanations and therefore stands as self-evident, or 2) the warrants and/or backings for one argumentation become the data or warrant for subsequent, more sophisticated arguments. When either of these instances occurs and no member of the community rejects the argumentation, a classroom mathematical practice has become established.

Our instructional planning about students' conceptual development was informed by research on students' learning about rate of change in general and about differential equations in particular (Thompson, 1994; Rasmussen, 1999; in press) and by our understandings of the Realistic Mathematics Education instructional design theory (Gravemeijer, 1999). Our analysis of the psychological processes involved in understanding of the space of solution functions served as background for the current research, which focuses on the social processes that fostered student's mathematical growth. Cobb and Yackel's (1996) interpretive framework served as our theoretical basis for coordinating individual student's conceptions and classroom mathematical practices.

Results and Discussion

Our analysis of the data corpus suggests students' creation of a structured space of solution functions can be understood in terms of what we call "the structuring the space of solution functions practice." This classroom mathematical practice entails the following two co-emergent aspects: taken-as-shared ways of reasoning about patterns in slope fields for autonomous differential equations (i.e., differential equations that do not depend on time) and taken-as-shared ways of reasoning about the implication of these patterns for graphs of solution functions. In this section we elaborate the processes by which this classroom mathematical practice emerged and became established in the classroom.

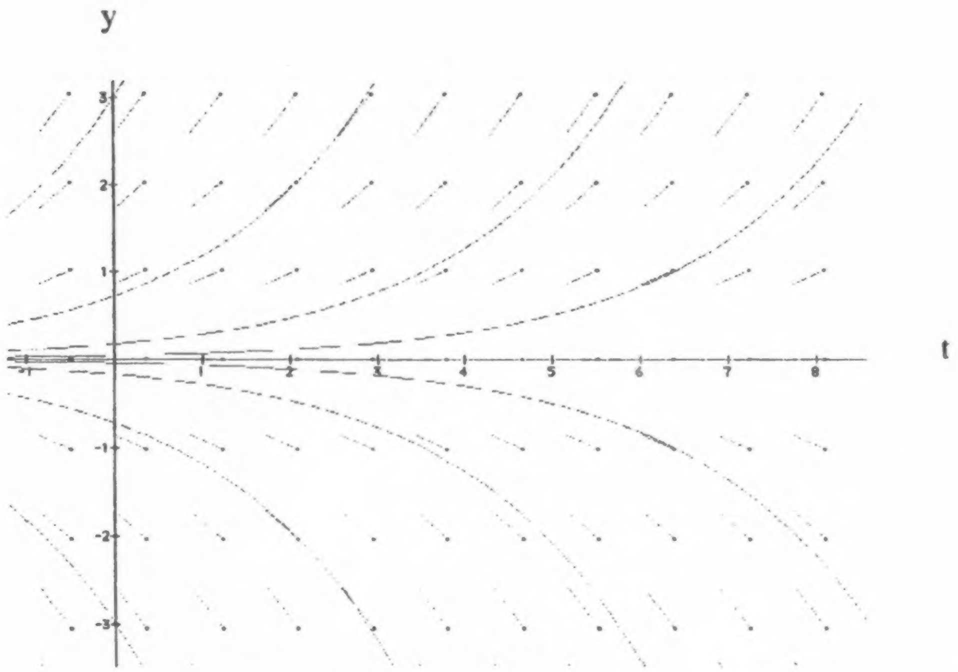


Figure 2. Slope field and representative solution graphs for dy/dt = 0.5y.

From an expert's perspective, a slope field for an autonomous differential equation has slope marks that are the same for any fixed value of the dependent quantity. As a result, the graphs of solution functions are shifts of each other along the axis of the independent quantity. For example, Figure 2 shows a slope field and several graphs of solution functions to the autonomous differential equation dy/dt = 0.5y. The reader already familiar with differential equations will most likely "see" the invariance in the slope marks and the way in which graphs of solution functions are structured.

In contrast to conventional approaches where such structuring and organizing might be explained and illustrated by the instructor in a linear progression, we found that the two aspects to the structuring the space of solution function practice actually co-emerged as a result of *students'* mathematizing activities. In the paragraphs that follow, we illustrate the co-emergence and constitution of the two aspects of this practice by analyzing students' argumentations in whole class discussions on the 2nd, 4th, 9th, and 12th days of class.

On the 2nd day of class students made predictions about the growth of a rabbit population over time under the assumptions that there are no predators, resources are unlimited, and the rabbits reproduce continuously (no differential equation was provided). After some discussion students agreed that if there were initially 10 rabbits (where 10 is scaled for say, 10,000) then a graph of the number of rabbits versus time would look like that shown in Figure 3a. The instructor then asked the class, "What if the starting data were 20 rabbits?"

[1] Arthur: It would go up quicker, it would increase quicker.

[2] Instr: It would increase quicker? Say a bit more about that.

[3] Arthur: There's more rabbits and there would be more babies. It's like further up the 10 curve.

[4] Randy: Just look at the graph you already have. When you're at 20 it's already a more steep increase.

[5] Arthur: Yeah, it would be like the same thing.

As a pedagogical tool, the instructor recorded students' reasoning with slope mark notation as shown in Figure 3b. These markings are crucial here as they are introduced as a way to notate students' reasoning and are meant to promote students' reinvention of the idea of a slope field rather than beginning with a complete slope field constructed by the instructor.

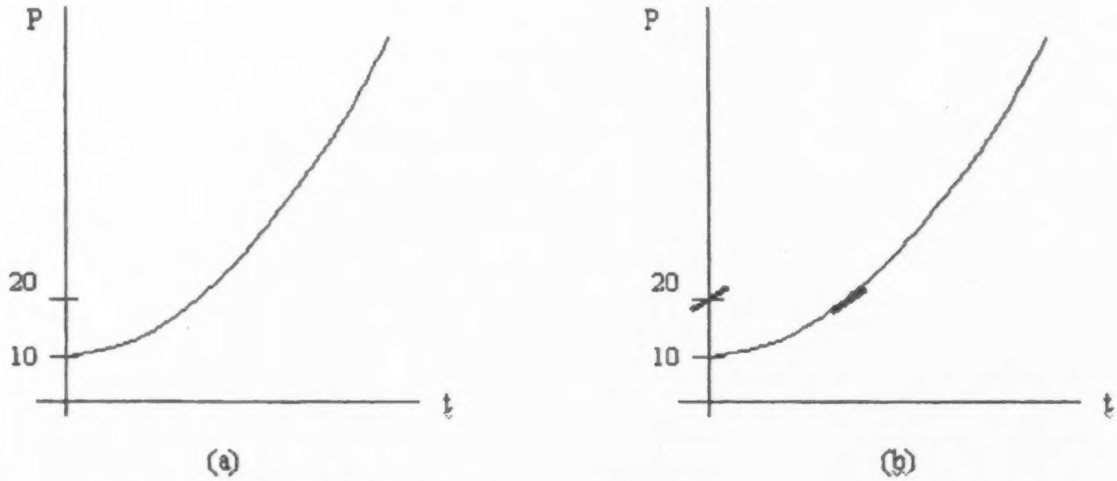


Figure 3. Growth of a rabbit population over time.

In this episode, Arthur and Randy concurrently initiated reasoning about patterns in rate of change and patterns in graphs of solution functions. First, Arthur claims that it [the slope at initial condition 20] would increase quicker. In response to the teacher's prompt to be more specific, Arthur provides some data to back up his claim. He elaborates that there would be more rabbits, hence more babies at a starting point of 20, and then states that "it's further up the 10 curve." Arthur's last statement, "further up the 10 curve" is, for Arthur, data that supports his claim that the slope increases quicker. However, one might ask what "it's further up the 10 curve" has to do with the claim being made. An elaboration of that issue would constitute a warrant that explains what

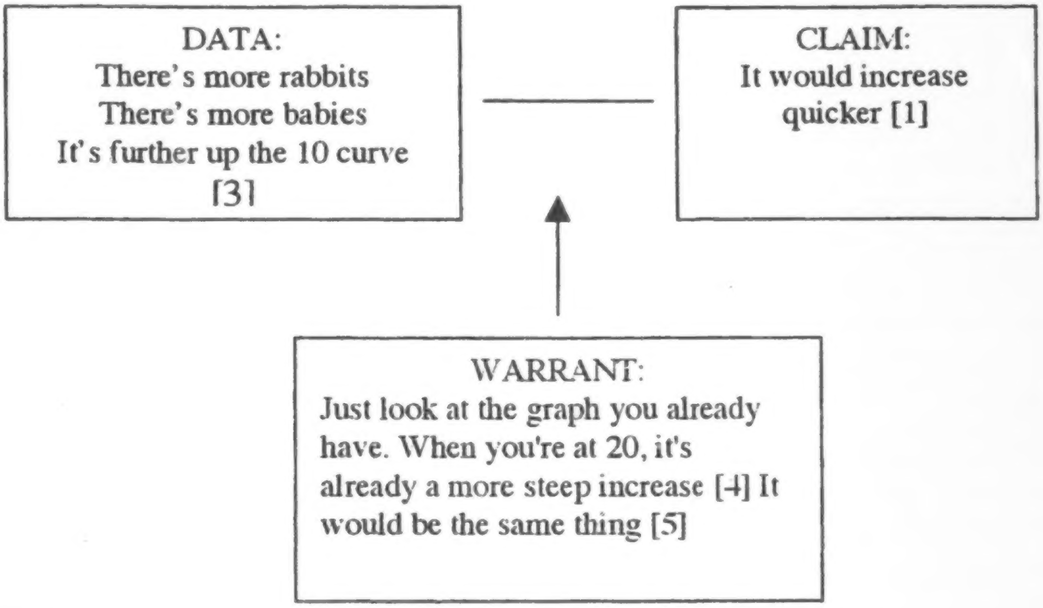

DATA:
There's more rabbits
There's more babies
It's further up the 10 curve
[3]
CLAIM:
It would increase
quicker [1]
WARRANT:
Just look at the graph you already
have. When you're at 20, it's
already a more steep increase [4] It
would be the same thing [5]

Figure 4. Structure of Arthur and Randy's argument.

Arthur's data has to do with the claim he made. The teacher, in this instance, did not need to push Arthur to articulate the warrant for this argument because Randy clarifies the data without prompting in line 4. More specifically, Randy explains that one can look at the graph already present (the 10 curve) and see a more steep increase at P=20 on that curve. Implicitly he is saying that the slope at P=20 on the 10 curve is similar to the slope at P=20 on the 20 curve. Arthur further elaborates the warrant by explaining that the slopes are the same (line 5).

Using Toulmin's model of argumentation, the argument that emerges in this instance can be diagrammed as in Figure 4. Although no backing was present in this argumentation, one might ask why the slopes at P=20 should be the same for two different starting points. In other words, the legitimacy of the warrant can be called into question. The backing in this case might consist of the fact that these are solutions to an autonomous rate of change equation. Although this type of backing did not emerge in this episode, it was brought in during subsequent discussions. We claim that this episode illustrates the first instance in which the classroom mathematical practice of structuring the space of solution functions emerged. As we will see, the math practice becomes established during subsequent episodes in which students continue to reason about the relation between slopes at particular populations at various times.

During a whole class discussion on day 4, students discussed their reasoning on a task where they were given a partially filled out slope field for the differential equation $dP/dt = 3P(1 - P/100)$ and asked to investigate graphs of solution functions corresponding to different starting scenarios. In support of their ideas, students generated argumentations that suggested that the emergence of patterns in slope fields and the implication of these patterns for graphs of solution functions emerged concurrently. For example, Joaquin argues for patterns in the slope field as follows.

> Joaquin: When we started at 10 rabbits (points to population of 10 at time zero on his drawing), and we got to, say the three years or whatever that it went by and we finally got to 30 rabbits (draws in slope mark at population 30 on the solution corresponding to the initial condition of 10), even though we started off with 30 rabbits over here (gestures to the slope at initial condition 30), it had the same slope as that 10 did at time like two years.

Joaquin's explanation is similar to the previous contributions on the second day in that he concludes that one can merely take the slope of the tangent line at P=30 on the 10 curve and indicate the same slope at P=30 for t=0. The support for this is that the slopes will be the same. At this point in the classroom such support was beyond justification; no one challenged the fact that the slopes were the same. Building on Joaquin's reasoning, Arthur contributes to the co-emergence of both aspects of the structuring the space of the solution function practice by arguing that since the slopes are the same for a set value of the dependent variable, *all* graphs of solution functions are shifts of each other along the t-axis.

> Arthur: Another thing we found out that like since all the graphs, slopes are the same, it's just like you're sliding this whole graph over one. Like going over here, towards 15 (gestures to grab the curve for the initial population just above 0) it's like the exact same thing, if you slide it over one more time you get the 30 graph, another time, you get the 45, so if you know the graph, you can kinda predict what happened in the past, a little bit before your time zero cause the graph is the same for all of them, you just pop it back for whatever your time interval was between the different 15 and 30 populations.

In his argumentation, Arthur claims that one can shift not merely individual slopes from one curve to another, but rather the solution function itself. Using Toulmin's model, the data for this argument is that all the slopes are the same. The collective argumentation has evolved from copying individual slopes to shifting the entire solution function itself. In other words, shifting slopes point by point, since the slopes are the same at those points, has dropped out of the argumentations and has been replaced by shifting entire solution curves. Methodologically speaking, we can see that the warrant from the previous argumentations [the slopes are the same] has shifted to serve as data for a more complex argument [shifting functions, not just slopes]. This gives further credence to our claims that shifting slopes along the t axis was taken-as-shared in this mathematical practice.

During whole class discussions on day 9 and day 12, students give arguments where the two aspects of the structuring the space of solution function practice are used as evidence for their conclusions. In terms of argumentation, there is a shift in the role played by the statements involving these ideas. This shift in role provides empirical support for the claim that the structuring the space of solution functions practice was taken-as-shared. For example, on day 9 the students debate whether a certain graph of an exact solution function to a particular differential equation is "steeper" than the graph of a certain Euler method approximation. In arguing their positions, students use slope field patterns and implications of these patterns as support for their conclusions. On day 12 students discuss whether the graphs of two solution functions to a Newton's Law of cooling differential equation will ever touch each other. For example, Paul argues they would not as follows.

> Paul: I kinda thought of it like this, well we know from some of the stuff that we've already done, those two graphs, same graphs just shifted, shifted along the horizontal by however many time increments. So I guess you could kinda think of it as like they're parallel at every point. Parallel or they have a parallel tangent at like every point because those slope lines are the same across whatever point you're at along the y, I guess. So if the graphs are just shifted over from each other, why would they ever touch? If they are indeed the same graph.

In Paul's argumentation, he uses the idea that graphs of solution functions are shifts of each other to draw a new mathematical conclusion (that they never touch). This is significant in that he uses claims that were debated previously ("we know from stuff that we've already done") to draw new conclusions. Further, no one in class rejected his explanation and during subsequent class periods, this type of reasoning was beyond justification. This fits our evidentiary criteria for establishing the structuring the space of solution functions practice. This mathematical practice arises from "re-inventing" the slope field and its associated meaning rather than decoding the meaning of a fully formed slope field from the beginning.

In conclusion, the research reported here has both practical and theoretical implications. Practically, our research illustrates that when students are engaged in instruction that supports re-inventing conventional representations out of mathematizing experiences, slope fields and graphs of solution functions can and do emerge from their mathematical activity. Further, this research indicates that reasoning about patterns in slope fields and the implications of these patterns for graphs of solution functions emerges simultaneously in students' argumentations, not linearly as previously thought. Theoretically, our research furthers methodological approaches that analyze communal aspects of learning. In particular, we have used Toulmin's model of argumentation to analyze the emergence of classroom mathematical practices in a differential equations context. Additionally, we have identified a new criterion for determining when mathematical practices are taken-as-shared, namely that the supporting evidence in an argumentation shifts roles [from warrant and backings to data and warrants] in subsequent, more sophisticated arguments. These practical and theoretical findings add to the growing research base in both the teaching and learning of differential equations and methodological aspects of documenting learning in social context.

References

Cobb, P. (2000). Conducting classroom teaching experiments in collaboration with teachers. In R. Lesh & E. Kelly (Eds.), *New methodologies in mathematics and science education*. Mahwah, NJ: Erlbaum.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *Journal of Learning Sciences, 10* (1/2), 113-163.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31* (3/4), 175-190.

Cobb, P., Yackel. E., & McClain, K. (Eds.). (2000). *Symbolizing and communicating in mathematics classrooms*. Mahwah, NJ: Erlbaum.

Habre, S. (2000). Exploring students' strategies to solve ordinary differential equations in a reformed setting. *Journal of Mathematical Behavior, 18* (4), 455-472.

Meira, L. (1995). The microevolution of mathematical representations in children's activity. *Cognition and Instruction, 13* (2), 269-313.

Rasmussen, C. (1999, April). *Symbolizing and unitizing in support of students' mathematical growth in differential equations.* Paper presented at the 1999 NCTM Research Presession, San Francisco, CA.

Rasmussen, C. (in press). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior*.

Smith, D. (1998, August). *Renewal in collegiate mathematics education.* Paper presented at the International Congress of Mathematicians, Berlin, Germany.

Stephan, M., Bowers, J., Cobb, P., & Gravemeijer, K. (under review). *Supporting students' development of measuring conceptions: Analyzing students' learning in social context.* Monograph under review.

Thompson, P.W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179-234). Albany, NY: SUNY Press.

Toulmin, S. (1969). *The uses of argument.* Cambridge: Cambridge University Press.

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity.* Cambridge: Cambridge University Press.

Yackel, E. (1997, April). *Explanation as an interactive accomplishment: A case study of one second-grade mathematics classroom.* Paper presented at the annual meeting of the American Educational Research Association, Chicago.

Zandieh, M., & McDonald, M. (1999). Student understanding of equilibrium solution in differential equations. In F. Hitt & M. Santos (Eds.), *Proceedings of the Twenty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 253-258). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Note

[1] The research reported here was supported by the National Science Foundation under grant No. REC-9875388. The opinions expressed do not necessarily reflect the views of the foundation.

A TEACHING PROPOSAL ABOUT RATIO AND PROPORTION WORKED WITH STUDENTS OF ELEMENTARY SCHOOL

Elena Fabiola Ruiz Ledesma
CINVESTAV-IPN. MÉXICO
leslieruiz2000@hotmail.com

Marta Valdemoros Álvarez
CINVESTAV-IPN. MÉXICO
mvaldem@mail.cinvestav.mx

Abstract: The present document is the report of research that deals with the topics of ratio and proportion whose great importance nowadays has been shown on the studies that the researchers of different countries have made in different decades. A reflection of the matter is done, its importance and effect are justified and the objectives are set. The theoretical and empirical antecedents of the matter are shown as well as the didactic and psychopedagogic expositions. In this way the importance of the cognitive aspects of a disciplinarian development and the importance of teaching the topics of ratio and proportion are taken into account. The teaching proposal that was practiced with a group of Mexican students of 6th grade of elementary education is presented. The elaboration of different tasks shows a progression, starting with the review of qualitative aspects for reaching the quantitative because the recognition of ratio as a relationship between quantities and proportion as the relationship between ratio was focused.

Introduction

An explanation about the problem of research

The problem of research that is reported consists in reviewing the strategies used by 6th grade students solving problems of simple and direct ratio and proportion and to determine how they structure their answers in problematic situations, which is the base of the design and the application of a teaching proposal about the topics that were previously mentioned, that was related to its elaboration and that was adapted to the school's program.

Justification of the problem

There are many reasons why this investigation was done. They are summarized as following:

1. In the curricular level there are themes introduced for the first time in the elementary school and the success achieved in their learning process lets the student progress in the understanding of concepts with which he will work in the following school levels. This is the case of the topics of ratio and proportion which teaching and learning starts in the elementary school and constitutes a base for learning fundamental concepts.
2. The experience lived by the author of this research, as teacher and counselor in the levels of elementary, secondary and high school education have let her

verify that teachers and students have difficulty solving situations that involve the concepts of ratio and proportion.

3. A previous study[1] with an exploratory character that Ruiz (1997) show, where they worked with 6th grade student and with teachers of the basic and medium levels, which showed that they could easily solve problems of lost value but not other kinds of problems that include ratio and proportion allowed to confirm the use of the simple rule of three as the only tool in its solution; in the case of teachers the use of this algorithm was successful but it did not happen with the students, furthermore there was no manifestation of the understanding of its use.

This previous study allowed us to see the transcendence of the school instruction submitted to a certain kind of courses through the use of an algorithm (the rule of three) in the solution of problems of lost value. It is shown that if teaching does not introduce the rule of three, this does not appear by itself.

It was also confirmed, in practice, the little importance that it has in the education committed choice of ratio and proportion. The 6th grade program includes the simple proportional variation in the majority of the parts that conforms it.

General objectives of the study

- To explore the strategies that a student uses to solve problems of ratio and proportion in order to recognize qualitative and quantitative components of the thinking linked to this topics in its different kinds of representation.
- To develop a teaching proposal that allows us to recuperate and enrich the strategies used by the student to face problems of ratio and proportion.

Theoretical Framework

After making a bibliographic review the relevant points of the study showed in this report were taken into account because they were compatible with it. It was found that many of the research made in the field of proportionality were focused in cognitive aspects of the student, as in case of Piaget and Inhelder (1972, 1978) and of the majority of the researchers of the 70´s and 80´s. Having reviewed the cognitive aspect as a part of a disciplinary development, it is very important for the teaching proposal that it is shown in this article, but at the same time, it was fundamental to have a support, psychopedagogic as well as didactic, taking from Coll (1998) and Freudenthal (1983) and Streefland (1993), respectively. It was pretended to do a conjunction between the mathematical reflection about ratio and proportion and the didactic aspects.

About Piaget, there are different situations exhibited through which he explored the proportional thinking, how knowledge is generated from the way in which children face to solve problems of proportionality. Other researchers as Noelting (1980), Karplus, Pulos and Stage (1983) and Hart (1988) also showed an exploration with diagnostic intentions, from the point of view of cognition, but differently from Piaget, they worked in the educational field and their interest focused on reviewing the strategies used by the students when solving problems of ratio and proportion with the intention of tracing the cognitive processes that they included. Some of there researchers give more importance to the learning process than others.

For Freudenthal, didactics are fundamental in the development of a program, understanding it as the constitution of mathematical objectives. The didactic analysis of mathematics refers to the analysis of the contents of mathematics, which is done for the organization that teaches them in the educative systems.

Coll (1998) points out the importance of giving didactic situations as problem-situations destined to progress the representations and proceedings of the students about some conceptual fields in the area of mathematics.

Coll gives special attention to the diversity of the needs, interests and motivations of the students; he gives importance to reviewing the objectives and contents of school education with the intention, among others, that the students "learn to learn", of the relationships between psychological development and the school learning, of the convenience of increasing a meaningful learning process in school, of the importance of the organization and giving sequence to the contents, of the transcendence of the kind of relationships that the teacher establishes with the students.

Methodological Aspects

To make the study, the use of different methodological instruments was required. They were designed to carry out with what was established in the objectives that are pointed out in this article. The sequence of their design is shown as follows:

Two phases manifested, interconnected, of the progression of the methodological instruments. The first one was the exploration phase, that is questioning and search of the knowledge that the student had. This was done through the *observations and the initial questionnaire*, even though the teaching sequence also allowed to obtain more information about the cognitive aspect of the student, what led to the inclusion of complementary activities to reaffirm elemental knowledge that the student demanded. In what concerns to the second phase that was the one of enhancement, it took place with the *teaching proposal and the interviews*. The observation as well as the Questionnaires gave a light for designing the teaching proposal which was constructivism-didactic; it allowed the students to give sense and meaning to the concepts of ratio and proportion, starting by getting deeper in the qualitative aspects of their thinking and giving passage to the quantification.

As a way of evaluating this proposal, a *final questionnaire* was used and after this one the *interviews* were designed. The interviews had two intentions: evaluation and feedback (Valdemoros 1997;1998).

Validation Performance

This investigation was carried out under a qualitative character so validation plays a fundamental role. Methodological instruments employed in the research, as well as the teaching proposal were validated.

The validation of the methodological instruments was carried out through a process of piloting by a triangulation of different sources of register and of crossed controls.

According to the teaching experience, systematic procedures of consistent validation in a global experience of preliminary tests about teaching models hat were already designed. This proposal was validated through anecdotic registers of the sessions that took place, this registers were done through videotape and/or tape recordings; some researchers of educative mathematics

Design and generalities of the teaching experience

The tasks of the exploratory questionnaire are included in situations that are close to the reality of the student. In the design of the teaching activities, it was taken into account the way in which the questionnaire tasks were made, as well as the strategies detected in the students. With this is shared what was worked by Streefland (1990) in the design he made of a course with activities referred to concrete situations of application.

Taking as a base what was obtained in the questionnaire, the design and the simultaneous test of the teaching activities were started.

The proposal was conformed by six teaching models, considering the definition that Figueras, Filloy, and Valdemoros (1987) give of teaching model[2], which were retaken in different sessions as they were required by the advance of the teaching process, which is similar to what Streefland (1993) points out in his Realistic Theory, referring to the strategy of change in perspective, that is to say, a model is created and it is exploited as much as possible from an idea and afterwards that model is retaken exploited from the perspective of another idea. The change in perspective is characterized by the exchange of a part of information in the problem situation that is being questioned. As a consequence, the possibilities for the reconstruction and production of problems become evident. As a result of the change that happens in the mental activities of the students, the reconstruction of problems becomes a real production of them, where the free production of real problems is supported by a complete change of the perspective with qualitative aspects of ratio and proportion through comparing and it is pretended to reach the recognition of ratio and proportion as relationships between two measures and to what refers to the equivalence of ratio as proportion.

In all sessions a previous planning was made about the following aspects: work of the student, work dynamic (individual, team work and/or the complete group), the intervention of the teacher-researcher, in the different moments of the work dynamic and feedback of what was worked with the students.

Teaching Models

One of the models that were used, the objectives that were followed, the pursuit of contents and the form in which it was pretended that they should be worked with the students is shown as follows.

Model 2. The world of Snow White and the seven dwarfs

In the second model the short story of children literature "Snow White and the seven dwarfs" is used in three sessions that correspond to the second, third and fourth of the teaching proposal. This model allows the use of different perspectives for working with it; one of them is the concept of "reduction", based on the experience of the scale drawing and of the photocopy machine. The students are required to make a comparison between Snow White's bed and the bed of the dwarfs, using arguments of qualitative character. The concept of reduction is worked through three ways: oral, written and through the use of drawings.

In the following session this model is used to deal with the concept of enlargement, also based on the idea of the photocopy machine and the scale drawing. In this case another piece of furniture of the dwarfs' house, the table, is used and the students have to choose from four options the table that corresponds to Snow White. In that choice they will use arguments of qualitative character.

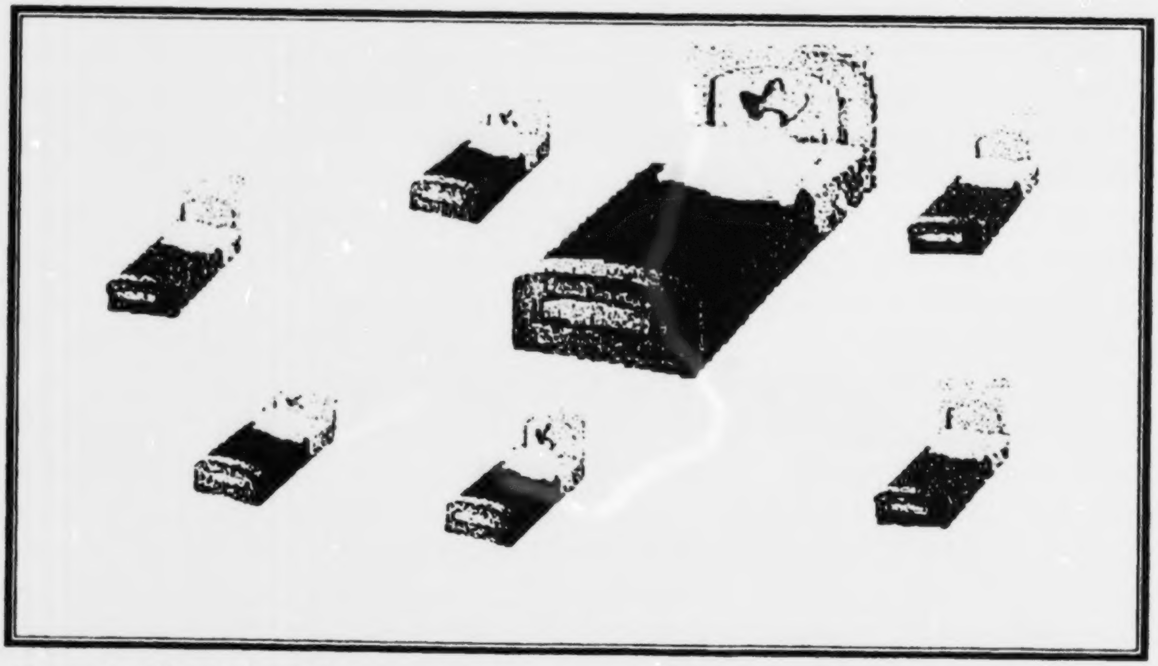


Figure 1. Material used in model 2. Snow White's bed and six little beds in which only one is the reduction of it.

In the fourth session, this model allows the student to verify the reduction and amplification of the figures using an instrument of measure in order to make comparisons of quantitative nature. In this work session the chart is introduced as a resource to organize information. Phrases such as "how many times can be included..." or "what part it represents of..."are used when making reference to the half and the double of a magnitude.

Treatment in what refers to the passage from the qualitative to the quantitative and preliminary results

The teaching proposal started with the qualitative. About this, Piaget (1978), through the experiments he did, points out that *the child acquires the qualitative identity before the conservation of the quantitative,* and he makes a distinction between qualitative comparisons and the real quantification. In fact, for Piaget the concept of proportion starts always in a qualitative and logical way before it is structured quantitatively. Streefland agrees with Piaget about the fact that the qualitative emerges before the quantitative, the new aspect is that Streefland takes it to the field of teaching and this contribution is the one used in the research that is refereed, because, first it is important to know how the students that finish their elementary education are organizing the qualitative components.

Streefland (1984, 1985) makes a research giving emphasis to the fact that the early teaching of ratio and proportion must start from qualitative levels of recognition of them and makes use of didactic resources that favor the development of perceptual models, supporting the quantification processes that correspond.

As qualitative reasoning evolves, what happens is that there is an advance in the thinking process and the child may integrate more elements for an analysis that allows him to consider different factors at the same time. It was pretended that the student gave sense and meaning to ratio and proportion and that gradually he could pass from the qualitative to the quantitative, without totally abandoning the qualitative. This passage was a process that was done through the various situations and the different moments. In an analogue way, Kieren (1988) points out the concrete passage from the concrete to the abstract, he points out that the intuitive is not totally abandoned.

The qualitative is based on linguistic recognition, creating categories of comparison such as *"bigger than..", and "littler than..."*. In the qualitative is included the intuitive, which is the information based on experience, on the empiric aspects, on senses. Afterwards appeared an order for making comparisons, the student used the phrases "major than..." and "minor than". About this Piaget (1978) points out that in the passage of the qualitative to the quantitative appears the idea of order without the appearance of the quantity, what he calls *intensive quantification.* Afterwards, the student used the measure for making comparisons, first putting a figure over another one and afterwards using an instrument of measure, conventional or not. In terms of Freudenthal the comparing elements are shown in two modalities: directs and indirect. The direct modality of comparing is when an object is placed over another object, while the indirect is when there are two objects (A and B) and a third element to compare (C).

In this way, an important step to get closer to quantification was that the student started to measure using natural numbers when making his comparisons, which is not the most important to emerge to ratio, as a proportionality linking expression, because what it is important is the ratio, not the comparison. However, measuring was an important instrument as requirement before using the multiplying operators.

Subsequently the establishing of ratios was achieved when the student established relations between magnitudes. First complete ratios when the student said that certain figure was 1 to 2 times long in respect to how long was another figure. And later ratios in terms of fractions.

In the teaching progression, students were able to recognize ratios as relations among quantities of a same dimension, as well as of two different dimensions; Freudenthal comments in this respect, that in teaching it is precise to take into account internal ratios and external ratios. Defining the first ones as established relations among different values of the same magnitude, and the second ones as relations among different values of different magnitudes. Finally, this proposal got sixth grade students to recognize proportions as relations among ratios.

Applying the initial questionnaire it was found:

- The students showed difficulty in correctly establishing relations between different measures. They had problems to work with the operator 3, and the some cases with the operator 1/2.
- They could to complete a table, but they didn't establish a relation with the task asked them.
- Some students incorrectly added; they didn't use multiplication. This result is like Hart, (1988) mentioned about the children who add.

The teaching proposal was already worked in the way described. During that, it was observed the evolution and maturation that the students had. Finally another questionnaire was applied, as well as interviews were made, to evaluate the reached achievements. In general one can say that the students understood the meaning of ratio as well as that of the proportion, and they demonstrated it to the power to solve with success different problems found in several situations. They used so much to the internal reasons as external. They were able to work indistinctly with the representation ways: to tabulate, numeric and the drawing in the tasks of proposed proportionality

References

Coll, C. (1998). Un Marco de Referencia Psicológico para la Educación Escolar: La concepción constructivista del Aprendizaje y de la enseñanza. In C. Coll,

J. Palaciosy, & A. Marchesi (Eds.), *Desarrollo psicológico y educación*. V:II. *Psicología de la eduacación* (pp. 435-453). Madrid: Alianza.

Figueras, O., Filloy, E., & Valdemoros, M. (1987). Some difficulties which obscure the apropiation of the fraction concept. *Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 366-372).

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* (pp. 28-33, 178-209). Reidel Dordrechet.

Hart, K. (1988). Ratio and proportion. In J. Hiebert & M. Behr (Eds.), *Concepts and operations in the middle grades* (Vol. 2). Reston, VA: National Council of Teachers of Mathematics.

Karpinski, L. C. (1965). *The history of arithmetic* (pp. 137-145). Ann Arbor, MI: Russell & Russell.

Karplus, R., Pulos, S., & Stage, E. K. (1983). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (Eds.), *Acquisitions of mathematics concepts and processes* (pp. 45-90). New York: Academic Press. *Journal of Mathematical Behavior, 4,* 25-36. Edmonton, Alberta, Canadá.

Noelting, G. (1980). The development of proportional reasoning and the ratio concept (Part I). *Educational Studies in Mathematics, 11 (2), 217-253.*

Piaget, J., & Inhelder, B. (1978). Las operaciones intelectuales y su desarrollo. In J. Delval (Ed.), *Lecturas en psicología del niño* (Vol. I, 70-119). Madrid: Alianza Editorial.

Piaget, J., & Inhelder, B. (1972). *De la lógica del niño a la lógica del adolescente.* Buenos Aires: Paidós.

Ruiz, E. F. (1997). Resolución de problemas a nivel primaria haciendo uso de la calculadora Math Explorer. *Resúmenes de la reunión Latinoamericana de matemática educativa. Comité Latinoamericano de matemática educativa* (p. 245).

Streefland, L. (1984). Search for the roots of ratio: Some thought on the long term learning process. Part I. *Educational Studies in Mathematics*, *15* (3*)*, 327-348.

Streefland, L. (1990). Free productions in teaching and learning mathematics In K. Gravemeijer, M. Van de Hunal, & L. Streefland (Eds.), *Context free productions tests and geometry in realistic mathematics education* (pp. 33-52). Utrecht: Research Group for Mathematical Educational Computer Centre State University of Utrecht, The Netherlands.

Streefland, L. (1993). The design of a mathematics course a theoretical reflection. *Educational Studies in Mathematics, 25,* 109-135.

Valdemoros, M. (1997). Recursos intuitivos que favorezcan la adición de fracciones: Estudio de caso. *Educación Matemática 9* (3), 5-17. México Iberoamérica.

Valdemoros , M. (1998). La construcción de la unidad en la suma de fracciones: Estudio de caso. In Hitt (Ed.), *Investigaciones en matemática educativa* (Vol. 2, pp. 465-481). México Iberoamérica.

Notes

[1] This previous study was only a source of information. The problems that conformed the activity were so common and they took to the use of the rule of three, so it was necessary to work with them using a guide of questions (taking the theme of the scale as an antecedent) and the heuristic strategy of the diagram.

[2] Teaching model: what includes meanings, in the technical language as well as in the current language, didactic treatments, specific forms of representation and the relationships that exist between them.

TOWARD CONSENSUS ON COMPUTATIONAL STRATEGIES FOR SINGLE-DIGIT MULTIPLICATION

Bruce Sherin
Northwestern University
bsherin@northwestern.edu

Abstract: The goal of this paper is to take some first steps toward forging consensus on a typology of strategies for single-digit multiplication. In order to create a relatively comprehensive typology, our work adopts two approaches. First, we have attempted to draw together and synthesize the conclusions of prior work by other researchers. Second, we have engaged in our own empirical efforts. As an outcome of these activities, we advocate for a particular stance toward the nature of computational strategies, namely that we treat strategies as patterns in computational activity. Furthermore, we argue for a two-tiered categorization scheme. In the first tier, we differentiated based on *patterns in represented quantities*. In the second tier, we differentiated based on *modes of representation*. Our emphasis in this paper will be on the category scheme associated with the first tier.

Introduction

Currently, there exists significant consensus concerning the kinds of computational strategies used by students for single-digit addition. The research community has largely agreed on the types of strategies, as well as on terminology for describing these types (see, for example, Fuson, 1992). However, although there is a growing body of research on single-digit *multiplication* strategies (e.g., Anghileri, 1989; Kouba, 1989; Lemaire & Siegler, 1995; Mulligan & Mitchelmore, 1997), no similar consensus exists.

The goal of this paper is to take some first steps toward forging consensus on a typology of strategies for single-digit multiplication. In order to create a relatively comprehensive typology, our work adopts two approaches. First, we have attempted to draw together and synthesize the conclusions of prior work by other researchers. Second, we have engaged in our own empirical efforts. Within the context of the work reported in this paper, the data corpus produced by our own studies will play two roles. First, we use this corpus to confirm the results of prior work, as well as to fill some of the gaps that exist. Second, we use examples drawn from this corpus to illustrate and argue for our points.

What is a Computational Strategy?

If we wish to synthesize prior work and reach some consensus on multiplication strategies, it is essential that we are clear about the criteria that will govern our convergence on categories. For this reason, we begin our discussion, in this section, by laying out our stance toward the nature of computational strategies.

The ontological assumption. The first assumption that underlies our stance is the strongest, and readers will likely find it to be the most controversial. When we speak of a *computational strategy,* we intend to be talking about *patterns in computational activity, viewed at a certain level of abstraction.* This is perhaps most clear if contrasted to an alternative stance, a view of strategies as *knowledge possessed by individuals.* Computational strategies, as we intend to be speaking of them, are not knowledge; rather, a computational strategy is a pattern in computational activity – a pattern in the steps taken toward producing a numerical result.

Much of the existing literature either takes the stance that strategies *are* knowledge, or closely ties strategies to hypotheses about knowledge. For example, some of the articles we will mention below (e.g., Mulligan & Mitchelmore, 1997) discuss the "intuitive models" that underlie students' understanding of multiplication. The nature of this underlying knowledge is certainly important, and research must continue to address these issues. However, we believe that a discussion of computational strategies can be profitably separated from an account of the associated knowledge.

Patterns in represented quantities. The ontological assumption directs our attention to patterns of activity. However, this assumption still leaves many important questions unanswered; in particular, it does not direct our attention to specific aspects of activity, and it does not tell us how we should "see patterns" in that activity. Suppose, for example, that we are faced with two episodes of computational activity. We need to be able to answer the question: Are the individuals enacting the same pattern of activity? If so, what is that pattern? As a first distinguishing criterion, we propose to look at patterns in *represented quantities, defined relative to the quantities given in the problem.* In this regard, the intermediate quantities, produced on the way to a final result, are particularly important. For example, Figure 1 shows the work of two students on a word problem that requires them to multiply 4x8. When producing the first solution, the student drew 4 boxes each containing 8 tally marks. Then he counted each of the tally marks, speaking aloud as he counted. Note that, in this computational activity, every quantity between 1 and the final result has been represented.

In contrast, consider the second solution to the same problem shown in Figure 1. This student started by adding pairs of 8s to get two groups of 16. Then he added these two 16s to get 32. In this solution, a very different set of quantities were represented. Most dramatically, many of the interme-

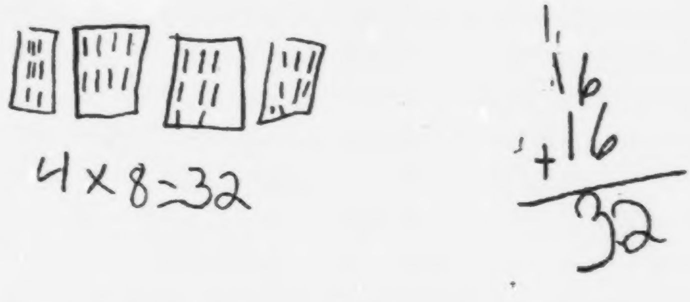


Figure 1. Solutions to a word problem that requires a student to multiply 4x8.

diate values between 1 and the final result did not appear. Thus, this pattern of activity differs from that in the first solution, and the computational strategy is, by our definition, different. There are some subtleties here. First, we will need criteria for judging when quantities are represented mentally. Furthermore, given the above account, there will still be multiple ways to describe the patterns in any given instance of computation. Thus, this specification still greatly underdetermines how we should "see patterns" in computational activity. In order to do better, we will need to say something about how cognitive resources are employed in getting from one quantity in the solution to the next. We will discuss this issue further in a more extensive report on this work (Sherin & Fuson, in preparation).

Modes of representation. Even when the patterns in computed quantities are the same across two solution episodes, there can still be important types of variation in the computational activity. One such type of variation relates to the *mode of representation* of quantities. During a computation, individual quantities may be represented on paper, verbally, on fingers, mentally, or in other external forms. Indeed, much of the skill in computation is located in techniques for managing these various modes of representations. For example, students learn specific techniques for producing drawings (such as the box and tally diagram in Figure 1), and they develop techniques for keeping track of quantities on their fingers.

For these reasons, we believe that it is important to distinguish computational strategies according to the modes of representation employed, and we do so throughout this work. However, we will treat this as a second-order distinction; we distinguish first based on patterns in the quantities represented, and second based upon the modes of representation employed. Due to reasons of space, we will only briefly discuss these second-order distinctions in this paper.

Context, Data Sources, and Modes of Inquiry

This research was conducted as part of the Children's Math World (CMW) Project. During the last three years, we have been engaged in the development of full-year curricula for third and fourth-grade mathematics. Single-digit multiplication was among the topics addressed in our learning activities.

Over the lifetime of the project, we have worked closely with a number of classrooms and engaged in a range of data-collection activities, including frequent classroom observations, teacher interviews, and student interviews. The work described in this paper draws primarily on the student interviews that were conducted throughout our three years of work. We conducted 166 student interviews that focused, at least in part, on single-digit multiplication strategies. Table 1 gives an overview of when, during the course of the project, these interviews were conducted. As described in the table, we conducted our first interviews, with 37 students, at the end of Year 1. The observations collected from these interviews were treated as pilot data.

During the second year of the project we engaged in our most extensive and systematic interviewing pertaining to multiplication. We conducted interviews with a substantial population of third graders before and after their classroom engaged in units focused on single-digit multiplication. In addition, we conducted interviews with a smaller sub-group of students at two mid-points during the multiplication units.

Table 1. Student interviews conducted concerning single-digit multiplication.

Year	Grade	When?	# classes	# interviews
1	3	End of year	1	22
	4	End of year	1	15
2	3	Start of year	4	69
	3	Mid-point 1	2	8
	3	Mid-point 2	2	7
	3	After instruction	3	45

The schools in which we worked varied significantly along the dimension of socio-economic status as well as race and ethnicity. Of the four classes that were studied extensively during Year 2, two were third-grade classes in suburban schools with student backgrounds ranging from working class to middle class. A substantial number were ESL students. The other classrooms were in urban schools with primarily minority populations, with working-class and poverty backgrounds. Many students were from Latino populations, with several other backgrounds represented.

Tasks. Throughout the project, our own goals in conducting the student interviews was quite varied. In addition to conducting research concerning fundamental issues in the learning of mathematics, we have also been concerned with improving the learning activities that comprise the Children's Math Words curriculum. Thus, the nature of the tasks employed in our interviews was dictated by many practical concerns, as well as by a multiplicity of research interests. Included among our tasks were multiplication word problems, and tasks in which students were asked simply to multiply two numbers (e.g., "What is 9 times 5?").

Analysis methods. All of the interviews were videotaped, and the videotapes were digitized. Our analysis of computational strategies then proceeded through several phases of coding of the data. During the first phase, researchers on the CMW project systematically coded and recoded student work on all relevant tasks in the Year 2 portion of the data corpus. This cycle of recoding proceeded until the team had reached convergence on an analysis scheme. Throughout this process, our coding was done within an online database directly linked to the digitized videos of the interviews. This digital system allowed for rapid comparison across the hundreds of instances of computational behavior contained in our data corpus, and facilitated convergence on a set of categories.

This systematic work was an important part of the project and, taken together with the work of other researchers, helps to validate the scheme we will propose. However, we should emphasize that the results of this coding do not constitute the primary result of this work. Because our interviews were conducted within a particular context (students in Children's Math World, at particular schools, etc.), the frequencies of strategies are largely specific to this context. In this work our goal is a systematic description of the *range and variety* of students' computational strategies, rather than any particular assertions about the frequencies with which these strategies appear. As stated above, we will use our observations, in combination with those presented by previous researchers, in order to document and illustrate this range.

Toward Consensus

We now present the category scheme for which we will argue. Table 2 presents our first-tier categories (across the top) in relation to the category schemes employed by a selection of prior studies. A comment on the selection of studies for inclusion in this list is merited. Some of the prior research on multiplication strategies has not shared our concern with describing the full range and variety of student strategies. For example, there have been lively debates concerning how best to model the retrieval of multiplication facts (e.g., Baroody, 1997; Koshmider & Ashcraft, 1991; LeFevre & Liu, 1997; Lemaire, Barrett, Fayol, & Abdi, 1994), and detailed analyses of the progression from other strategies to retrieval (e.g., Lemaire & Siegler, 1995). In many of these retrieval-focused analyses, computational strategies are simply combined into two large categories, *retrieval* and *other.* In contrast, the primary concern of the present work is with opening up the "other" category.

Table 2. Our Canonical Categories in Relation to the Categories of Researchers Who Distinguish Among Non-retrieval Strategies

	Count-all		Count-by	Additive Calculation	Rapid Response		Hybrid
Mulligan & Mitchelmore, 1997	Unitary (direct) Counting	Rhythmic Counting	Skip Counting	Additive Calculation	Multiplicative Calculation		
					Known multiplication Fact		Derived Fact
Kouba, 1989	Direct Representation (only use of physical objects)		Transitional Counting	Additive or Subtractive	Recalled Number Facts (includes derived facts)		
Anghileri, 1990	Unary				Binary		
	Count-all Count-on	Rhythmic Counting	Number Pattern	Addition Facts	Known Fact		
Lemaire & Seigler; 1995	Counting-sets of-objects (tally marks)	Repeated-Addition			Retrieval	Writing Problem	
Cooney et al., 1988	Counting				Memory Retrieval	Rules (0 & 1)	Derived Fact

For that reason, we have selected for inclusion in Table 2 only studies that have, as a primary concern, describing the range of computational strategies for single-digit multiplication. In the table, the top 3 studies strongly share our concern with documenting the range of non-retrieval strategies. The bottom two papers were selected to illustrate studies that fall into an intermediate category. Cooney and colleagues (1988) were largely concerned with the transition to retrieval, while Lemaire and Siegler were interested, more generally, in aspects of "strategic change." Nonetheless, in both these cases, the authors did take some steps to open up the "other" category.

We should also mention that these research studies varied significantly along other dimensions, such as the tasks employed and the populations studied. For example, some of the studies included in Table 2 looked only at students solving word problems (e.g., Mulligan & Mitchelmore, 1997), while others looked only at purely numerical tasks (Lemaire & Siegler, 1995). Still more dramatically, in some cases students were encouraged to use paper-and-pencil during solutions, or were given manipulatives (Anghileri, 1989) while in other cases they were prohibited from making use of any external supports (e.g., Cooney et al., 1988). Because of these significant differences in approach, we should expect variation in the strategies observed.

We now present our own framework, making comparisons to the prior research where appropriate. Our framework has a specific form that is reflected in our presentation below. We begin by, first, laying out a set of what we call *canonical strategies*: count-all, count-by, additive calculation, and rapid response. These canonical types differentiate strategies along the dimension of *patterns in represented quantities*, discussed above. We furthermore understand strategies in terms of two types of variations on these canonical types. First, there is variation in the *mode of representation*, as discussed above. Second, there are *hybrid* strategies that are formed by combining two or more of the canonical types. In the remainder of this section, we describe the canonical strategies. Then, in the section that follows, we discuss the two types of variation.

Count-all. Our first canonical strategy is *count-all*. The key distinguishing feature of count-all strategies is that all values between 1 and the total are represented by the student during the computation. For example, Figure 2 shows a student's written work for a word problem in which there are 4 boxes of pencils, each containing 8 pencils. When solving this problem, the student drew the diagram shown in this figure, and then counted aloud from 1 to 32.

Looking at Table 2, it is evident that this canonical type appeared, in some form, in all of the studies included in the table. There was, however, some variation in how this strategy

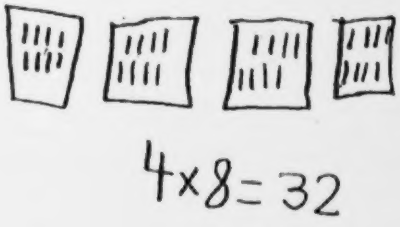


Figure 2. Student drawing for a count-all solution. Taken from Year 1, mid-point 1.

was treated by individual researchers. For example, Cooney and colleagues have a more encompassing category that, in the manner of the retrieval-focused research, combines count-all with other strategies. In contrast, three of the other research teams have split our count-all category into two separate categories. Anghileri (1989) and Mulligan and Mitchelmore (Mulligan & Mitchelmore, 1997) both have two types here, one of which is "rhythmic counting." In rhythmic counting, all of the values between one and the total are spoken aloud, but the value at each multiple is verbally stressed. So, for example, if a student were multiplying 3x4 they would say: "One, two, three, **four**, five, six, seven, **eight**, nine, ten, eleven, **twelve**." We include rhythmic counting within our count-all category because, even though some values are stressed, all values are represented between 1 and the total.

Within this category, Lemaire and Siegler (1995) differentiate between "counting-sets-of-objects" and "repeated addition." In counting-sets of-objects, there is an external representation that supports the student in counting from 1 to the total. Lemaire and Siegler refer to all other count-all strategies as "repeated-addition." In our scheme, this would be described as a difference in mode of representation, and thus does not appear as a distinction in our canonical types.

Count-by. In a *count-by* solution, the student skips values between 1 and the total, and only the multiples of the multiplier or multiplicand are represented. For example, in order to multiply 4 and 8 a student might count by 8 four times saying: "eight, sixteen, twenty-four, thirty-two." Note that none of the values between 8 and 16 are represented (verbally or otherwise). Furthermore, at each step, the 8 to be added on is not represented. All that is represented are the elements of the count-by-8 sequence of numbers.

Across the literature, this canonical type has been given a number of different names. Within Table 2 it is referred to as "skip counting," "transitional counting," and "number pattern." As mentioned above, Cooney et al. have a large, encompassing category, called "counting," that includes this type of strategy.

Within our table, only Lemaire and Siegler do not have a category that can be easily aligned with our count-by type. Although it is not possible to draw any strong conclusions here, it is worth noting that the students in Lemaire and Siegler's study were French 2nd graders. It is at least possible that students in this population simply do not make use of this strategy.

Additive calculation. *Additive calculations* are like count-by based strategies in that only some intermediate values between one and the total are represented. For example, Figure 3 shows student work, again for the word problem with 4 boxes, each containing 8 pencils. The student began by writing "8" four times. Then he formed pairs of 8s to get two 16s. Then he added the two 16s using multi-column techniques.

It is worth taking a moment to see how our criteria can be applied to differentiate additive calculation from count-by-based strategies. In the example shown in Figure 3, it is clearly the case that the pattern of represented numbers is different than the pattern that would be produced by a count-by solution. Note, for example, that 24 never appears and that 16 is represented twice.

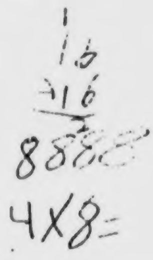


Figure 3. Student drawing for an additive calculation. Taken from year 1, mid-point 1.

However, there are cases that are somewhat less clear. Suppose, for example, that the student had begun by grouping just the first two 8s to obtain an intermediate result of 16. Suppose, further, that the student then used multi-column techniques to first add on one 8, then to add on another 8. The pattern of intermediate results would then be the same as a count-by sequence. Is this, then, an instance of the count-by category of strategies?

To see this is not a count-by strategy, we have to pay attention to all of the quantities that are represented as the solution proceeds, not just the intermediate results. As a count-by strategy is enacted, the elements of the associated count-by sequence and the number of groups are the only quantities that are represented. However, the situation is different in an additive calculation. As each intermediate result is calculated, a number of quantities may be represented. These include, for example, the value to be added on, the total in the tens column, and the amount to be carried.

Referring to Table 2, the additive calculation strategy appears in 4 of the 5 studies, with three papers having categories that are truly parallel. With regard to the literature, one other point deserves mention here. Some of the literature on the learning of multiplication has been concerned with whether repeated addition is the "intuitive model" that underlies some student understanding of multiplication. In contrast, our assertion that additive calculation is a fundamental type of multiplication strategy is not intended to be a claim about students' intuitive models. As stated in our ontological assumption, the primary concern of this paper is with strategies as enacted computational activity, and not with the underlying structures that generate these models. To see the magnitude of the difference, notice that a repeated-addition model could underlie a count-all strategy, just as it may underlie an additive calculation. Thus, the distinguishing feature of additive calculation strategies cannot be that they are based upon a repeated addition model; rather the distinguishing feature must be the pattern in the computational activity.

Rapid response. Our final canonical type, *rapid response,* encompasses strategies that other researchers have variously called "retrieval" and "known fact." Given our discussion of features that define canonical types of strategies, it is not clear that rapid response is a strategy in the same sense as our other canonical strategies. There is not much of a pattern of "computational activity;" rather, a product is just associated with two given quantities, and no intermediate quantities are produced overtly during the solution.

All of the studies included in Table 2 include this type of strategy, although there is some variation across papers in how the category is formed. Both Kouba (1989) and Mulligan and Mitchelmore (1997) have an extended category that includes our rapid response, as well strategies that we would describe as "hybrid." And the two retrieval-focused papers in Table 2 split the rapid response category into two sub-categories. Lemaire and Siegler (1995) distinguish the case in which the student writes the problem (e.g., "4x8") from the case in which the student simple states the result without writing. And Cooney et al. (1988) highlight "rule"-based strategies that they associate with products of 0 and 1. For example, a student may simply know the "rule" that any number multiplied by 0 gives 0 as a product.

Other strategies. We note only briefly that a few special-purpose strategies have been omitted from our discussion. For example, students sometimes learn "trick" or pattern-based strategies for dealing with multiplications by 9. These are omitted for reasons of space.

Dimensions of Variation

Hybrids. In this part of the paper, we present a brief discussion of the ways in which actual enacted strategies will be variations on the ideals associated with the canonical types. The first type of variation we refer to as *hybrids,* strategies that can be understood as a combination of two or more of our canonical types. For example, in one type of hybrid that we observed frequently, students use a count-by or rapid-response strategy to get partway to a total, and then employ count-all to finish the computation.

These hybrid strategies do appear in the literature, though they are often treated differently than in the present study. Most importantly, the studies listed in Table 2 only identified a single hybrid strategy —"derived fact"— and this strategy was treated as an extension of a retrieval-like category. In a derived fact strategy, a student starts with a recalled product and then adds or subtracts to derive a final result. Note that our hybrid category includes these derived facts, but also many other strategies, corresponding to the reasonable compositions of our canonical types.

Representational variety. As we discussed earlier, there is an important dimension of variation that is not captured by our canonical types; namely, there can be great variation in how quantities are *represented* during a computation. Quantities can be represented on fingers, on paper, verbally, mentally, or aloud, and each of these media can be employed in multiple ways. As stated above, we think of this dimension as producing a second-order distinction in our category scheme.

Variability in representational mode is most prominent in count-all strategies. During a count-all computation, a student must keep track of 5 quantities. Two of these quantities, the multiplicand and the multiplier given in the problem, do not change as the computation proceeds. However, the three remaining quantities do change. These are: (1) the current value within the counting off of the multiplicand, (2) the current value within the counting off of the multiplier, and (3) the current value of the total. Because there are so many quantities to keep track of, there is room for great representational diversity in count-all strategies.

Conclusion

In this paper, we have attempted to lay the foundations for consensus around a set of computational strategies for multiplication. We advocated a particular stance toward the nature of computational strategies, namely that we treat them as patterns in computational activity. Furthermore, we argued for a two-tiered categorization scheme. In the first tier, we differentiated based on *patterns in represented quantities*. This gives rise to our canonical types. In the second tier, we differentiated based on *mode of representation*.

Throughout our discussion, we have not attempted to address a number of central questions. For example, we did not discuss typical learning progressions, nor did we discuss some of the more frequent difficulties that students encounter. We will discuss these issues, and elaborate on the issues discussed in this paper, in a forthcoming paper (Sherin & Fuson, in preparation).

References

Anghileri, J. (1989). An investigation of young children's understanding of multiplication. *Educational Studies in Mathematics, 20*, 367-385.

Baroody, A. J. (1997). *The development of third-graders' mental multiplication.* Proceedings of the Annual Meeting of the American Educational Research Association, Chicago, IL.

Cooney, J. B., Swanson, H. L., & Ladd, S. F. (1988). Acquisition of mental multiplication skill: Evidence for the transition between counting and retrieval strategies. *Cognition & Instruction, 5* (4), 323-345.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243-275). New York: Macmillian.

Koshmider, J. W., & Ashcraft, M. H. (1991). The development of children's mental multiplication skills. *Journal of Experimental Child Psychology, 51* (1), 53-89.

Kouba, V. (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal for Research in Mathematics Education, 20* (2), 147-158.

LeFevre, J.-A., & Liu, J. (1997). The role of experience in numerical skill: Multiplication performance in adults from Canada and China. *Mathematical Cognition, 3* (1), 31-62.

Lemaire, P., Barrett, S. E., Fayol, M., & Abdi, H. (1994). Automatic activation of addition and multiplication facts in elementary school children. *Journal of Experimental Child Psychology, 57* (2), 224-258.

Lemaire, P., & Siegler, R. (1995). Four aspects of strategic change: Contributions to children's learning of multiplication. *Journal of Experimental Psychology, 124* (1), 83-97.

Mulligan, J., & Mitchelmore, M. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education, 28* (3), 309-330.

Sherin, B. L., & Fuson, K. C. (in preparation). *Multiplication strategies and the cycle of cultural appropriation*. Unpublished manuscript.

PART-WHOLE CONCEPT UNDERSTANDING IN A POPULATION OF 2ND AND 3RD GRADERS

Dorothea A. Steinke
NumberWorks
steinke@rt66.com

Abstract: Second grade students (104) and 3rd grade students (106) in a lower socio-economic-level urban school were assessed for understanding of the part-whole concept in one-on-one interviews at the beginning of the school year as part of a yearlong study of the effectiveness of concept-targeted music instruction in increasing part-whole understanding in this population. The percentage of students lacking part-whole concept understanding appears to be at least 60% of 2nd graders and 40% of 3rd graders. Standardized test scores show about a one-grade-level lag in mathematics performance for the two groups. That difference is in line with research indicating that lower-socio-economic-level students start and remain a year behind their middle-class age peers in mathematics understanding. Implications of these findings are suggested.

Objectives

A yearlong classroom-based research study is currently under way to test the effectiveness of concept-targeted music instruction in increasing the mathematical understanding of second and third-graders in a high-minority (Hispanic, Native American, African American) school in a low-income urban area. However, a stand-alone evaluation of the pre-test data can also yield important information about the alignment of curriculum and textbook expectations with student readiness in this population. In particular, do the implicit expectations of mathematics textbooks that children grasp the part-whole concept by the beginning of 3rd grade track with reality?

Theoretical Framework

Steffe and Cobb's (1988) 3 Stages model of young children's development of number sense (Perceptual, Figurative, Abstract) provided the framework for one pre-instruction assessment in the aforementioned study, the other being a standardized test (Iowa Tests of Basic Skills, 1996). Steffe and Cobb provide a concise and practical model based on mathematical and physical behaviors rather than on the vocabulary of word problems. Steffe and Cobb observed that children's responses to addition or missing addend questions were marked at different Stages by the way that hands and fingers were used and by the pattern of counting, as summarized below:

- Stage 1 (Perceptual): Children 1) show a number on their hands as a group of fingers ("block" fingers) raised all at once; 2) count visible objects only; 3) always count from 1 when joining two sets.

- Stage 2 (Figurative): Children 1) show number by raising one finger after the other in sequence; 2) may substitute an action (touching "pretend" objects in a geometric pattern; sequential fingers; head nodding) or a vocalization (saying number words aloud or subvocally) or a mental image (observed by eye movement) for an "invisible" or hidden number of objects; 3) can "count on" from one of the numbers to find the sum of two sets.
- Stage 3 (Abstract): Children 1) are able to separate numbers into parts either mentally or by reorganizing finger patterns; 2) stop "counting on" at the given "whole" in missing addend problems; 3) can keep track of how many items were counted (fingers or number words) when counting "up to" or "down from"; 4) show confidence that the answer given is right; and 5) may be able to explain how the answer was arrived at.

Steffe and Cobb also noted that marks of an earlier Stage, particularly "block" finger patterns, continue to be used at later Stages.

It has been proposed that the "gate" between the Perceptual and Figurative Stages is the understanding of the "equal distance of 1" between neighboring whole numbers; and the "gate" between the Figurative and Abstract stages is the part-whole concept (Steinke 2000). Kamii's studies (with De Clark, 1985; with Joseph, 1989) and Ross (1989) also provide data that indicate the extent of lack of the part-whole concept from 2nd grade up. Steinke (1999) has found that some adults also lack the part-whole concept.

Methods

Based on Steffe and Cobb's (1988) 3 Stages model, this researcher conducted one-on-one interviews with students. The questions (see Table 1) were chosen to distinguish the student's grasp of the "gate" concepts. The questions were given and answered orally. The students used no paper or pencil. The "Visible Circles" were of yellow poster board and two inches in diameter. The numerals were written on white unlined 3x5 cards. Two white paper plates were used to contain and delimit the objects representing each addend.

The children were asked: 1) to count more than 20 circles (one-to-one correspondence); 2) to find the sum of two small sets of visible circles (Stage 1: Perceptual); 3) to find the sum of two sets, one a set of visible circles and the other set represented by a numeral (Stage 2: Figurative); and 4) to find the amount in a subset when the complete set is given as a numeral and one subset is presented as visible circles ("missing addend" problem) (Stage 3: Abstract). This assessment is similar to that referenced by Pearn (2000), except that Pearn used Steffe's earlier 5-Stage model (see: Wright, 1994) rather than the 3-Stage model used here.

At Stage 1, yellow circles were placed on each plate. At Stage 2, children were told that one of the plates has "hidden" or "invisible" circles. A card with a numeral was placed on that plate; yellow circles were placed on the other. A strong effort was made to make the children aware of the change from the Stage 2 questions (find the whole, i.e., 9+6=?) to the Stage 3 questions (find the "missing part", i.e., 7=3+?) (see Note 1). At Stage 3, one plate was empty, one plate contained yellow circles, and a card with the numeral of the "whole" was placed on the table above the two plates.

The interview questions were structured to indicate the student's Stage of number sense development based both on the kind of question asked and the physical signs observed. The questions were refined over a period of years in 300 interviews with children pre-K through 6th grade and about 100 interviews with adults. In these earlier interviews, additional Stage 1 questions were used with younger children (pre-K and K). By 2nd grade, however, responses for addition sums of 10 or less can be influenced by memorized number facts so that more questions with smaller addends give little or no additional information about Stage. They merely lengthen the time required for the test. Longer test time could affect how well children stay focused on the activity.

The Stage 2 questions use addends less than 10 whose total is more than 10 so that only "counting on" or substituting (or rote memorization) will yield the answer. The Stage 1 child attempting to sum these numbers runs out of fingers to represent the numbers as a "block." Students' familiarity with the "teen" numbers in Stage 2 questions may be a concern at the start of 2nd grade but should have little effect at the start of 3rd grade.

For the missing addend (Stage 3) problems, the first question, with a whole less than 10, can also be answered by rote. However, it does establish the student's awareness of the question: Does the child add the two given numbers and/or use the visible circles as the answer, or does the child start with the whole and reorganize fingers to find the "missing part"? One question is needed that goes beyond potential memorized "facts" for 3rd graders, hence 25 = 7 + ?. Using a whole greater than 20 and forcing the crossing of the "decade" is intended to initiate reorganization of finger patterns and/or to force mental breaking of the whole (or of the given addend) into parts coexisting with the whole.

Data Sources

In August and September 2000, this researcher interviewed 104 2nd grade and 106 3rd grade students using the method described. This is at least 95% of the 2nd and 3rd graders other than Special Education students at one Albuquerque elementary school. The number is less than 100% because some parents chose not to have their child participate in the interview. Each interview took five to eight minutes and was videotaped for later verification of the researcher's notes written during and immediately following each interview.

Results

Table 1 is a count of actual incorrect responses based solely on the students' final answers. In cases where the visible circles were miscounted, the answer was counted correct if it was correct using the number the student gave for the visible circles. That is, miscounting did not automatically result in an incorrect answer. Final determination of Stage, which will be reported at a future date, depends not only on the answers given but also on review of the videotapes for indications of strategy and physical signs of Stage, especially the use of fingers.

In Table 1, the quantities shown to students as visible circles are indicated by **BOLD** type. The quantities shown to students as a visible numeral on 3x5 card are indicated by normal type. The answer is indicated by a question mark.

The interviews found students at all three Stages in both 2nd and 3rd grade based on answers to the problems and interviewer notes. In fact, five 2nd graders and three 3rd graders were still at the Pre-perceptual stage, unable even to add two small collections of visible circles. Some of these students were transferred to Special Education classes later in the year.

A large number of 2nd graders (29%) and 10% of third graders were still at Stage 1, needing visible physical objects other than their fingers to find the answer when asked to add two numbers under 10 whose sum was greater than 10. When the required answer was 11 or more, these students answered incorrectly. They showed one or more of the physical signs of Stage 1.

Based only on student answers to the first Stage 3 question (7=3+?), 63% of 2nd graders and 50% of 3rd graders lack part-whole concept. In fact, six 2nd graders and ten 3rd graders who correctly answered one or both of the later Stage 3 questions missed the first one (7=3+?). The reason for this may be that the child missed the change in the question from "find the whole" (add two sets) to "find the missing part" (find missing addend). When the students who correctly answered a later Stage 3 questions are excluded, that still leaves 60 2nd graders (58%) and 43 3rd graders (41%) who were unable to correctly answer any of the Stage 3 "find the missing part" questions. In other words, this many students are still at Stage 2.

Table 1. Student Errors on Interview Problems

Stage	1	2	2	3	3	3
Problem	3+5=?	7+4=?	9+6=?	7=3+?	14=6+?	25= 7+?
Gr. 2[a] Errors	5	30	45	66	79	92
Gr. 2 % Errors	5%	29%	43%	63%	76%	88%
Gr. 3[b] Errors	3	11	15	53	65	90
Gr. 3 % Errors	3%	10%	14%	50%	61%	85%

[a] Grade 2: n = 104; [b] Grade 3: n = 106

As a check on the interview results, the Math Concepts and Math Problems sections of the Iowa Tests of Basic Skills (Form M) [ITBS] were also administered. For this test, the teacher or monitor reads each question aloud and the students mark the answers directly in the test booklet. This eliminates some of the variability of student reading skills and errors caused by having to mark separate answers sheets. For each grade, the average Grade Equivalent Scores were: 2nd grade - Grade 1.3 on Math Concepts, Grade 1.5 on Math Problems; 3rd grade - Grade 2.0 on Math Concepts, Grade 2.3 on Math Problems.

Conclusion

Students in a low socio-economic, high-minority urban school were assessed for number sense development using questions based on Steffe and Cobb's 3 Stages model. Additionally students were tested using the Math Concepts and Math Problems sections of the ITBS Form M. The ITBS results lead one to believe that students at the school make about a year of progress in math in the course of a school year. However, they are still about a year behind the standardized test norms. This is to be expected based on Hughes' (1986) report that found working-class children were about a year behind their middle-class peers in mathematical development. A more recent study of infants and toddlers (Black et al., 2000) indicates that this lag begins well before children enter school.

The interviews reported here confirm this gap. The proportion of 3rd graders lacking the part-whole concept at the beginning of the school year, even by the most favorable and forgiving standards, is about 40%. Since the first question could have been answered by rote, the true proportion is probably closer to the 60% who missed the second question. The proportion of 2nd graders lacking the concept is even higher, about 60% and 75% respectively. These children are far short of the "Mastery goals for numerical understanding for all children" which Fuson et al. (2001) propose and which are developmentally achievable.

The results from the interviews suggest that lack of the part-whole concept plays a role in low mathematical performance against "norms" probably by 2nd grade and surely by 3rd grade. Failing to take into account where students "are" developmentally in math exacerbates students' difficulties in math learning at school. This is in addition to other factors that affect these students' ability to learn (lack of stimulating home environment, lack of parental involvement, substance abuse in the home).

Students' difficulties start with the math materials. Many, if not virtually all, math textbooks expect the part-whole concept to be learned during 2nd grade or even before. None of the elementary textbooks reviewed to date by this researcher explicitly teach the part-whole concept in the 2nd or 3rd grade children's text when these children are ready to learn it.

Teacher preparation is also a factor. Primary-grade teachers may be unaware of the true developmental sequence of young children's math learning. They may assume that subtraction follows addition because the textbook has it that way. They may not realize that they can teach graphing, data collection, geometry and other topics until their children show themselves developmentally ready (i.e., have the part-whole concept) for place value. Also, they may think of math in terms of "skills" rather than concepts, an idea that textbook chapter headings and student workbook titles reinforce.

The research study of which these assessments are the first step is attempting to make students aware of part-whole relationships through another avenue: concept-targeted rhythm experiences. It is proposed that physical experience of the concept in music will transfer to and facilitate awareness of the concept in number relationships. Such transfer of learning, if successful, has far-reaching implications for the psychology of learning in general and the psychology of mathematics in particular.

Endnote

[1] Following is the text of that portion of the interview in which the question changes from "find the whole" to "find the missing part":

> This time, I have some hidden circles, but I don't know how many. (Interviewer indicates an empty paper plate.) (Interviewer puts out another plate containing 3 yellow circles.) I have some circles here. How many? (Student counts visible circles and responds.) And someone told me that altogether I have 7 circles. (Interviewer places a 3x5 card with the numeral 7 above the two plates approximately on a line of symmetry between the two plates.) That is, these circles (Interviewer touches plate with 3 circles.) and these circles (Interviewer touches empty plate.) altogether (Interviewer makes circle in the air over both plates with a closed hand.) make 7 (Interviewer touches card with numeral). How many hidden circles are there? (If the child answers 7 or zero, the interviewer repeats the question and the motions.)

References

Black, M. M., Hess, C. R., & Berenson-Howard, J. (2000). Toddlers from low-income families have below normal mental, motor and behavior scores on the revised Bayley Scales. *Journal of Applied Developmental Psychology, 21* (6): 655-666.

Fuson, K. C., Grandau, L., & Sugiyama, P. A. (2001). Achievable numerical understandings for all young children. *Teaching Children Mathematics*, *7* (9), 522-526.

Hughes, M. (1986). *Children and number: Difficulties in learning mathematics*. Oxford: Blackwell.

Kamii, C., & DeClark, G. (1985). *Young children reinvent arithmetic: Implications of Piaget's theory*. New York: Teachers College Press.

Kamii, C., & Joseph, L. L. (1989). *Young children continue to reinvent arithmetic: 2nd grade.* New York: Teachers College Press.

Pearn, C. (2000). Mathematics intervention: The identification of year 1 students mathematically "at risk". In M. L. Fernández (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 175-180). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education. (ERIC Document No. SE 064 088)

Ross, S. H. (1989). Parts, wholes, and place value: A developmental view. *Arithmetic Teacher*, 36, 47-51.

Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

Steinke, D. A. (2000). Math, music and the part-whole concept: Linking Steffe and Cobb's model of number sense development to current music research. In M. L. Fernández (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (p. 399). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education. (ERIC Document No. SE 064 088)

Steinke, D. A. (1999). Adults' understanding of number: Three groups interviewed for Steffe and Cobb's 3 stages. *Journal of Adult Education, 27* (1), 24-29.

University of Iowa (1996). *Iowa tests of basic skills*. Itasca, IL: Riverside.

Wright, R. J. (1994). A study of the numerical development of 5-year-olds and 6-year-olds. *Educational Studies in Mathematics, 26,* 25-44.

MENTAL REPRESENTATIONS – THE INTERRELATIONSHIP OF SUBJECT MATTER KNOWLEDGE AND PEDAGOGICAL CONTENT KNOWLEDGE – THE CASE OF EXPONENTIAL FUNCTIONS

Günter Törner
Universität Duisburg
toerner@math.uni-duisburg.de

The exponential function ... it looks easier than what it is.

Introduction and the Research Field

Growth of mathematical understanding is the result of a comprehensive mental restructuring process. Thus observing and understanding growth first necessitates grasping the accompanying transformation process, which then implies taking stock of the respective initial and final states. Based on such analysis, one is capable of estimating and probably designing the results of possible learning supporting measures (or activities).

A central claim of the philosopher Wittgenstein is that thought is limited by language. Mathematics can be understood as elaborated language, whereby graphic elements are viewed as integral parts of this language. Development in this expert language is insofar relevant for mathematical thought and, vice versa, mode of use of this language presents us with insight into such thought. As Davis (1992) stressed, the 'study of mental representations' is a paramount didactical research task: "... *One knows very little about how someone thinks if one has no knowledge at all of his or her mental representations...*" In light of the above-mentioned background this thought is of central importance. It is beyond dispute that not too many results are available in the research literature, especially for topics of higher mathematics. Of course, mental representations can only be opened indirectly on the macro level. Inasmuch, even the adequate methodology of data gathering is a research topic of its own. Whereas subject matter knowledge of the persons in question can be observed through tests, such a data gathering can only provide limited results on the underlying representations. But how should we observe them?

Mental representations of mathematical objects are part of the comprehensive (student) conceptions on these objects. We distinguish between a *microstructure* and a *macrostructure* of mental representations. The papers in Confrey (1991) represent a comprehensive research contribution towards possible (micro-) mental representations of exponential functions; we however are primarily interested in researching the *macro-view* of the subject-specific network structures and their links to the outside. Besides criticism occasionally expressed here, we employ on the whole Pines' (1985, p. 101) idea of conceptual structure, namely that ... *the emphasis here is not on the ele-*

ments, although they are important to a structure, but on the way those elements are bound together.

Research Hypotheses and Research Questions

Our research is based on several initial hypotheses:

(A) The interrelationship of knowledge and beliefs. Mental representations are not primarily tied to 'hard' knowledge, but have substantial reference to 'soft' information (see e.g., Confrey, 1990). In this context, the open question of the interweaving of knowledge and beliefs becomes more important. . Our dualistic *knowledge – belief* view is probably reflected in the opposition of *formal* versus *informal knowledge*. Once again, it becomes evident that every separation of knowledge components from belief components must be considered artificial. Ernest (1989) considers the belief variable *'why what we do what we do'* especially significant for the quality of any mathematics education program. When we accept Schoenfeld's (1998) definition of 'beliefs' as '*mental constructs* that represent the codification of people's experiences and understandings', the correlation to mental representations is obvious.

If one asks for models of individual mental representations, then the guiding metaphor that serves our purpose well seems to us to be *a mathematical graph*. Both *vertices* and *edges* can represent subject-matter knowledge as well as corresponding beliefs. However, we do not believe that we are able to describe the 'mental landscape' by this alone. We cannot ignore that mathematical contents are perceived by prospective teachers on the background of (still virtual) classroom teaching situations.

(B) Subject-matter knowledge and pedagogical content knowledge. According to Shulman (1986), one differentiates between subject-matter content knowledge, pedagogical content knowledge and curricular knowledge. This approach has been scrutinized and improved by additional researchers (see in particular Cooney, 1994 and Bromme, 1994). The concept of pedagogical content knowledge has also been questioned by some researchers who have suggested that it is not a discrete category of knowledge at all, but inextricable from content knowledge itself. We consider it important to understand which role pedagogical content knowledge plays and, referring to the above-mentioned metaphor, it can be a vertice as well as an edge.

Pedagogical content knowledge is in some sense 'knowledge about', and there are many arguments (see (A)) for also considering again the respective beliefs, thus including *pedagogical content beliefs* from the beginning. One notes that mathematical subject-matter knowledge about exponential functions has a multiple-directed structure: What is a definition? What is the consequence? What is a sentence? What is a corollary? What is an application? Pedagogical content knowledge however can draw together various aspects, because they are equivalent. The proposition of a theorem can also serve as a definition, through which the previous definition receives the character of a theorem itself.

Following the categorization in Törner (2000), subject-specific information can be distinguished according to its *content extension* (C); the levels to be given here are insofar *global, domain specific* and *content-related.*

Thus the theoretical framework of our *research question* has been mapped out in outlines, namely registering possible mental representations of mathematical objects and, when possible, describing phenomena pointing to areas beyond the singular mathematical object.

Methodology and the Interview Participants

As the mathematics objects under consideration, the topic of 'exponential functions' was selected. This topic is first-semester subject matter, namely in the first-year-calculus lecture, and in other introductory courses, e.g., linear algebra and numerical analysis, exponential functions are dealt with again from different perspectives. In addition, exponential functions are taught in German school mathematics curriculum. The theme is 'exceedingly rich' (Confrey, 1995) ('ideal critical research site') and features a high degree of networking possibilities to other mathematical themes.

The information collection was the result of phenomenological and qualitative research. For this, open interviews were chosen. Interviews were recorded on video and later, result protocols were sent to the interviewees for comment.

Illuminating for the behavior of teachers for Schoenfeld (1998) are *'teaching-in-context'* situations in which in particular original influence factors for teaching mathematics arise. We were therefore interested in a comparable 'replying-in-context' situation, whereby we deliberately kept the interviewees uninformed about the intended interview. It was therefore not possible for the students to prepare themselves for the interview. To prevent the impression of an examination situation however, we pointed out the advantages of experiencing such a situation as a test without having to fear adverse consequences.

The persons who had volunteered to be interviewees were six prospective teacher students of the upper secondary grade level who were already in their third-year-course and had passed an intermediate examination after their fourth semester. It therefore can be assumed confidently that they will reach their teaching qualifications within three additional semesters.

Data Gathering

The six interviews were held in 2000. The participating students were presented the scope of the actual research beforehand in order to relieve especially the 45 minute conversation of the character of an exam. The contents of questions intended to initiate conversation and induce narrative reports were of the following topics:

- Where were you confronted with exponential functions for the first time? (type of introduction)

- In what way were exponential functions defined? At school? At university? What do you think of e^x?
- What do you understand by e? Which decimals of e are you familiar with?
- Name some defining properties of exponential functions (characteristics)!
- Compare the treatment of this theme in school and at university.
- Report on the inner-mathematical and outer-mathematical importance of exponential functions.
- Which historical aspects of exponential functions are you aware of?
- Are there information gaps about exponential functions that would like to fill in?

Along these technical guidelines intended to illuminate the network around the e-functions, prompting questions were asked for that were intended to provide spontaneous responses. We believe that students' conceptions can be confirmed only through methods that encourage students to be expressive and predictive:

- To what extent was each respective learning phase demonstrative from an affective perspective resp. to what extent are emotionally-charged memories noticeable in hindsight?
- What is each respective assessment based on? (through teaching personal, independent confrontation)
- Which evaluation motive in combination with the meaning of exponential functions are significant for them, and which central? Usefulness of exponential functions? Beauty of the relevant mathematical features? Mathematical centrality? High complexity?
- Did they experience the process of knowledge acquisition organically?

Following the reference in Calderhead (1996) that teachers' knowledge may be better represented in terms of metaphors and images, leeway for metaphoric assessment was provided. Thus we asked the following question:

- If the different functional classes were comparable with animals in a zoo, which species would be assigned to exponential functions?

Results

For the sake for brevity, only a few results can be referred to and summarized. For the same reason we abstain from relating the results to the 6 individual interview partners. The classification of the results into the terminology of concept images from Tall and Vinner (1981) can also not be responded to here. According to the above mentioned steps (A) and (B), we exemplary refer to the individual observed results; to prevent repetitions of the observations we categorize the comments according to content criteria whereby we limit ourselves to just a few positions.

The Euler number e. In particular the categories (A) and (B) are very strongly interwoven. Alone the decimal number e = 2.71 82 82... is known by all the students up to the numeral before the decimal point, two further students name two further numerals after the point. It is commented that there must be further numerals behind the point, one of the students gets tangled up in the infinity of decimal quotient development with the transcendental properties of e, which points to deficiencies in the understanding of real numbers. Vaguely represented is also information about equivalent mathematical definitions of the number e. The importance of this number is not doubted as such, nevertheless this circumstance is only indirectly reached: "*... Euler must have discovered this, he is famous...* ", without stating any precise details as to his historic achievement or as to his personal biography: "*... Don't ask me for dates.*"

What is further emphasized below is the interwoven nature of factually correct information by the students and quite reservedly expressed imprecise knowledge featuring on the whole belief character. Concerning pedagogical content knowledge, the statement of only one student strikes the core of this issue: "*... the Euler number is not really defined, it is constructed, it is a kind of natural constant in mathematics.*"

Definition of the exponential function to the basis e from a mathematical view. The graphic representation of the e-function does not present a problem to any of the students. However, three students have to correct their statements after the interviewer made objecting comments referring to the graph of the general exponential function a^x, with respect to the variance of the basis a, and to some characteristic function points. Only one of the students pointed out that the function e^x is qualitatively something different than 2^x. The response of the interviewer as to the function value at a prominent argument, e.g., the circle constant Pi, is answered by all the students as question easily solved by a (not available) pocket computer (if only it *were* available!). It does not appear to the interviewees to be particularly exciting that in-depth and historically relevant mathematical definition processes (the fundamental research aspect) underlie this (e.g., in the case of e^{Pi} for the basis e as well as for the exponent one is dealing with infinite decimal series).

Representation of the exponential function. The observations make clear that the students uncritically associate the e-function with a monotone increasing graph *(iconic representation)*. The answers become uncommiting when it comes to stating characteristic parameters for the specific growth behavior ('exponential growth')! This representation is insofar of only limited evaluation reliability (for solving mathematical tasks for students).

Not one student articulates that exponential growth on the one side stands in contrast to exponential convergence towards zero on the other side.

All students share the reassuring view that there exists a tool, namely a pocket calculator, which represents the function *(enactive representation)*. This rather comfortable representation entails the danger of an almost negligent simplification of viewing the exponential function e^x in a naive sense as an exponential function with simply a slightly more complicated basis. It is less this naive representation as such which has to be categorized as reserved but moreover the uncritical, mathematically insensitive use of such a reductionistic view.

The observations feature another form of representation of the exponential function: the *symbolic representation* via $f' = f$. A students states: "... *This occurs only once...*" All test persons (two candidates however only after prompting) recalled the property of derivation $f' = f$. None independently noted that this differential equation could possibly characterize the e-function axiomatically, thus providing a sensible symbolic representation.

The mathematical observer is however sobered when viewing very subjective reasons associated with this property. It is the *mnemotechnical simple structure of the differential equation*, which remains in the mind: "... *I will be able to tell this to my grandchildren in fifty years...*" Thus, it is not the mathematical centrality, the beauty of the presentation or similar assessment criteria, which is predominant. "...*You simply cannot make a mistake when differentiating...*", was one comment.

The mentioned three representations of the test persons are only weakly interlocked. Only two students explain the relation between the central differential equation property $f' = f$ and the limit value definition of the Euler number e.

The equivalent approaches usually dealt with in the academic study of mathematics on the series definition or the limit value definition, possibly through an axiomatic characteristic, do not play a primary role in the reports. It is only of secondary importance to the students that mathematics makes possible various different terminological approaches for the definition of its objects.

Pedagogical content knowledge. From the author's perspective, an integral role is played by the pedagogical content knowledge as well as the pedagogical content beliefs. These pedagogical content beliefs have a networking function, they represent orientation information extending beyond the mathematical context. In particular *the relevance aspect of exponential functions* is to be mentioned here. It appears that in the presentation of mathematical contexts on the exponential functions, this aspect is only partially taken into account. Only one test person, who had however enrolled for physics as a second major proved an incomparable comprehensive and highly networked pedagogical content knowledge concerning the relevance of exponential functions.

On the relevance of the exponential function. One hardly needs to present any evidence here on the importance of the exponential function for other fields within mathematics in the narrow sense (inner-mathematical aspect) and also for other sciences drawing heavily on mathematics (outer-mathematical aspect). Inner-mathematical centrality is hardly mentioned in the statements. Due to the fact that the exponential function in school is "... *a completely different one...*" than the one in academic studies ("... *that was the e-function from school...*", we conclude that the integration has not yet been sufficiently consolidated yet. "... *As Euler, who achieved a lot in mathematics, developed the exponential function, it should be allocated a central role in mathematics"*.

All the participants initially emphasize the outer-mathematical relevance aspect, but further statements here remain often non-committing when it comes to exactly describing facts. This observation is based on the unconventional question of naming a zoo animal as a representative of the function class of all functions in the zoo.

By means of relating a zoo animal to an exponential function, information was intended to be elicited on the role of this function in comparison to other functions. The answers of the students are: "... *no small animals since the growth of the function is large... but, one could also imagine rabbits, which increase their population quickly! Isn't there the famous mathematical problem [Fibonacci problem]...* " or "... *maybe the human being, who exists everywhere...*" Another student proposed: "... *the lion!*" Then he hesitated: "... *But the lion does however not occur everywhere, only in Asia and Africa...*" Finally it was remarked: "... *the giraffe with its long steeply rising neck reminds me of graphs... or maybe also a bird which can rise above everything else...*"

It is common practice to speak of the so-called exponential growth, which links onto a differential equation condition. Ideas that alterations can be measured by derivations are not evident and point to a weakly developed basic understanding of this mathematical term. "...*The exponential function stands for exponential growth... what that exactly means I cannot say...*" The information presented here is almost wholly of belief character, e.g., when one refers to the statements of other persons or bases one own statement on the basis of another person: "...*Mr. S. [a university lecturer] reported once in an preparatory university mathematics course on growth of fish populations during the First World War... My fellow-student, who studies engineering, often employs the exponential function in the complex rabbit task of Fibonacci.*" Some narrations also have almost episodic character: "...*Our teacher always says: Look there comes an e-function again"*.

Emotional loadings of mathematical topics. Through the description of the interview topic at the beginning of the interview, a spontaneous situation comparable to that of a test was created. All participants confirmed that the information about the topic of the interview immediately caused emotional reactions within them which encompassed the entire spectrum, from 'dismay' to relief; the mean of such a qualitative distribution can more likely be accounted in the negative range.

The candidates all only had vague memories of the introduction of the subject 'exponential functions' in school curriculum. The visual image of the run of curves remained, whereas the term e^x primarily played the role of indicating a special function in their memories. Besides this iconic representation, the enactive representation of e^x as the name of a calculator button is often referred to, especially when asking about e^p. Two test persons mentioned the correlation to natural logarithms as the respective inverse function in reference with the symbolic representation. Here it cannot be overlooked that 'mathematical bridges' also emit lasting 'negative affective charges' (from the context of logarithms) onto the e-functions.

Conclusion

Overall the results the author was sobered by the interviewees' weak and modest subject matter knowledge of exponential functions. The same applies to detailed information: precise definitions, characterizing properties, importance for application, importance of the Euler number e, correct sketching of the graph a^x dependent on the basis a, the asymptotical behavior at infinity, intersection with the y-axis, etc.

When the fact that beliefs are held with varying degrees of conviction is deemed constitutive, the information that was hesitantly presented by the test persons has overtly belief character. Additional questioning tended to increase uncertainty and it is often left unanswered how possible doubts could be cleared up in a mathematically uncritical fashion. Very often, it is related to authority figures ("*...we learned that in our course...*"; "*... a student told me...*"). This indicates that the knowledge is from unreliable non-academic sources or justified by reference to learning phases in which one vaguely experienced the contents.

The experiment exposed again that the difference between knowledge and beliefs is rather theoretical, in analysis, dividing lines can hardly be drawn, even the interviewees themselves did not articulate differences between provable, theoretically present mathematical facts and approximate, imprecise memories. Gaps registered as deficits in the eye of the observer were accepted as natural by the test persons and not considered disturbing, even the fuzzy character of some information did not cause uncertainty. Inasmuch a discussion about subject matter knowledge always includes inventory taking of subject matter beliefs.

Metaphorically speaking, the experiment however provides evidence that the knots in a cognitive knowledge network carry emotional charge in which the previous experience of knowledge acquisition are stored. Cognitively neighboring elements appear to radiate onto one another with respect to their affective charges. One student interestingly reported of the e-function being burdened by the even more complicated logarithm function: "*... this pair are inseparably linked...*"

When teaching mathematics (in a beginners lecture), a generally understood 'pedagogical context knowledge' apparently is normally only allocated to a minimal role. Mathematically constituted networks which are substantial for experts (e.g., test developers) are only taught weakly. Networks receive supplementary stability only through interdisplinary work or through a specific changes of perspective. Students having multi-modal (mathematically equivalent) representations of the mathematical object 'exponential function' is the exception to the rule. The action character of knowledge (enactive representation) is often dominant. Possibly the deficits in subject matter knowledge are not only covered by an insufficient pedagogical content knowledge, but even favored because respective gaps in the metaknowledge do not result in some pressure in further questioning.

References

Bromme, R. (1994). Beyond subject matter: A psychological topology of teachers' professional knowledge. In R. Biehler, R. W. Scholz, R. Strasser, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 73 - 88). Dordrecht: Kluwer.

Calderhead, J. (1996). Teachers: Beliefs and knowledge. In D. C. Berliner & R. Calfee, (Eds.), *Handbook of educational psychology* (pp. 709-725). New York: Simon & Schuster, Macmillan.

Confrey, J. (1990). A review of the research on student conceptions in mathematics, science and programming. In C. B. Cazden (Ed.), *Review of research in education* (Vol. 16, pp. 3-55). Washington DC: American Educational Research Association.

Confrey, J. (1991). The concept of exponential functions: A student's perspective. In L. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 124-159). New York: Springer.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of changes, and the multiplicative unit. *Educational Studies in Mathematics, 26*, 135-164.

Cooney, T. (1994). Research and teacher education: In search of common ground. *Journal for Research in Mathematics Education, 25*, 608-636.

Davis, R. B. (1992). Reflections on where mathematics education now stands and on where it may be going. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 724-734). New York: Macmillan.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In C. Keitel, P. Damerow, A. Bishop, & P. Gerdes (Eds.), *Mathematics, education and society* (pp. 99-101). Paris: UNESCO Science and Technology Education Document Series No 35.

Ernest, P. (1999). Forms of knowledge in mathematics and mathematics education: Philosophical and rhetorical perspectives. *Educational Studies in Mathematics, 38*, 67-83.

Even, R., & Tirosh, D. (1995). Knowing that and knowing why: Subject-matter knowledge and knowledge about students as sources of mathematics pedagogical knowledge. *Educational Studies in Mathematics, 29* (1), 1-20.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed*.), Handbook of research on mathematics learning and teaching* (pp. 147-164). New York: Macmillan.

Pines, A. L. (Ed.). (1985). Towards a taxonomy of conceptual relations. In L. West & A. L. Pines (Eds.), *Cognitive structure and conceptual change* (pp. 101-116). New York: Academic Press.

Schoenfeld, A. (1998). Toward a theory of teaching-in-context. *Issues in Education,* 4 (1), 1-94.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15* (2), 4-14.

Tall, D., & Vinner, S. (1981). Concept images and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D.A. Grouws (Ed.), *Handbook of research on mathematics learning and teaching* (pp. 127-146). New York: Macmillan.

Törner, G. (2000). Structuring mathematical belief structures – Some theoretical considerations on beliefs, some research questions and some phenomenological observations. In M. L. Fernández, (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 499-508). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

DOES TEACHING MATTER? EXPLORING THE RELATIONSHIP BETWEEN TEACHING AND THE GROWTH OF STUDENTS' MATHEMATICAL UNDERSTANDING

Jo Towers
University of Calgary
towers@ucalgary.ca

Focus of the Study

This paper presents some of the findings of a three-year research study that explored the relationship between teachers' classroom interventions and the growth of students' mathematical understanding. In investigating this relationship, other researchers have considered such areas as the nature of teachers' talk in mathematics classrooms and particularly their use of questions (e.g., Martino & Maher, 1994), the nature of students' talk in mathematics classrooms (e.g., Ball, 1991), and the nature of students' understanding of mathematics (e.g., Pirie & Kieren, 1994; Sierpinska, 1990). Most recently, and of most relevance to my own work, researchers have begun to coordinate investigations of teachers' actions with those of students' learning (e.g., Cobb, Boufi, McClain & Whitenack, 1997). Building on these works, this paper presents episodes of classroom data that demonstrate how teaching contributes to the growth of students' mathematical understanding.

Theoretical Framework

This study draws upon an enactive theoretical framework. The enactive approach was first postulated by Varela, Thompson and Rosch (1991) and is drawn from a diverse collection of recent and ancient thought, including Buddhism, continental philosophy, biology, and neuroscience. Enactivist theorists situate cognition not as problem solving on the basis of representations, but as embodied action. Significantly for my study, such a shift enables understanding to be seen as a continuously unfolding phenomenon, not as a state to be achieved.

Methodology

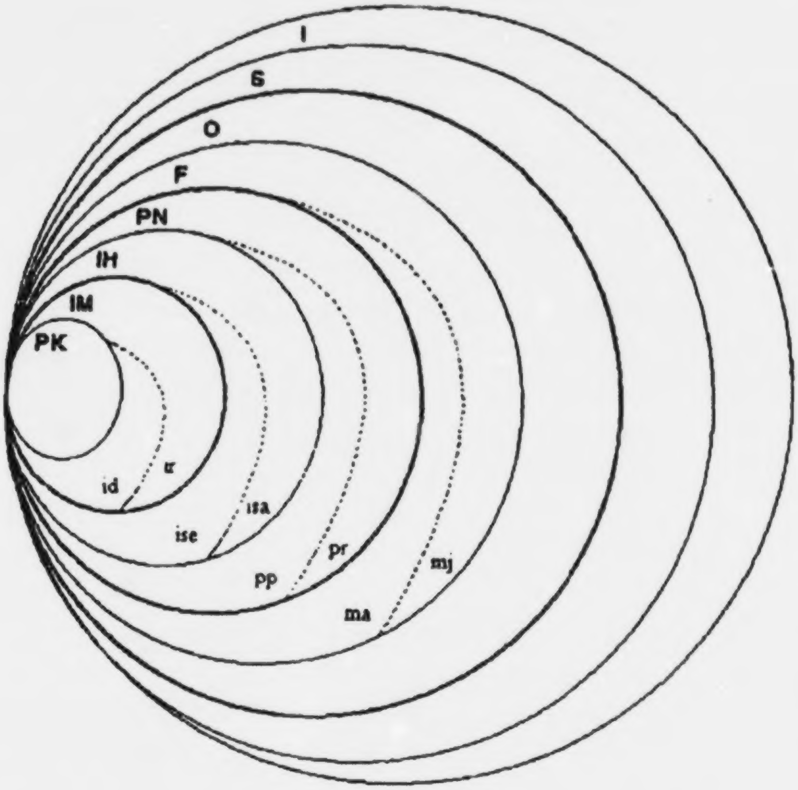
I adopted a qualitative case study approach in order to provide a rich and detailed analysis, and a deep and comprehensive description and interpretation, of the processes of classroom interaction leading to the growth of mathematical understanding. As part of the study two 'cases' were documented. The first case concerned data collected in my own classroom at a time when I was a full-time teacher of mathematics in a small, rural, British secondary school. Students of (North American equivalent) Grades 6 and 7 were participants in this strand. The second case concerned data collected in a large, urban high school in Canada, a single mathematics teacher and a group of her Grade 9 students being the focus of this strand. In order to develop an understanding of each classroom environment I engaged in detailed analyses of video-recorded lessons, field-notes, copies of students' work, my own journal entries, and video-recorded interviews with the Canadian teacher and with the student participants in both strands of the study.

Analysis of Mathematical Understanding and Teachers' Interventions

In order to understand the nature of the unfolding mathematical understandings of the students in my study I adopted the Dynamical Theory for the Growth of Mathematical Understanding (Pirie & Kieren, 1994) as a theoretical tool for analysis. It shares and is intertwined in the enactivist view of learning and understanding as an interactive process. The theory, developed to offer a language for, and way of observing, the dynamical growth of mathematical understanding, contains eight potential levels for understanding. A diagrammatic representation or model is provided by eight nested circles (Figure 1). Within each level beyond Primitive Knowing are complementarities of acting and expressing (such as *image doing* and *image reviewing*, etc.), further elaborating the process of understanding. It is beyond the scope of this paper to define and describe all of these modes of understanding and I refer the reader to Pirie and Kieren's work for further explanations of these terms (e.g., Pirie & Kieren, 1994). A familiarity with Pirie and Kieren's work is not essential to a reading of this paper, however it is important to understand the following general features of their theory.

The nesting of levels and the associated traces of students' understanding, illustrate the fact that growth in understanding need be neither linear nor mono-directional. Each layer contains all previous layers and is included in all subsequent layers. This set of unfolding layers suggests that any more formal or abstract layer of understanding action enfolds, unfolds from, and is connected to inner less formal, less sophisticated, less abstract and more local ways of acting. Growth occurs through a continual movement back and forth through the levels of knowing, as the individual reflects on and reconstructs his or her current and previous knowledge. This shift to working at an inner layer of understanding actions is termed *folding back* and enables the learner to make use of current outer layer knowing to inform inner understanding acts, which in turn enable further outer layer understanding. The theory suggests that for understanding to grow and develop, folding back is an intrinsic and necessary part of the process.

I created traces, known as mappings, of the students' growth of mathematical understanding using an adaptation of a model described by Pirie and Kieren. Appendix A shows one such mapping for one of the students in my study, Kayleigh. Similar mappings were created for all the students in the study. Readers will notice that I have moved away from using the embedded rings to represent the various modes of understanding, however this is not a conceptual shift. I continue to recognise the modes as embedded within one another. My rather more linear representation began as a pragmatic move resulting from the difficulty I began to face as I tried to fit a representation



Key

PK	Primitive Knowing
IM	Image Making
IN	Image Having
PN	Property Noticing
F	Formalising
O	Observing
S	Structuring
I	Inventising
id	image doing
ir	image reviewing
ise	image seeing
isa	image saying
pp	property predicting
pr	property recording
ma	method applying
mj	method justifying

Figure 1. The Dynamic Theory for the Growth of Mathematical Understanding.

of a student's understanding over the period of several months onto the diagrammatic form favoured by Pirie and Kieren. However, rather than restricting my analysis, I feel that this development enabled patterns to unfold that may not have been so evident on the conventional mapping diagram. For instance, on completing Kayleigh's mapping my attention was immediately drawn to the pattern of growth of understanding revealed in the fourth lesson in the sequence. I began to wonder what forms of teaching interventions might have occasioned such growth, and so I began to concentrate my attention on the teachers' interventions.

Teachers' Interventions

I developed fifteen intervention themes to describe the teachers' actions-in-the-moment[1]. Though some of these themes have been defined and described elsewhere (Towers, 1998; 2001), the proposed paper focuses on several themes that have not previously been presented. I drew together the traces of the students' understandings with the teachers' interventions as I had characterised them in terms of themes and paid particular attention to turning points in the pathways of growth of understanding revealed by the traces – those moments when a student extended to an outer mode of understanding, or folded back to an inner one. My attention was drawn to episodes in the data of *repeated folding back*, those times when a student repeatedly folded back and extended his or her understanding of the mathematical concept being studied within a relatively short period of time, often one lesson or one task within a lesson (see Lesson 4, Appendix A). I reflected on the teaching that occurred during these episodes in terms of the intervention themes I had identified and noted that particular intervention themes were associated with the phenomenon of repeated folding back. These themes are *Shepherding*, *Inviting*, *Retreating*, and *Rug-pulling*. Shepherding is an extended stream of interventions directing a student towards understanding through subtle coaxing, nudging and prompting. Inviting is the suggesting of a new and potentially fruitful avenue of exploration. Retreating is a deliberate strategy whereby the teacher leaves the student(s) to ponder on a problem. Rug-pulling is a deliberate shift of the student's attention to something that confuses and forces the student to reassess what she or he is doing, and often results in a return to Image Making activities by the student.

The Relationship Between Teaching and the Growth of Students' Mathematical Understanding

The data on which I focus here was collected in my own classroom. The particular class was equivalent to North-American Grade 6, and the student on whom I will focus, Kayleigh, was a relatively strong student in this mixed-ability class. Although Kayleigh worked with a partner (Carrie) in this lesson, for reasons of clarity in the space available I will refrain from discussing Carrie's understandings and contributions in this paper. In fact, Carrie spoke infrequently during this lesson, though as I have described elsewhere that does not necessarily imply a lack of understanding on her part (Towers, 1996). In the particular lesson on which I will focus (Lesson 4, Appendix A), I introduced an open-ended problem to the class by describing a situation in which a gardener wished to construct a square pond in her garden and frame the pond with 1m x 1m paving stones (see Figure 2). I worked with the class as a whole to introduce the problem and frame the difficulty – how might the gardener work out how many paving stones would be needed for any size of square pond (to begin with the pond itself was to be whole-units in dimension, e.g., 3x3 or 14x14). The class was then given squared paper to assist with drawing the ponds, if required, and they worked in pairs on a solution. By the end of the lesson Kayleigh and her working partner had moved beyond square ponds to rectangular ponds and then L-shaped ponds, each time generating correct algebraic expressions for the relationship between the size of the pond and the number of paving stones required to surround it. Kayleigh's pathway of understanding was by no means linear, however, as can be seen in Appendix A, and her growth of understanding was marked by repeated folding back. The posing of an open-ended question, including the prompt to explore alternatives and extend the problem further, is the kind of teaching intervention that I have called *Inviting*. In this case, the

Inviting intervention appeared at the beginning of the lesson and was directed to all students (and hence has some of the flavour of the kind of intervention Simmt (1996) calls "variable-entry"), however it should be stressed that Inviting interventions are not confined to the category of teachers' lesson-opening gambits and may describe other interventions that teachers make, either to groups or single students, at any moment in a lesson.

In the following excerpt, Kayleigh has already generated a correct algebraic expression linking the size of a square pond to the number of paving stones needed to surround it (see Figure 2) using one variable to represent the length of the side of the inner pond, and has moved on to consider rectangular ponds belonging to the "family" shown in Figure 3. She has worked for some minutes with her partner and has generated an expression for this new family of ponds, again using one variable to represent the length of the inner pond (note that only one variable is needed as this family of ponds has a fixed width).

K: Kayleigh, C: Carrie, J: Jo (me, the teacher)

K: We've got the formula.

J: That was quick. You've got your chart straight away.

K: We haven't, we haven't drawn it because we knew what we were doing [*referring to the diagrams*].

J: All right.

K: It's times by two and add six.

J: It certainly is. How did you do that?

K: Well, I just looked at it and I saw this one [*indicating one of the entries on her chart*] and I thought times it by two and add six 'cause that's pretty obvious.

C: Yeah, 'cause that's [inaudible]

J: Ah, right.

K: And then I thought well I'll check it so I don't [*pause*]

J: OK

K: And then [inaudible]

J: So, it was nothing really to do with the diagrams then? You didn't actually count squares in a certain way?

K: No, I didn't.

J: OK. That's fine. Now then, I want you to, because that formula is fine for those, but what would you do if the rectangle I suggested was this size? [*Begins to draw a 3x7 pond and surround it by paving stones*]

K: Well, you'd have to do *n*. You'd have to do two times three which is six. So it'd be *n* times six. *n* six. Six *n* times whatever. Add no, *n* six *n* add…

J: [*Completes the inner pond*] So that's your middle.

K: Well you'd get the length [*pause*]

J: I'm sorry to make a mess of your paper when you've done it all so beautifully.

K: Well you'd get like. After you'd just had it like that [*indicating an inner pond of dimensions 1x7 by covering part of the diagram with her hand*] it would have been…

J: Ah, yes I know that.

K: If you'd just had it like that it would have been seven, so it would have been seven, what the *n* would have counted as seven.

J: It would.

K: But if you had [*uncovering the entire diagram*] and you'd have to do two, you'd have to do the length, you'd have to do a width as well.

J: You will. So that's your next problem, girls. I want a formula to work for any size of rectangle, that way [*indicating length on the diagram*] or that way [*indicating width on the diagram*].

At this point I moved away from this pair of students and left them to work on the problem. This is an example of Retreating—a deliberate decision to leave students to ponder on a problem. I also want to suggest that this extract contains an example of Rug-pulling. In offering the diagram of the 3x7 pond I was attempting to disrupt Kayleigh's current image of the problem, which was centred on families of ponds for which it is possible to generate an algebraic expression linking the size of the pond and the number of paving stones needed to surround it using only one variable. For example, the first two families of ponds she had investigated are those for which the first two members are shown in each of Figures 2 and 3 (although as the above transcript shows, it should be noted that Kayleigh had not drawn any diagrams for the rectangular pond family shown in Figure 3 and had simply visualized the ponds, gen-

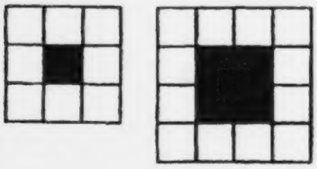


Figure 2. Rectangular ponds.

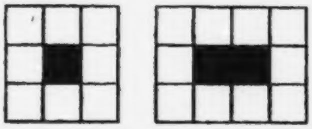


Figure 3. Rectangular ponds.

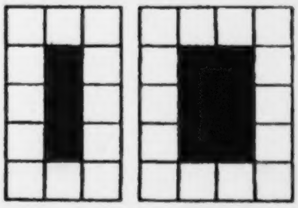


Figure 4. Rectangular ponds.

erated a table of values linking pond size with number of surrounding paving stones and produced a correct algebraic generalization for this relationship). In each of these cases, only one variable is required to represent the inner pond (in Figure 2 because each inner pond's length and width are equal and in Figure 3 because the width of this family of ponds remains constant). In offering the 3x7 pond I hoped to challenge Kayleigh's certainty, possibly confuse her, probably prompt her to return to Image Making activities to build a more complex image, and hopefully help her understanding to grow. As can be seen from the above transcript, before I had even finished drawing the 3x7 pond and paving stones Kayleigh appeared to have anticipated my strategy and was already talking about having "to do a width as well" as a length. Upon hearing this statement I felt confident in Retreating, which I did immediately, sure that Kayleigh had understood the implications of the dimensions of the diagram I had drawn. This was not to be, or at least not immediately. Again without drawing additional diagrams, but using the diagram I had drawn as a prompt, Kayleigh and her working partner created a table of data relating the size of the inner pond to the number of paving stones surrounding it for the family of ponds for which the first two members are shown in Figure 4.

The students concluded that they still only needed one variable in their algebraic expression linking these two sets of data because the width of the pond remained constant at 3 units. So, although I had intended my diagram to provoke Kayleigh and her partner think about expressions involving two variables, their reading of my diagram provoked no such shift in understanding - at least, no immediate shift. The videotape shows the pair breezing through the collation of data (although they did not draw any diagrams they did cover and uncover the diagram I had drawn to reveal successively larger ponds as they generated their table of data) and generating an algebraic expression involving one variable to fit their data. They paused at this point and Kayleigh raised her hand to attract my attention. Before I arrived at their desk, however, the following exchange took place (they are using *n* to represent the length of the inner pond and *w* to represent its width):

K: Or, it could otherwise be written as [*pause*]. Now this may be 'cause she might do [*pause*]. If you had more than three on the thing it would be [*pause*]. If you had more than three in the width of the rectangle inside it would be two *n* add *w*.

C: Yeah, yeah, yeah.

K: Add two *w*. Add two *w* add two. If that makes sense [*writing* $2n + (2w + 2)$].

C: I think it makes sense.

As the teacher, as I arrived at their desk I was confronted by a confusing array of information. Their final algebraic expression, $2n + (2w + 2)$, did not match the values in their table (because in this case they had chosen to collect data relating the total

332

number of squares in the pond to the number of surrounding paving stones rather than relating the *dimensions* of the pond to the number of surrounding paving stones), however I did not even notice this anomaly at first. I was, of course, fully expecting to see an expression involving two variables, as this is the problem I thought I had pointed towards before Retreating, so I immediately focused upon this expression, however the expression they had created was incorrect. There followed a complex conversation as the three of us tried to interpret each other's understandings of the information before us. Eventually, we agreed upon a new expression with an additional set of parentheses, $2n + (2\,(w + 2))$. As I retreated a second time I was fairly sure that Kayleigh understood why the second set of parentheses was needed to accurately represent the relationship between the inner pond and its paving stones, but I ignored (deliberately, though with the teacher's usual unease at leaving an incomplete or incorrect expression unchallenged) the fact that the final expression at which we had collectively arrived contained a superfluous set of parentheses. Retreating leaves room for student discovery but also student error, and a teacher needs to decide when something would be better addressed another day.

Though in this instance Kayleigh was not confused by my intervention, as my data reveals is often the result of a Rug-pulling intervention, I do consider my prompt of the 3x7 pond to be a Rug-pulling intervention as it prompted Kayleigh to reassess her use of variables and thereby extend her image of the problem (though her recognition during the episode transcribed above that she needed two variables to represent the "width as well" as the length seemed initially to be forgotten and then later recalled, as though she suddenly remembered her conversation with me and realised what I'd been prompting her to consider). My intervention also succeeded in provoking Kayleigh to fold back to Image Making activities, as she used my diagram as an aid to her thinking where previously she had not found it necessary to use diagrams as an aid.

The kind of teaching explored in this example falls within an intervention style I have called Shepherding. Here the teacher points the student towards understanding through a more subtle nudging, coaxing and prompting than is common in more traditional modes of teaching (such as those I call *Showing and Telling* wherein the teacher is the predominant speaker in the classroom and teaching is understood as the giving of information, and *Leading* wherein the teacher involves the students in the activity through frequent questioning but teaching continues to be understood as directing the students to a particular answer or position). In my research I have represented Showing and Telling, Leading, and Shepherding as broad intervention *styles* in contrast to the remaining twelve intervention strategies of which Inviting, Retreating and Rug-pulling are three examples. Intervention styles, as I define them, are broad practices that appear extensively within a particular teacher's activity. The two teachers in this study tended to draw upon one of these three styles predominantly and the others less frequently. An intervention strategy, on the other hand, I characterise as a brief intervention, and the two teachers appeared to have a repertoire of many of these strategies upon which to draw. The three intervention styles, Showing and Telling, Leading, and Shepherding, could be used to describe all of the teaching I studied, and many of the other themes fall within their descriptions. Hence I was able to cluster the intervention strategies, Inviting, Retreating, and Rug-pulling within the intervention style, Shepherding. I identified similar intervention clusters for the styles Showing and Telling, and Leading (see Towers, 2001).

In order to understand the relationship between the teaching interventions and the growth of Kayleigh's understanding during this lesson it is important to consider Kayleigh's growth of understanding during other lessons in this teaching sequence and to consider the intervention styles and strategies prevalent in those lessons. The reader will note the particular pattern of the trace of Kayleigh's growth of understanding during the lesson on which I have focused (Lesson 4) in contrast to surrounding lessons. Analysis shows that the teaching in those surrounding lessons falls predominantly within the intervention style I have called Leading. In those lessons the mathematical tasks were not open-ended (in contrast to the "Ponds" activity I have described here). As the teacher, I tended to intervene more frequently during the surrounding lessons and those interventions were more directive in nature. My explanations to students were more frequent during those surrounding lessons, and, although there was still lots of student discussion as was the norm in this classroom, students were generally not encouraged to extend the task as set, in contrast to the "Ponds" investigation where I actively encouraged students to consider alternative possibilities as they investigated the geometrical and algebraic relationships. Though I only had space here to include one brief example of the growth of understanding associated with Shepherding and its related cluster of intervention strategies, it should be noted that the data indicate that a pattern of repeated folding back within the trace of a student's growth of understanding (which, recall, Pirie and Kieren (1994) suggest is necessary for growth of understanding) is consistently associated with Shepherding interventions.

Significance of the Study

This study explicitly addresses the growth of students' mathematical understanding through an exploration of students' mathematical knowing and how teaching contributes to the bringing forth of that knowing. By combining an analysis of students' growing mathematical understanding using the Dynamical Theory for the Growth of Mathematical Understanding with an analysis of teaching using the intervention themes I have developed, I am able to contribute to the existing body of knowledge concerning the relationship between teaching and the growth of students' mathematical understanding. Based on the data I collected during this study I suggest that teaching does matter; that a student's growing understanding, like Kayleigh's, is dependent on, though not wholly determined by, a teacher's interventions, and that interventions consistent with a Shepherding style are likely to promote the growth of students' mathematical understanding.

It should be emphasized that the intervention styles and strategies I have identified serve as more than a categorization of teachers' questions, which is the most common form of analysis of teaching found in the literature. These intervention themes constitute a broad vocabulary for describing teaching. Chazan and Ball (1995) have called for a more complex, explicit, and contextualized characterization of the roles teachers play in classroom interaction, and for a more precise means of describing the telling that teachers do. I offer these intervention themes as a response to Chazan and Ball's call for a new vocabulary, and as a suggestion to teachers for alternative teaching strategies. As such, this work has implications for the teaching and learning of mathematics, may also inform the teaching of other subject disciplines, and may contribute to current understandings of teaching that inform teacher preparation and development.

References

Ball, D. (1991, November). What's all this talk about "discourse"? *Arithmetic Teacher*, 44-48.

Chazan, D., & Ball, D. (1995). *Beyond exhortations not to tell: The teacher's role in discussion-intensive mathematics classes.* Craft Paper 95-2. East Lansing, MI: National Center for Research on Teacher Learning, Michigan State University.

Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education, 28* (3), 258-277.

Martino, A., & Maher, C. (1994, April). *Teacher questioning to stimulate justification and generalization in mathematics*. Paper presented at the annual meeting of the American Educational Research Association. New Orleans, LA.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics*, *26* (2-3), 165-190.

Sierpinska, A. (1990). Some remarks on understanding in mathematics. *For the Learning of Mathematics*, *10* (3), 24-36.

Simmt, E. (1996, April). *Children and parents and mathematics: Occasioning in mathematical cognition*. Paper presented at the annual meeting of the American Educational Research Association. New York.

Towers, J. (1996). False impressions: Listening for understanding in the mathematics classroom. In E. Jakubowski, D. Watkins, & H. Biske, (Eds.), *Proceedings of the Eighteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 527-532). Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Towers, J. (1998). Telling tales. *Journal of Curriculum Theorizing, 14* (3), 29-35.

Towers, J. (2001). *Teachers' interventions and the growth of students' mathematical understanding*. Paper presented at the 2001 annual meeting of the American Educational Research Association. Seattle, WA.

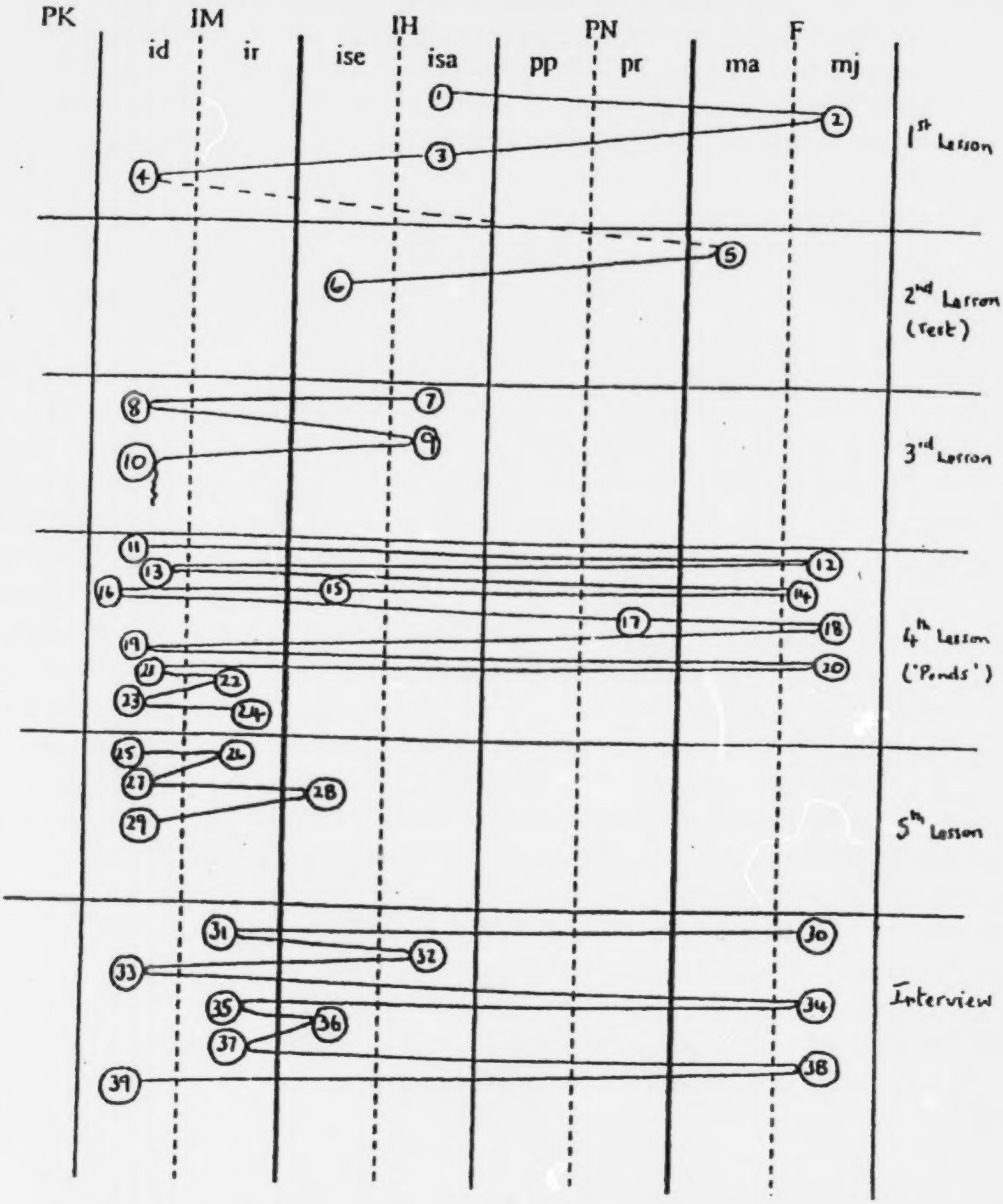
Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge, MA: MIT Press.

Note

[1] The intervention themes I identified were: *Showing and Telling*, *Leading*, *Shepherding*, *Checking*, *Reinforcing*, *Inviting*, *Clue-giving*, *Managing*, *Enculturating*, *Blocking*, *Modelling*, *Praising*, *Rug-pulling*, *Retreating*, and *Anticipating*.

APPENDIX A

KAYLEIGH'S MAPPING DIAGRAM


PK
IM
IH
PN
F
id
ir
ise
isa
pp
pr
ma
mj
1st Lesson
2nd Lesson (test)
3rd Lesson
4th Lesson ('Ponds')
5th Lesson
Interview
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39

BEST COPY AVAILABLE

THEORY, VIDEO AND MATHEMATICAL UNDERSTANDING: AN EXAMINATION OF WHAT DIFFERENT THEORETICAL PERSPECTIVES CAN OFFER

The "μ-Group"
University of British Columbia

Introduction

Mathematical understanding is an intriguing and complex area. In particular the basic question of how to teach mathematics so that students are thought to 'understand' a concept or topic continues to challenge us. As 'learning with understanding' has become more of a focus in the arena of education greater attention is being paid to what is involved in the understanding of a mathematical concept. It is only by knowing, not only what mathematical understanding is, but how it comes about and develops, that we can begin to consider ways to enable and facilitate a greater and deeper understanding in our learners. Through being able to define, explain and account for understanding we hope that teaching and learning will be a more effective process for the participants.

We are presenting one long session of linked papers which consider the notion and nature of mathematical understanding and the use of video as a data source. In recent years a number of different theories and approaches to considering mathematical understanding have been put forward and developed, and a growing number of researchers are making use of classroom video. We will present, discuss and contrast a variety of key ideas in this area and consider their usefulness and applicability in analysing the mathematical actions of pupils working in school classrooms. The session will bring together and demonstrate the collaborative work of the teachers and mathematics educators in the "μ-Group" at UBC.

The first paper is an introductory one on how we work with videotape in our endeavours to get at what goes on when children are thinking and acting mathematically. At the actual presentation this will be followed by the showing of a videotape of a group of children working on a mathematical task, (in these proceedings we offer a brief description). The section of tape was chosen almost at random from our collection of videos, since we believe that almost any classroom video, when carefully analysed has the potential of allow insights into aspects of the teaching and learning of mathematics. This is not to say, of course, that we do not have specific questions in mind when we make the decisions of how and what we will capture on tape, but it points up the value of revisiting taped classrooms from new perspectives. A number of researchers and teachers will then analyse and discuss what has been shown, from a variety of different specific theoretical perspectives that lend themselves to enriching our understanding of what we are seeing. The perspectives will be deliberately diverse, covering analysis of the mathematics in the task, children's misconceptions and epistemological obstacles, the complexity of emerging mathematical reification, teacher interventions and understanding as a dynamical process. Our intent is to alert the reader to the fact that certainly not one, and indeed not even five, different viewpoints can tell us "all" about these children's mathematical learning. At the conference these papers will be followed by a more general discussion in which we will invite the audience to comment on the approaches that have been taken and their contribution to our knowledge of how the understanding of mathematics grows for children. In these proceedings, the papers follow one-another with a comprehensive bibliography will be given at the end of the set of papers.

The Video Clip

The segment we will show is taken from a series of tapes made by one of the μ-group using a fixed video camera focused on three grade 5 children, Allan, Veronique and William[1]. In this extract they have been presented with the three pyramids constructed from multilink[2] cubes illustrated in Figure 1, and told that these are the first three pyramids in a sequence.

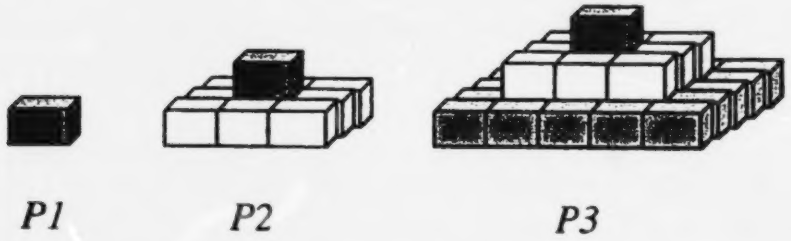


Figure 1. The three given pyramids.

The children are working from the worksheet in Figure 2 and the extract we are analyzing begins with Veronique reading:

> *Using any of the materials that are available, can your group find out how many more cubes will be needed to build a rectangular pyramid that is four levels high.*

Veronique points out that the second pyramid has nine cubes as a base, the third has 25 and states that the fourth will have "seven times seven", "which is forty-nine". William builds the fourth pyramid by adding cubes all over the third pyramid in a covering. (see Figure 3)

Before William has finished building, Veronique and Allan have determined that nine times nine is eighty-one, in answer to the next part of the worksheet. William points out that they have not counted the cubes he has used to cover the third pyramid, then he leaves the physical models and joins the others in calculating eleven times

Rectangular Pyramids

Introduction to Rectangular Pyramids

(demonstration and discussion with teacher)

How many cubes are in pyramid #1 (1 level high)?

How many *more* cubes are in pyramid #2 (2 levels high)?

How many *more* cubes are in pyramid #3 (3 levels high)?

Group Project #1:

Using any of the materials that are available,
can your group:

- find out how many *more* cubes will be needed to build a rectangular pyramid that is **4 levels high**?
- find out how many *more* cubes will be needed to build a rectangular pyramid that is **5 levels high**?
- find out how many *more* cubes will be needed to build a rectangular pyramid that is **6 levels high**?
- *without* actually building, **guess** how many *more* cubes your group will need to build a rectangular pyramid that is **7 levels high**.
- find out by **testing and explaining**, why your group's guess was correct/incorrect.
- To build a rectangular pyramid that is 7 levels high, you need to use this many more cubes:
- *without* actually building, **guess** how many *more* cubes your group will need to build a rectangular pyramid that is **8 levels high**.
- find out by **testing and explaining**, why your group's guess was correct/incorrect. To build a rectangular pyramid that is 8 levels high, you need to use this many more cubes:

Figure 2. The activity sheet of mathematical challenges involving rectangular pyramids that was presented to the students.

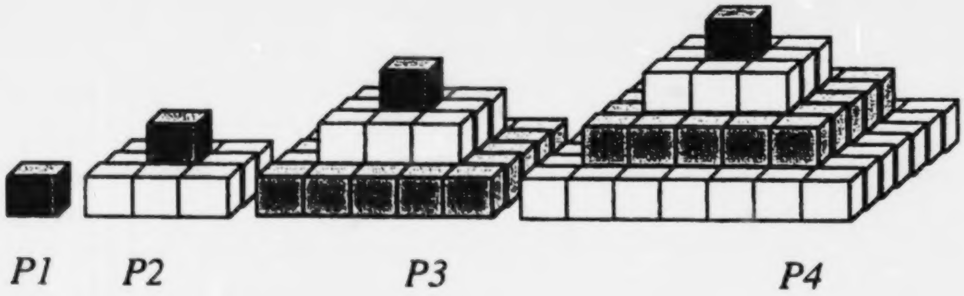


Figure 3. The first four pyramids.

eleven. Allan states that they are not allowed to use a calculator when asked to "guess" the answer for the seventh pyramid and writes in vertical form on the chart paper (see Figure 4), thirteen times thirteen. Allan and Veronique working together are unable to do the two-digit multiplication and use the calculator to produce the answer. The three children then make several attempts to calculate fifteen times fifteen on the chart paper and put their final answer of two hundred and seven as their "guess". They obtain the correct answer on the calculator and Veronique tries again to do the written calculation.

Paper 1: Analysis, Lies and Videotape

Susan E. B. Pirie
University of British Columbia

There are many researchers who work with video tapes and few who write about what they actually do when they are doing this work. This can be problematic if it is assumed that each works as the others do. I contend that 'who we are, where we place the cameras, even the type of microphone that we use governs which data we will gather and which we will lose' (Pirie, 1996, p.553) and as early as 1966, Byers reminded readers that it was not the camera which took the pictures, but the person operating it. The possibilities concerning the ways in which videos are taken will, therefore, be set out, and the reasoning behind some of the decisions that are made by the "μ-Group" when faced with the hard choices confronting all those who use a camera will be given. Erickson (1992) notes that the use of video is particularly useful when 'the distinctive shape and character of events unfolds moment by moment, during which it is important to have accurate information on the speech and non-verbal behaviour of particular participants in the scene' (p.205). This is undoubtedly the case when we are trying to access the mathematical understanding that is being built by children. As it was for Mercer and Hammersley and undoubtedly others, the facility to revisit the data, to be able to look again and again and again at a particular action or

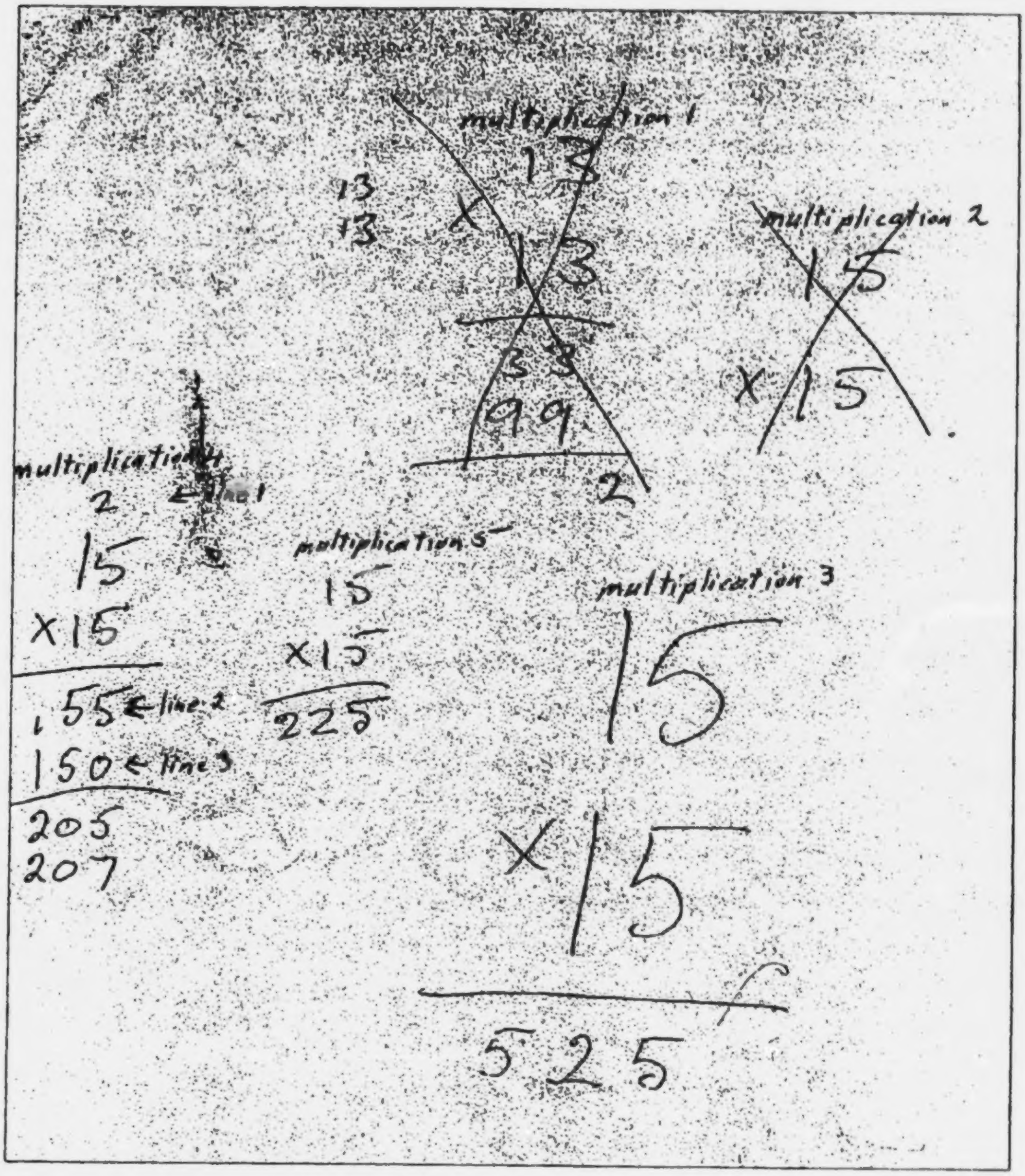

multiplication 1
13
× 13
33
99
2
13
13
multiplication 2
15
× 15
multiplication 4
2 ← line 1
15
× 15
155 ← line 2
150 ← line 3
205
207
multiplication 5
15
× 15
225
multiplication 3
15
× 15
525

Figure 4. The working of the pupils.

sequence of actions, is a significant factor in our decision to use video (Mercer, 1995) because one is freed 'from the limits of the sequential occurrence of events in real time.' (Erickson, 1992, p.209)

How the task of analysing the growth of mathematical understanding, through the examination of video data, is undertaken by the "μ-Group" at UBC, will also be made explicit, and its rationale argued for. The thorny question of "to *transcribe* or not to transcribe" will be addressed (Johnson, 2000), and how and why theory is used to inform the analyses will be illustrated. The need to 'hold in abeyance interpretive judgements about the function (meaning) of the actions observed' will be explored and cognisance taken of the danger that 'especially in the early stages of fieldwork these interpretive inferences can be faulty.' (Erickson, 1992, p.210). Bottorff (1994), too, notes that video does not necessarily provide the contextual information which might be necessary for the process of interpretation. The reader will, thus, be advised of these and other inherent and hidden dangers in this type of research. There is not room in this brief paper to reference the research which we have studied in which video tapes are used in other academic fields such as medicine and computer science, but our methods are built from the considerations of these writings.

As with all research, the initial "question" or focus of interest drives any project we undertake. However, since we form a coherent group focusing on mathematical understanding, we have taken the decision that video-taping, and any artefacts produced by the participants during the taped sessions, will always need to form a part of our data set, though not necessarily the totality . Stimulated recall, interviews, follow-up sessions etc. can all add to the information on which we base our analyses[3]. Questions of whether to use one or multiple, fixed or roaming cameras are made on a case by case basis, bearing always in mind that video can only be viewed in real time and our method of analysis requires a large number of re-viewings. We take cognisance of the fact that we cannot hope to get the full picture, but also take steps to maximise the data as described below.

We currently use digital tapes, Mac computers and VPrism software, to make (eventually!) communication between the members of the group simpler. We are still learning how to exploit the maximum power from this combination. Prior to this investment we used Hyperqual, C-Video and the pause button on a VCR, and before that a clock, the stop button and pencil and paper. Whatever the technology, we have not changed our basic method of analysis.

The first step is to familiarise ourselves with the tapes, even if we have been the one actively recording in the classroom. We watch the tape right through, *without pausing*, on three of four occasions to familiarise ourselves with the lesson[4] *as a whole*. We bring no particular conscious lens at this point, merely careful watching and listening skills. This is important because understanding is a social act and the environment co-emerges with the students' interactions as they unfold (Davis, 1996). We need to try to comprehend the context, therefore, within which the growth is taking place. Where we feel it necessary, we will follow-up with interviews, video-recall sessions, or further video-taping to gain the best appreciation of the context, actions and speech that is available to us, bearing in mind Bottorff's comment above. Our next step is what we call "creating a timed activity trace". This time the focus is very specific - and hard to adhere to. Depending on the length of the tape and the density of the physical activity, we select a time interval of 2 or 3 minutes, and starting at the beginning of the tape, we stop it every, say, 2 minutes, regardless of what is taking place. We note the time and write a very brief description of what is happening at this point, *with no judgmental comments whatsoever*. It is a description of exactly what we see, with no inferential remarks such as "He is trying to ..." or "She seems to have ...."or even "A confusing diagram on the board ...". We force ourselves to use simply factual description such as "He writes..", "She says...", The teacher draws...". VPrism is a godsend for this stage as it allows us to have the video playing and type text simultaneously on the screen. The purpose of this activity is two-fold. It allows the researcher easy access to remembered events - "I need to check exactly what Mary said just after the teacher dropped the chalk." - because one remembers events more easily than exact words. It also provides a vital *neutral* tool for colleagues working with the same data, by providing common reference events to situate discussions of interpretations of student's learning.

Note that there has been no mention of transcribing. This is because we do not transcribe our tapes except for short extracts, at the very end of the research process, for the purpose of writing about the research. There are three reasons for this decision. Firstly an early transcript can "cement" the hearing of speech, by recording it, temptingly permanently, neatly typed and organized. Transcripts have definitely been shown to inhibit a new hearing of what was said, and thus to lead to a misinterpretation of the mathematical actions. A more detrimental temptation is to work from the transcript while performing the analysis. The purpose of working with video tapes is to be aware constantly of the environment in which the lesson is taking place. Of course, the use of transcripts is much more cost-effective in terms of both time and money - every researcher does not need a VCR and a copy of the tape, and reading is far faster than video-watching. I have no problems with those who work in this way, *if* they are *honest* with their readers and make it very clear that the data on which the analysis was performed were transcripts and not constant re-viewing of the tapes. We do not claim that a video can ever capture "everything" that is relevant, but it provides a different, and in my opinion richer, data base than a written version of the talk that took place, even if the latter is annotated with descriptions of actions.

Our data, therefore, as we enter the analysis phase are the tapes, the written, descriptive, "timed activity traces", any artefacts produced by the students or teacher, and field notes of any board writing or relevant activity that took place off camera. Only at this point do we let the "lens" of theory influence our viewing. We view the lesson for episodes, moments, comments, etc., that have potential interest and mark these on the timed activity trace. Then begins the time-consuming, arduous but rewarding task of scrutinising in minute detail the understanding of the students through the perspective of the appropriately chosen theory. This entails many, many further viewings, of course, and much to-ing and fro-ing within the tape for which the original timed activity trace is vital. Until this point the researcher's job can be a lonely one, but from here on the involvement of "informed others" can be a positive addition to the attempt to make sense of what we see. By "informed", here, I mean knowledgeable in terms of the theory, rather than the video. Any researcher, however scrupulous, can be blind to "subtle nuances of meaning that occur in speech and non-verbal action - subtleties that may be shifting over the course of activity that takes place" (Erickson, 1992) or, indeed, even to glaring instances of action and voice through familiarity with the data. An "informed other's" fresh, first take on the understanding of the children can provoke discussion and new vision, or act to confirm judgements already made. Again, VPrism gives us added power. There is the possibility for each viewer to code the data separately, but within the same document, so that different perspectives can be easily gathered and compared. Here lies the strength of a group such as the UBC "μ-Group".

The final stage, of course, is to review the tape once more in the light of the written analysis. Unless one is scrupulously honest, it is possible to get carried away with the tale one has extracted and wants to tell, and gradually drift away from the minutiae of the actual data. It is all to easy to write about what you have noticed and omit some of what you have seen. Interpretive research can never tell THE story - but then there is never only one story to be told about the complex world of teaching and learning.

The purpose of the following collection of papers is to illustrate how, working with the same 16 minute segment of video tape, using the method of analysis outlined above, but through the lenses of different theories concerning the growth of children's understanding, we can produce pieces of a jigsaw puzzle that together combine to tell a much fuller story of the mathematical understanding of the children than any single account could do.

Paper 2: Analysis and Structure: Using Schoenfeld's Table to Help Understand Students' Understanding

Katharine Borgen
University of British Columbia
John Oliver School

Setting the Stage

While children are working on an assigned piece of mathematics, it is often assumed that they know the background material necessary to perform the work. This, however, is not always the case and it may cause unanticipated difficulties as students may not recall all the necessary concepts, or may recall them incorrectly. It may lead the students to concentrate on a completely different aspect of mathematics than had been the teacher's intent. This appears to have been the situation in the following discussion.

In viewing a video of three 6th grade children asked to determine "How many more blocks would be necessary to build a pyramid [of a certain height]", it was observed that they had two very different geometric concepts for doing this. One concept involved adding cubes all around the pyramid and thus building it up from the outside. This complex method was used by William, the least assertive of the three children, and, as it was not understood by Veronique, the most assertive, it was dropped in favour of her less complex method. This method involved simply making a new base and inserting it under the old pyramid. The new base was two cubes longer and two cubes wider than the previous base. Thus, a pyramid 1 level high had a 1 cube base. A pyramid two levels high, required a base 3 cubes by 3 cubes, so 9 more cubes were required. For three levels, the base was 5 cubes by 5 cubes, requiring 25 more cubes. This process worked well for the students and they seemed *certain* that it was correct. However, when they reached a pyramid of 7 levels, requiring a base of 13 cubes by 13 cubes, their recalled number facts failed them. They attempted to apply a multiplication algorithm, but quickly abandoned it in favour of a calculator to obtain the solution that 169 more cubes would be required.

At this point, also, their instructions asked them "to *guess* (italics added) how many more cubes you will need to build a rectangular pyramid that is 8 levels high". They determined that they would need a base 15 cubes by 15 cubes. However, they decided that using the calculator was, in essence, cheating, so, even though they "knew" the answer, they felt that they had to "guess". Their "guessing" was related to obtaining the solution of a multiplication question which was not the intent of the assignment, but one that led to an interesting discussion, revealing a lack of understanding of both the concept of estimation and of the multiplication algorithm. It is my intent to concentrate on their application of the multiplication algorithm, particularly, the understanding of it by Allen, the third member of the group.

Schoenfeld's Model for Discussion of Allen's Understanding

In order to analyse a situation, it is desirable to have a framework within which to work. Also, in order to understand what a student understands about the mathematics under consideration, it is usually necessary to first determine what mathematics is involved and what underlying principles must be in place to understand it. Schoenfeld's levels of analysis and structure (1989) provide such a model. By first looking at the mathematics involved, Schoenfeld tries to "understand as much as possible, what the mathematics looked like from the student's point of view" (p.106). His table allows one to start with the "chunks" of knowledge that need to be in place to understand the mathematics, and breaks this down to smaller, structures that form the supporting domain of knowledge, always comparing the mathematicians understanding with that of the student. I decided that this model provided a suitable method with which to discuss Allen's understanding of the multiplication algorithm. Following is my understanding of the mathematics involved and of Allen's understanding of it.

A. The Concept of Multiplication of Two-Digit Numbers

Murray and Olivier (1989) identify four levels of understanding two-digit numbers and note that these are identifiable by the strategies applied and that each level relies on full understanding of previous levels:

- Pre-numerical: The student understands the whole numeral but not the significance of place value. (13 is a number, not 1 ten and 3 ones.)
- Numerical: The student understands "numerosity" and can "count on". (5 plus 3 can be thought of as 5 -- 6, 7, 8.)
- Decimal decomposition understood: This may be used to simplify a problem. (25+12 becomes 20+5+10+2 then (20+10)+(5+2))
- Place value is understood: 10 becomes an iteratable unit. (64 is thought of as 6 tens and 4 ones)

Calculations are meaningless unless a student understands the "how-manyness" of the number and a student must reach level 4 understanding in order to algorithmically manipulate numbers meaningfully. This level of understanding allows for operation on the digits as "units" for convenience, with place value applied later (Murray & Olivier, 1989).

A situation is perceived as "multiplicative when the whole is viewed as resulting from the repeated iteration of a one-on-one or one-on-many correspondence" (Natais & Herscovics, 1989; p. 21). Natais & Herscovics (1989) describe a two tier model of understanding multiplication:

Understanding of preliminary physical concepts: This includes intuitive knowledge of the difference between multiplicative and non-multiplicative situations as well as procedural understanding of the logico-physical situations that change it from additive to multiplicative, and the ability to abstract concepts.

Understanding of emerging mathematical concepts: This implies the appropriate use of procedures, the ability to abstract the reversibility and invariance of the operations, and leads to the ability to formalize the axiom symbolically.

Although the model seems linear, the procedure is not, and the second level of understanding can be developing before the first has been completed. Mulligan and Mitchelmore (1997) agree and note that students create intuitive models and that understanding of the multiplicative process proceeds from unitary counting to repeated addition (including rhythmic and skip counting) to application of an algorithm.

Children should observe a variety of contexts for multiplication and by seeing different models for multiplication, such as equal groupings, allocations (2 apples for each of 3 children), or equal displacements on the number line, understanding of the multiplication algorithm becomes easier (Anghileri & Johnson, 1988). Anghileri and Johnson (1988) indicate that operations with larger numbers can only be understood if students have developed a basic understanding of the properties of commutativity, associativity and distributivity, have knowledge of the basic number facts, and have the ability to visualise subsets of a group. Murray and Olivier (1989) note that one can begin to teach the multiplication algorithm using the concept of repeated addition, but, that if a student is to understand the algorithm, comprehension of place value and knowledge of the multiplication facts are necessary. As well, understanding of factors and multipliers of 10 are also essential.

In summary, then, what is needed in order to apply the multiplication algorithm with understanding to two digit number is:

- Understanding of two digit numbers (as identified above by Murray and Olivier above).
- Ability to recognize a situation as multiplicative.
- Knowledge of number facts, including products of 10.
- Some comprehension of place value.
- Awareness of subsets.
- Basic understanding of multiplicative properties.
- Ability to abstract ideas.

B.1. Allen's Understanding of the Multiplication Algorithm

Allen displayed a good ability to view spatial relations. After initially admonishing William for the manner in which he was building his pyramid, adding cubes to the outside on each layer by saying:

"What are you doing? That's supposed to go *underneath.*"

"That's supposed to go *underneath!*"

"I know what you're doing. You're fooling around."

He was willing to take a closer look at it and wondered "why is he putting it at the side?" He seemed to be watching what William was doing, and suddenly realised "how" William was building his pyramid.

"Oh, yeah. I see I see what he's doing."

"Yes. I know I know. Its all our fault."

His willingness to follow the method of building a new base, instead, seemed to rest, therefore, not on his inability to understand William's method, but rather on the simplicity in determining the number of cubes needed using it compared to the complexity using William's method.

Fairly early in the discussion, Allen indicated that he understood the concept of number and the basic multiplication facts. When asked what 7 times 7 was, he verified Veronique's answer of 49 by skip counting:

"7; 7, 14, 21, 28, 35, 42 Yeah its 49."

Later, when they had to determine 11 time 11, he replied:

"11 times 11 is ... You don't know that?"

"One twenty one"

Veronique does not totally believed him and states:

"That's Allen's if its wrong."

Allen also indicated that he had probably memorized his multiplication facts to 12 times 12 because when the were trying to do 13 times 13, he stated:

"12 times 12 equals one hundred and ... No, that's different."

But he could not recite 13 times 13.

B.2. The Process of Multiplication?

Reference in this section will be to Figure 4. When the students attempted to multiply 13 times 13 on a paper, Allen did not exhibit an understanding of place value as needed in this situation or of the multiplication algorithm. He indicated that a 3 in the unit's column multiplied by a 1 in the ten's column gave a 3 in the unit's column *(multiplication 1)*, and that a 1 in the ten's column multiplied by a 3 in the unit's column gave a 3 in the ten's column. When Veronique interrupted, stating that a 1 in the ten's column multiplied by a 3 in the unit's column gave the 3 in the ten's column, and when the two 3's in the one's column were multiplied, it gave a 9 in the unit's column, he agreed. Then, with no explanation, he also wrote a 9 in the ten's column and started to add. He went no further with this as Veronique decided to do the calculation on the calculator, obtaining the solution 169. He looked back at the paper, noted that it was wrong and crossed it out. No one questioned where they went wrong.

For the next pyramid, the students realised that they needed to calculate 15 times 15. William wrote a little 15 times 15 *(multiplication 2)* then crossed it out, writing the large 15 times 15 below it (*multiplication 3)*. Meanwhile, Allen seemed to be considering the algorithm and, looking off into the distance in a thoughtful way, verbalized that he thought he knew it. He seemed to be trying to form a mental image or abstraction of the process. Veronique went to the paper and indicated that "1 times 5 equals 5 *(she wrote the 5 in the hundred's column, then adds 25 after it, multiplication 3)* but then indicates that she made a mistake and wrote 15 times 15 again *(multiplication 4)*. This, along with William's comments distracted, Allen and he seemed to lose track of what he was thinking gave a big sigh and made no further attempt to think about the multiplication. However, William's movement toward the calculator prompted him to remember that they were to "Guess, just make a wild guess." At this point, 'guessing' became guessing the solution to a multiplication question, and was not necessarily related to guessing the number of cubes needed for the next pyramid.

Allen thought that they should just guess 250, then 173. He gave no reason for these guesses, just that they should use them. Veronique, however, wanted the 'right' answer and her question of "How do you multiply? … Like the double numbers?" (referring to two digit numbers) seemed to prompt Allen to think once more about the algorithm. Veronique, who was continuing to work on the paper, then asked "How do you get twenty-five? You know 25 here?" *(pointing to multiplication 4, line 1, where at the time, there were no numbers)*. She then said, "Oh, I know." And she wrote the little 2 as a "carry" above the one in the ten's column *multiplication 4, line 1)* and the 5 in the unit's column *(multiplication 4, line 2)*. Allen then noted that "1 times 5 equals 5. Put it there" *(pointing to multiplication 4, line 2, ten's column)*. He ignored Veronique's "Doesn't that equal 7? 5 (*counts with fingers))* 6, 7?." *(pointing to the two that had been carried and 'tap' counting it on)*. Allen countered with "What are you doing? Why did you put that 2 up there?" and Veronique stated that "You have to carry it over". Allen ignored this. Veronique stated that she had known how to do "this"(presumably the multiplication) last year, but has forgotten and Allen suggested that he had done it just a few minutes ago, possibly referring to when he obtained the answer 250. William agreed with him, saying "You did. You got 250". Again, Allen wanted to guess, but Veronique persevered in trying to multiply. She stated that the 5 in the upper unit's column times the one in the lower ten's column gave a 5 in the ten's column and wrote this *(multiplication 4, line 3)*. She then followed by writing in the hundred's column *(multiplication 4, line 3))* since the 1 times 1 gave 1. She was not sure whether she should add or subtract, and Allen told her to add.

Allen once again indicated a good knowledge of number concepts when he looked at the addition and, while Veronique was trying to calculate the sum, he quickly determined the answer mentally and told her to write "two - oh - five". Interestingly, once they had "done the multiplication" and were doing the addition, Allen remembered the "carried 2" and added it in the correct column, the ten's column, and said the answer, "225". This time, however, Veronique ignored him and added it into the unit's column, obtaining the incorrect answer of 207. When the students calculated the correct answer of 225 on the calculator, Allen did not go back to his correct answer, or even realise that he had given it. This seems further evidence that he did not really understand the multiplication process. William pointed to the answer of 525 *(multiplication 3)* and stated that "You were right here except … that should have been a 2". This was ignored. Veronique indicated that she thought that she knew how they got the wrong answer, wrote 15 times 15, (*multiplication 5*) but after a brief exchange, she simply wrote 225 as the answer and said "I found that out but I don't think that's the right way". Allen did not remember that the correct answer was 225 but was assured by the other two that it was. This would seem to indicate that his solution of 225 earlier had not been based on a solid mathematical knowledge or understanding, but in his words was "just a wild guess".

Conclusion

Although Allen had a good knowledge of the basic number facts, he did not understand the multiplication algorithm. It would appear that he had not connected the multiplication algorithm with his concept of place value and/or the closely related notion of products of ten. He was thus, not able to abstract the necessary relationships that would allow him to make meaning of where and what numbers should be placed in the different columns. His work was consistent with findings by Angheleri and Johnson (1988) that indicate that, with young children, errors made in multiplication are not consistent, but are influenced by the numbers used. Allen was working beyond his memorised numerical facts, but more importantly, perhaps beyond intuitive numerical understanding. He may not have been able to visualise the "how-manyness" of the solution, or to make any concrete connection with the algorithm that had previously been taught. This suggests that the teaching of an algorithm may lead to senseless manipulation of numbers if it is taught before a student has developed an intuitive understanding of the procedure.

Paper 3: Exploring the Notion of Epistemological Obstacles

Sitaniselao Stan Manu
University of British Columbia

Theoretical Framework: Epistemological Obstacles

Educators and researchers, over the years, have discussed the notion of "understanding" quite extensively. Alongside this notion, they have also mentioned the conflict between the students' conceptualization and the new ideas that the teacher hopes will be learned (Nickerson, 1985; Davis & Vinner, 1986; Sierpinska, 1990). *Misconceptions* and *epistemological obstacles* are two different aspects of this conflict. An example of an epistemological obstacle occurs between considerations of relativity and Newtonian mechanics. The latter works fine (and worked for 100s of years) for everyday explanations of movement, but does not explain the problems that Einstein was looking at. Newtonian Mechanics was an obstacle to understanding relativity. In other words, when a conceptualization that "worked" no longer holds good. In this analysis, we will focus on the notion of epistemological obstacles, because, as Sierpinska has put it, "depth of understanding might be measured by the number and quality of acts of understanding one has experienced, or by the number of epistemological obstacles one has to overcome." (1990, p. 35) This "overcoming" is crucial when the new knowledge contradicts the old, because the old knowledge can become an obstacle to learning the new (Sierpinska, 2000). In interpreting Brousseau's (1997) work, Sierpinska wrote:

> Epistemological obstacles are those whose limitations are related to the very meaning of the mathematical concepts. A mathematical concept has many levels of generality and abstraction. Each level and aspect has its limitations, and if one thinks of a concept in a meaning that is not appropriate for the given context or problem, then this way of thinking functions as an obstacle and one makes mistakes or cannot solve the problem. (Sierpinska, 2000, p. 84).

This notion of epistemological obstacles, central to Sierpinska's work in the last few years (1987, 1990, 1992, 1994, 2000), is not itself a new one. Nickerson (1985) cites Miyake (1981) concerning some evidence that changes "tend to occur when a person is struggling to figure out an aspect of a process that has not been understood yet" (p. 221), and Sierpinska cites Lindsay and Norman (1984) by saying that a student stands against any revision rather than rebuilding his or her previous constructed 'system of interrelations' (1990). Sierpinska insists, however, that what is needed is to "reject parts of our old knowledge, or reorganize it, or generalize it and recognise that it had a limited domain of application" (2000, p. 83). In so doing, the student feels as if there is something, an obstacle, blocking the mind. Sierpinska also proposes that some acts of understanding are acts of overcoming epistemological obstacles while some may turn out to be acts of acquiring new obstacles (1990). To overcome such obstacle, "a mental conflict is necessary and [that] therefore, the didactical situation (Brousseau, 1986) in its two dimensions - cognitive and social – must favour these students" (Sierpinska, 1987, p. 371). I intend to use this perspective as a theoretical framework through which to analyse the video of children's mathematical working.

The Task: Constructing a Four-Levels Pyramid

In this task, three students, Allan (A), William (W), and Veronique (V), attempt to find how many more cubes will be needed to build a four-levels pyramid given the set of pyramids shown in Figure 1. They were not taught formally about this type of problem, or about sequences before, and therefore they were only expected to come up with their own intuitions of how the next pyramid (P4) will be constructed.

Almost immediately after Veronique read the "Task Sheet", William said, "All right, let's just start adding." In fact, it is a reference to physically "adding" cubes to the third pyramid, as we will come to see when we analyse his process of constructing the pyramids, not a reference to "adding" numbers. The children are asked to find "how many more cubes (emphasis in the original worksheet) will be needed to build a rectangular pyramid that is four-levels high" (P4, figure 3). Veronique appears to notice a number progression along the bottom layers, and hence she starts by observing the bottom layer of the second pyramid (P2):

V: It has 9. Right. 9. Three. One, two, three, four, five (*counting P3 bottom rows*).

That's adding two. So, five plus two equals? (*she's now projecting the next pyramid* (P4) *bottom layer rows*).

A: Oh well…five, six, seven. Seven.

V: So, seven times seven equals? (*looking at Allan*)

W: I don't know (*laughing and starting to build*)

V: 49?

A: 7. 7, 14, 21, 28, 35, 42, 49. Yea it's 49.

[section omitted]

V: Its how many more cubes

W: Well, we have to build it.

A: We need some cubes … some more cubes

W: Let's find out.

[section omitted]

V: But isn't that already correct?

W: We don't know that

A: 49?

W: We don't know that.

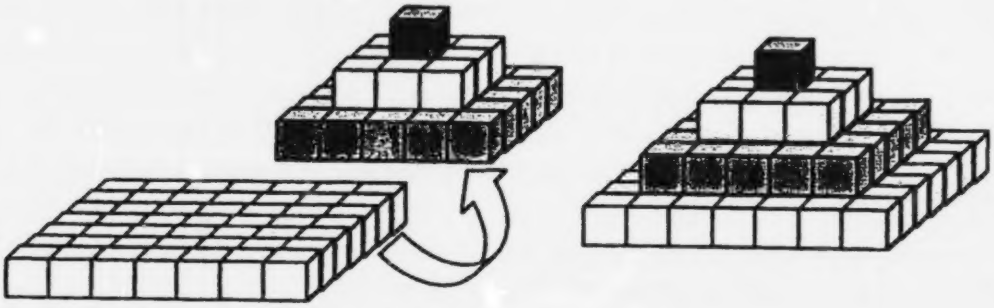
A: 49!

V: That's correct!

W: We don't know that!

Here, William appears not to rely on Veronique's and Alan's intuitions. He needs concrete proof of the final solution, which to him is to simply build a four-levels pyramid. Not only that, William's concept of the process of constructing the four-levels pyramid (P4) is totally different from Veronique and Allan's conceptualisations. Veronique suggests building a seven by seven bottom layer and then attaching it underneath the constructed pyramid (P3) to get the next one (P4, Figure 5) if William insists on building.

This is where William isolates himself: what he sees is a method of adding cubes to all sides of the constructed pyramid (P3) as if to 'expand it' to the next one (Figures 6). Unfortunately, Veronique and Allan, who later admits to seeing what William is doing, can not, at this stage, see William's constructing process either. They keep



P4

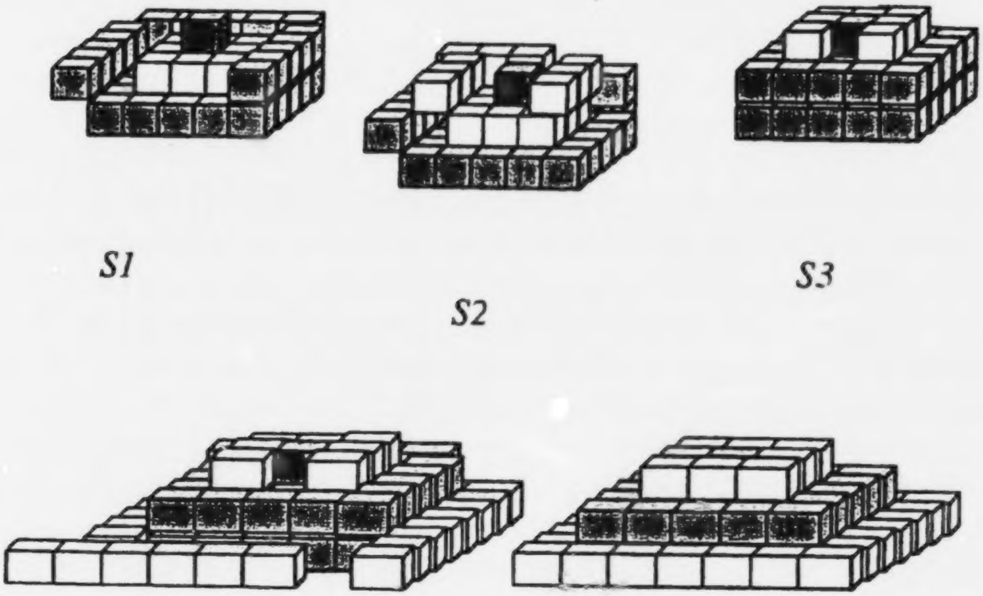
Figure 5. Veronique's layers-constructed pyramids.

on pressing William that "it's supposed to go underneath". Still, William insists "No, you'll see what it is. You'll see what I'm doing."

William completes the construction of the required pyramid after a few adjustments, but soon faces a problem in counting how many more cubes he used.

A: Okay, how much more do we have?

W: No. Wait, wait, wait, wait.



S1

S2

S3

S4

S5

Figure 6. William's sides-constructed pyramids.

V: Okay.

A: How much do we have more?

W: Oh, no. We have to add another layer.

(Because, after adding cubes to all sides, he gets the structure in S5: Figure 6)

A: No you don't. 1, 2 … (*counting the layers*)

V: 1, 2, 3, 4, 5, … (*counting the length of the base side*) it's only down here is how much we need.

William, do you know there's a much simpler …?

William it's only six this way (*one side is short*).

You need to add one more group of 7 there.

A: (*to William*) Then you need to put some more up here (*pointing to the top*) Right here. Like that.

V: The answer is 49

W: We don't know that

A: (*to Veronique*)Yeah. Just write … 7 times 7 equals 49

[*section omitted]*

W: (After correcting his constructed pyramid) Hey guys. Look! I did it!

V: 1, 2, 3, 4, 5, 6, 7. (*counting one side*) 1, 2, 3, 4, 5, 6, 7. (*counting the side perpendicular to it*)

7 times 7 equals 49. Ta-da!

W: What about these? (*pointing to all the side-cubes he added*)

A: It says "how many", "how many more".

W: I know, but what about these up here? (*pointing to the side-cubes near the top*)

A: You don't count them.

W: What about this? We added this. (*waving the cube from the top*)

A: Oh yeah,

W: We added these. And we added these. (*again, he points to all the side-cubes he added*)

From this excerpt, we see clearly how to interpret the phrase William made at the beginning when he said "All right, lets just start adding!": he "added" to cover the sides of the constructed pyramid (P3, and see Figure 6). For Veronique, it was clear that a seven by seven layer is the needed amount to build underneath the constructed pyramid (P3, and see Figure 3) to form the four-levels pyramid (P4) required.

Discussion and Analysis

The extract above demonstrates a mental conflict within William, and between the conceptions of the three students. Hence, we see an epistemological obstacle occurring for William. Allan struggles and figures out the processes used by both Veronique and William. However, Veronique and William's conflicting processes of constructing a four-levels pyramid act as an obstacle to each other's understanding. Veronique has quickly seen the pattern of adding square bases with sides of odd numbered cubes. Her method of seeing is no obstacle to a numerical generalisation of added cubes. Obviously, William could possibly have understood what Veronique was talking about if she had physically demonstrated her mental construction of a fourth pyramid. But she did not do this and therefore William held fast to his initial conceptualization of his process of 'covering all sides' construction. His method was not one that would easily lead to the production of a numerical pattern in the number of added cubes from one pyramid to the next in the sequence. In fact in order to know how many cubes were added he wanted to count them.

Another aspect related to the notion of epistemological obstacles in this analysis deals with the students' attitudes. Veronique appears as what Sierpinska might call an "intuitive empiricist": a conviction that her mathematical result is intuitively acceptable. She sees the pattern of increasing bases (Figure 7) as a number problem, and having made that abstraction she is able to figure out how many more cubes are needed or how the next pyramid of the sequence will look, without even doing the action or the process of construction.

She is convinced of the correctness of her solution and makes no attempt to understand William's method. William, on the other hand, appears to be what Sierpinska might call a "discursive empiricist": where the source and motive of his mathematics rests in the verification of the problem and its application (Sierpinska, 1987). He relies on the action, and in order for him to figure out the solution, he has to build it. For him, covering the sides is necessary in order for him to find out how many more cubes are

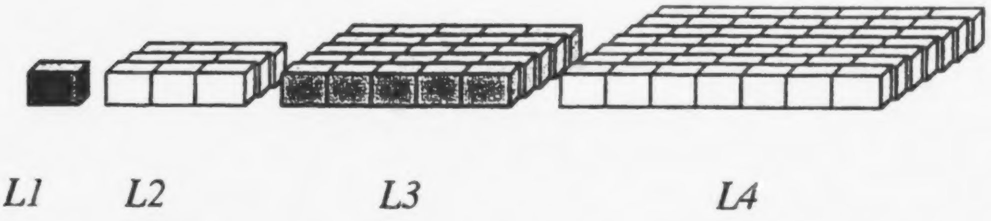


Figure 7. Veronique's patterns of bottom layers.

needed. He is unhappy with accepting Veronique's numerical approach without seeing it demonstrated physically. Indeed, Sierpinska (1987) states that:

> If the construction is not made impersonal then its actualisation is inconceivable.... If, on the contrary, the construction is no longer tied to the person performing it, then the actualisation is theoretically conceivable (p.385).

William is not able to move from his specific actions to a mental numerical conception of the process and thus in Sierpinska's words its "actualisation is inconceivable." He is unable to move on without further physical activity. In the actual episode he abandons his method, and although showing no evidence of understanding Veronique's way of thinking joins in the numerical activities of multiplication.

Final Remarks

We have seen from this perspective how epistemological obstacles do not necessarily arise in the same way for different children. William needs either Veronique's or a teacher's help to overcome this obstacle. More importantly, for shared understanding there is a need for the individuals to reorganize his/her current knowledge and recognize that it had a limited domain of application, as Sierpinska has suggested (1987). This is what happened with William; he could not reorganize his mental construction in a way that made it "impersonal", and could not actualise the mathematics without having to do the action. In addition, attitudes also play an important part in overcoming such obstacles such as those that we have seen in this analysis. As Sierpinska (1997) has suggested, attitudes towards mathematical knowledge might turn out to be very serious obstacles, and therefore, serious consideration must be taken to overcome it.

Paper 4: Making Decisions: Analysis Through a Teacher's Eyes

Donna Jenner
University of British Columbia
Annieville Elementary School

When does one intervene when a group of students are working together? What guides the teacher's decision to either stand back and observe students wrestling with a problem or to step in and become part of the conversation? Teacher intervention is not unlike an intricate dance characterized where one individual leads while the other follows but with the role of lead dancer alternating. What factors influence this back and forward movement of conversation? Teacher interventions have the potential to be either highly constructive to the group dynamics and learning of the group but may, however, be detrimental. My analysis of the videotape is centred on teacher's actions with a group of three students. Specifically, I am looking through a lens that attempts to bring into focus how the teacher's lack of intervention impacts the growth of the group's and/or the individual's mathematical understanding. I will also pose some hypotheses about how timely interventions may have impacted the students' actions, emerging understanding of the problem, strategies used to solve the problem and the final answers.

To guide my analysis, I will be drawing on research that has examined teacher interventions and the growth of students' mathematical understanding in interactive classroom settings. The lens I use for analysis will be that of Jo Towers' studies of how teachers' and students' actions and understandings in the classroom co-evolve through interactions and evolving understanding (Towers, 1998a) . Specifically, I will use intervention themes developed by Towers to describe teacher "actions in the moment" or interventions with students to identify the impact that teacher interventions and non-interventions have on the growth of three children's mathematical understanding. The Pirie-Kieren theory (Pirie and Kieren, 1994) and the enactivist notion of 'co-emergence' elaborated by Davis (1996) will also be used to analyse teacher actions that may occasion the growth of mathematical understanding in an interactive, co-emergent classroom environment.

In this video scenario, the teacher has gathered three intermediate children in one room for a problem solving session. All possible materials the group could require to complete the task are in the room and the children are free to select any of the manipulatives and tools. The teacher outlines a task to the three students and then leaves the room. This is a deliberate teacher intervention theme that Towers (1998a) describes as *retreating*. The teacher's intention is to leave the students to ponder the problem, work through a strategy or strategies and arrive at a reasonable solution. The teacher deliberately sets up a situation where she does not play a role in how the group functions. The way the problem will be solved is left solely to the group.

Distinct roles begin to emerge immediately. Allan acts as the negotiator or conscience of the group, continually reminding the group of their responsibility as group members, admonishing them to "solve as a group" while imploring the others to cease working and think independently. Initially, William is an active, contributing player in the group but is quickly relegated to a subservient role by Veronique - the powerful, dominant voice among the three children. It has been over a year since the students have worked together and the dynamics in the group resemble what one would expect to see in younger children. In fact, Allan's repeated plea, "we need to work in a group" and the lapses into giggles and sometimes, silly behaviour imply that the three children are experiencing some difficulty working effectively in the group situation. A teacher hearing some of the behaviour likely would remind each of the three what it looks and sounds like to be an effective and productive group. Towers (1998a) would describe this intervention as *managing*, a teacher intervention that reminds students of the task requirements and the expected group behaviour. Without such a reminder the group often wanders off task and behaves in silly ways.

The assumed roles are quickly reinforced by how the children respond to each other's ideas and actions. It seems that the teacher's absence from the group's activity has implications for how each child experiences the activity. Veronique, Allan and William's foci are usually limited to their own ideas and how they will use the cubes or other tools to represent their personal thinking. They are often unable to step outside their own way of thinking to consider each other's ideas. As the group begins to consider the task, Allan tries to get the group to focus on using the cubes to build one model to illuminate the group's ideas while Veronique definitively proclaims she has the answer. At the same time, William tries to initiate another strategy, one that is quickly dismissed by Veronique. Although William continues to try and convince the group that he has a feasible approach, his attempts are eventually squashed.

A teacher's intervention may have changed how William experiences this activity. When totally frustrated with the group, William actually says to the camera that he needs his teacher's help. If the classroom teacher had been part of the action, the children may have been encouraged to focus their attention on strategies that were different from their own. The group's activity may have become coordinated in such a way that William would have been heard and involved in a more personally meaningful way. Towers (1998a) describes this type of intervention as *shepherding* where the teacher tries to direct individuals to understanding by subtle nudging, coaxing, and prompting. If the teacher had invited William to explain his thinking to the group, a situation would have been created from which the teacher may have been able to occasion the group's growth of understanding. As well, a reminder by the teacher of the need to hear all ideas could have kept William engaged in the activity. Without this intervention at a critical point in the activity, William ceases to be an active partner in the groups' efforts. William eventually gives up a viable approach to building and calculating the number of cubes in a rectangular pyramid when the group continues see his strategy as just "fooling around".

In the midst of the action, the children frequently ask each other "what are you doing?" but the question never gets answered. Each child merely continues acting independently of the others. The Pirie-Kieren (1994) theory provides a way of considering the emergence of understanding in a way that emphasises the interdependence of all participants in the environment. In this aspect, the theory shares the enactivist view of understanding as being an interactive process (Davis, 1996). The filter of these theories brings clarity to the lens focused on these children's activity. What if the teacher had encouraged Veronique, William and Allan to answer the "what are you doing" questions? Teacher questioning may have given voice to their personal thinking in ways that could construct meaning for the group. A brief interaction between William and Allan has the potential to begin building some mutual understanding of what William is trying to show the group about his pyramid. William tries repeatedly to get the group to pay attention to his strategy. At one point, Allan begins to take note of his strategy and says "Oh, yeah. I see, I see what he's doing" to which William replies "Now do you know what I'm doing?" Unfortunately, Veronique 'muscles' her way into the action and what might have been a moment where the two boys shared some understanding is lost. Refocusing the group's attention on William and Allan may have sustained their conversation long enough for the boys to begin moving together towards a shared understanding of the strategy and solution. A teacher noting the beginning of the dialogue would foster the boy's connection while preventing Veronique from diverting Allan's attention away from William's strategy. Veronique's interruption and subsequent silly behaviour ends what could have been a fertile interaction between the boys.

The children's comments, questions and actions reveal a lack of understanding of each other's actions - they don't "see" what the others are doing. As a result, they continue to engage in parallel activities disconnected from each other's. By *inviting* (Towers, 1998a) individuals to explain their ideas, teacher questioning may have helped to group to see a shared approach to the task in a manner that is meaningful and engaging for all three children.

As the scene unfolds, the children often come to hasty conclusions about the correctness of their own or others' actions. The children do not offer explanations when challenged about the accuracy of their conclusions and solutions nor do they offer proof for their solutions. Veronique emphatically states she has the correct answer because she "know(s) how to". William repeatedly tells her "we don't know that". In other words, the group hasn't seen any proof that would prove that Veronique's solution was accurate. A simple teacher query "tell the group how you found the answer" posed to Veronique would have asked her to examine and justify the accuracy of her solution. Such questioning would also allow the group to be privy to thinking that up to now has been hidden. In a sense, the teacher's probing would help make an object of Veronique's ideas allowing the group to wrestle with the "proof".

Later in the video, the group comes up against a situation where the absence of the teacher allows the group to resolve the difficulty themselves and thus keep moving toward a final solution. When Allan reads the instructions again: "find out by testing and explaining why your group's guess was correct/incorrect", the students realise they have missed part of the task, estimating. The group now struggles to find a way to offer an estimate when they believe they have a correct answer. The students are confused, do they have an answer or a guess? And how do they resolve this? In fact they do not expend very much time or mental energy working through the dilemma. They simply move forward towards finding the final total of cubes. They consider it simply too late in the game to worry about an estimate and are not in any case sure that it is important. The explicit discussion between the students reveals this: as articulated by William, they "just want to see the answer, then we'll guess". The important task in their eyes is to find the answer and they will just make up an estimate later. The teacher arriving on the scene may have delayed this decision and diverted their attention to the need and value of an estimate. In this situation the group may not have moved along as quickly to their calculations.

When calculating the final total of cubes, differing strategies are used by Veronique and Allan and the children continue to struggle with what is meant by "guess" and "testing and explaining" on the work sheet. The students use images for two-digit multiplication that are inaccurate and incomplete. The inaccuracies are compounded by the fractured discussion between the students. Towers (1998a) describes an intervention called *rug-pulling* that could have engaged the students in a fruitful, re-examining of their strategies for multiplication. In this intervention, the teacher acts deliberately to shift the student's attention to something that forces the student to reassess what he or she is doing. Teacher intervention with Veronique and Allan would direct each one to focus on their own understanding of two digit multiplication. This is an opportune time for the teacher to probe their thinking. Pirie and Kieren (1994) describe such an interaction and process as helping the students to *fold back* and examine their mental *images*. In a rug-pulling intervention, Towers (1998a) contends that students often return to image making activities. Pirie and Kieren (1994) describe these activities as a process where students begin to create new, more accurate and useful images. By interacting with the group, the teacher could have helped individuals to fold back and so engaged the children in examining and modifying their images. As a result, they may have together developed an understanding for multiplication that was more useful and meaningful.

This article has been an attempt to analyse, through my "teacher's eyes", a problem solving session involving three students. The lens of theory which I focused on these children's activities has allowed me to portray potential new scenes. These new scenes are those created by looking at how teacher interventions may have altered what actually happened. These new scenes may have produced situations where the teacher and three students could have interacted together in ways that prompted the emergence of new understandings.

Paper 5: Rectangular Pyramids, Objects of Thinking, and Mathematical Understandings

Jennifer Thom
University of British Columbia

Background

Sfard's earlier works which focus on the nature of mathematical conception (1991, 1992; Sfard & Linchevski, 1994) illuminate mathematical *understanding* as operational knowings and condensed, reified mathematical knowledge. In Sfard's latest works however, she examines the nature of mathematical *thinking* and brings forth an enactivist perspective by characterizing mathematical thinking as circular, non-linear, and complex (1997, 1998, in press; Sfard & Kieran, 2000). Mathematical thinking is viewed as "communication"; an internal *and* external cognitive process that allows individuals to make sense for themselves and with others *how* and *what* they are understanding about the mathematics at hand. Moreover, like conversations, mathematical communication is fluid, co-emergent, and exists in the act of engaging. Sfard contends that mathematical objects *do not* precede communication, but rather, they are brought into being through the development of our conceptual understandings; a result of our own internal thinking and the communicating we do with others. Mathematical objects then, emerge as a result of our communication and not the other way around. We do not start with mathematical objects and then communicate, we communicate and from such individual or collective interaction, mathematical objects evolve and in turn, shape our future communications.

Integration and Application

This paper focuses on the video analysis of a group of three fifth grade children's solving of the rectangular pyramid task taken from a perspective that integrates notions from Sfard's model of mathematical conception and her recent works concerning mathematical thinking as communication. Providing the reader with descriptive examples of the individual and collective workings of the group of children, this interpretative analysis addresses the following questions:

What are some examples of the "object[s] of attention" (Sfard, in press) which emerged from Allan, Veronique, and William's mathematical communication about the rectangular pyramids? And, how did the nature (i.e., operational or reified) of these "objects" shape the children's subsequent physical, verbal, and mental thinking about the rectangular pyramids?

Examining the Children's Mathematical Communication and Understandings

One Pyramid, Two Operational Conceptualizations

The videotaped session for this study opened (prior to the extract shown at PME-NA) with the children's teacher presenting them with three 3-D models of *rectangular pyramids* that were made out of multi-link cubes. These structures consisted of a one-level, a two-level, and a three-level pyramid which were referred to as pyramid #1, #2, and #3. Before she left the room, the group was asked to identify the number of cubes in the first pyramid and then the number of *additional* cubes in the second and third pyramids. Two distinct *objects of thinking* surfaced as a result of the children's individual actions of physically handling the structures while solving for the second and third tasks. Holding the second pyramid and pointing with his finger, William counted each of the eight outer cubes *around* the base and then the top cube, concluding, "I think it's nine". He repeated this action with the third pyramid. By counting the sixteen outer cubes around the structure's base, then the eight outer ring of cubes in the middle level, and ending with the cube on top, he got a total of twenty-five cubes additional cubes. For William, we can see that the "outer, ring-like" layers of cubes which included the top-most cube became the 'object' *of* and *for* his further understanding. William's physical finger pointing not only specified that it was the outer rings of cubes and the top cube which he was 'seeing' but also, it is his communication with himself about this object that enables his counting strategy.

Identification of a second, completely different object occurred when Veronique and Allan were solving for the third rectangular pyramid. Each, on their own, used the pyramid's *horizontal bottom* layer as the object with which they determined the number of additional cubes. They both used their finger to point and count the bottom layer's width as five cubes and its length as five cubes. After doing this, Veronique and Allan gave responses of "twenty-five" more cubes. The fact that they both performed identical actions at different times suggests that both of them were multiplying the base of the pyramid's length and width together in order to get the total number of cubes in the third layer.

Although William's enumeration method is distinct from that of Veronique and Allan's, both are "processes" which the children used to determine the number of additional cubes (the "product"), and thus, are taken to be mathematical understandings of an operational nature (Sfard, 1991, 1992; Sfard & Linchevski, 1994).

One Pyramid, Reified and Operational Thinking

Sfard's notion of "reified" knowledge is described as a structural way of thinking (1991, 1992; Sfard & Linchevski, 1994). Formal and abstract in nature, this type of understanding enables one to view the mathematics at hand as an "entity" in itself and not simply a "product" or a "result" of an action like that observed in operational ways of knowing. For example, a structural form of thinking related to the rectangular pyramid could be one's ability to conceptualise the series of rectangular pyramids as "patterned growth" as opposed to an operational manner of thinking such as each *separate* pyramid to be "increasing in size as a result of having x more cubes added to it". Furthermore, "reified" knowledge is mathematical understanding that is flexible and versatile. It enables one to move back and forth as needed, between formal and informal thinking of the mathematics.

In the very beginning of the session, we get a glimpse of what could be considered a reified mathematical object when Veronique did not count or study the second rectangular pyramid but simply smiled, passed it to William, and responded with "nine" more cubes. Here it appears that she did not need to perform any mathematical action, but simply *knew* the bottom of the third pyramid to "be" nine cubes. We see a more elaborate example of this from Veronique again when the group began to solve for the fourth pyramid and she tells William to stop building because, "okay, I know how to". Here she did not think it was necessary for the group to build the fourth pyramid in order to solve the task of finding out how many more cubes than the third pyramid it would need. In her step-by-step explanation to Allan and William it is clear that her mathematical object was that she "saw" the pyramids as 'patterned growth' and through her communicating of this, attempted to bring Allan and William to her place of knowing.

V: It has nine. Right. Nine... *[referring to the total number of cubes that she has just counted in the bottom layer of the second pyramid]* Three. *[sweeps her finger across the length/width of the bottom layer]*

One, two, three, four, five *[counting the cubes in the length/width of the third pyramid's base]*. That's adding two, so, five plus two equals...?*[smiles, patronizing Allan]*

W and A: Oh well...five, six, seven, seven!

V: So seven times seven equals...

W: Forty-nine.

A: Seven, seven, fourteen, twenty-eight, thirty-five, forty-two, forty-nine. Yeah, its forty-nine.

Given the fact that these children are only in the fifth grade, it cannot be expected that Veronique would use symbolic language or generate an expression such as, *If* L *=the number of cubes along one side of the previous pyramid, then the extra cubes needed for the current pyramid* $(L+2)^2$. We can see however, that she identified the pattern of growth in the length of each successive pyramid's horizontal base to be an increase of two cubes, and by multiplying the value by itself, the number of additional cubes can be found.

Despite Veronique's success in leading the group to the solution of 49 more cubes, William insisted, "Let's get some building blocks" and Allan got a container of multilink cubes to begin building the fourth pyramid. When Veronique repeatedly asked the boys, "isn't that [49] already correct?", William answered back each time, with an increasingly louder voice, "we don't know that". From these instances it is likely that Veronique was the only one who saw the pyramid's growth as being the repeated increase of two in the length of the base and therefore, the only one who was actually thinking with this mathematical object. Nevertheless, Veronique demonstrated the flexibility of her mathematical object when William went to build the fourth pyramid and she was able to move her thinking into a more informal mode and re-present her object of thought, "7 x 7" as a horizontal fourth layer made of "seven groups of seven [cubes]". Here Allan and Veronique use the same object of 'conversation'; a 7x7 horizontal layer, but Allan's understanding is considered to be an operational knowing and not reified knowledge. Veronique's reason for building with cubes was to re-present her thinking and accommodate Allan's need to produce the fourth pyramid.

Two Different Objects of Communication

It was not until the children were actually constructing the fourth pyramid that they realised they were not all thinking with the same conversational object to conceptualize the pyramid's growth. Veronique and Allan began to build a horizontal fourth layer by putting together seven rods of seven cubes each, while William attached cubes onto the existing third pyramid to form outer, ring-like layers. It was during their building that Veronique noticed William was not constructing the fourth pyramid in the same way as she and Allan. Veronique told him that he would have to remove the cubes he had attached and Allan questioned him by saying, "what are you doing?!" This sparked a lively debate about William's object of thought. As Veronique attempted to take apart his model, arguing, "It's supposed to be built under!" William objected to Allan and Veronique's disregard for his work and urged them that if they let him continue, they would "see" what he was doing.

Interestingly, Allan continued to watch William as he built the fourth pyramid and came to an understanding of William's method. This is confirmed by Allan's comments of "Oh yeah, I see, I see what he's doing", and his offer to help William: "You put three more on top. There." Through the physical emergence of two different building methods--Veronique and Allan's construction of a *horizontal 7x7 cube layer* that was to be added as the pyramid's base, and William's action of attaching cubes as *outer, ring-like layers* to cover the existing third pyramid--Allan was able to realise the two building methods as different but nevertheless, correct. Veronique on the other hand, appeared to 'accept' William's *finished* pyramid as correct but unlike Allan, it cannot be said that she was 'seeing' the pyramid as having outer, ring-like layers. It is more likely she considered that William just happened to produce a pyramid with the 7x7 horizontal base which was her original, reified understanding. This was evidenced when Veronique looked at William's pyramid, counted the length and width of its horizontal base as "one, two, three, four, five, six, seven. One, two, three, four, five, six, seven.", and then multiplied, "seven times seven equals forty-nine. Ta-da!". The fact that William repeatedly asked, "What about these? [the outer rings]" also suggested that he did not understand her as speaking of his object. As well, this episode also included Allan's total abandonment of William's approach, stating that "you don't count them [the additional, outer cubes]". From this point onward, Veronique's identified and "proved" (her word) pattern became the object that the *entire* group used to solve for the fifth through eighth rectangular pyramids; respectively, "nine time nine" or 81 more cubes, "eleven times eleven" or 121 more cubes, "thirteen times thirteen" or 169 more cubes, and "fifteen times fifteen" or 225 more cubes.

Pondering the Disappearance of William's Object of Thinking

But why did the object of outer rings not emerge again? Perhaps this was because William and Allan did not develop a structural way of thinking about the cubes in the pyramid's outer layers as being something like, "1 (top cube) add 1x8 (second layer) add 2x8 (third layer) add 3x8 (fourth layer)", the object remained in an operational form and dependent on the time-consuming activity of physically building each layer with the cubes. Secondly, because Veronique's mathematical object was already reified, it served as a more efficient tool to use for quickly solving the rest of the pyramid tasks. Unlike the concept of outer rings, the notion of the length of each successive pyramid's horizontal base to be an increase of two cubes, and multiplying the value by itself, was *not* dependent on the group having to construct each pyramid with cubes.

Conclusion

Through the integration and application of notions regarding Sfard's objects of attention and concept formation, it was possible to examine the conversational objects and mathematical understandings brought forth by Allan, Veronique, and William's verbal and physical mathematical "communication" about the rectangular pyramid. Here, two different objects were revealed regarding the children's conceptualisations of the rectangular pyramid's growth. These were interpreted as *horizontal* layers and *outer ring-like* layers. Veronique's communication to the boys of the pyramid's horizontal layers as an increase of two from its previous length and then multiplied by the value, indicated that her understanding existed as reified knowledge; that it was flexible and re-presentable in a 3-D cube model. Allan's use of the same conversational object was viewed as an operational form of knowing by his action of building with cubes. It is important to make clear that even though Allan used the same structural mode of thinking as Veronique after the fourth pyramid task, it is not possible from the video data to say whether or not he had reified the series of rectangular pyramids as *patterned growth* or, if he was *applying* the calculation as a means by which to *achieve* the solutions. The second object of thinking that surfaced during the children's communications was William's outer, ring-like layers. Like Allan's, William's understanding was also seen to be an operational knowing because it was dependent on his building with and counting of the pyramid's cubes.

Analysing the video data from a perspective which wove together Sfard's two theoretical areas also allowed for the exploration into the co-emergence between the children's conversational objects and their communication about the rectangular pyramid. This was demonstrated by the group's complex, fluid movement that began with Veronique's knowledge of the pyramid's patterned growth, into their physical building of the fourth pyramid using two different objects of thinking, to a debate over which object made the most sense, and returning to Veronique's notion of patterned growth as the mathematical tool to solve for the remaining tasks. Possible reasons for why a certain object, persisted, or disappeared altogether from the children's mathematical communications was interpreted as being connected to whether or not the children actually recognized the existence of the particular object and, if it was deemed useful to them in terms of their informal and formal thinking about the tasks.

Paper 6: Understanding as a Dynamical Process: How do They get to Where They are?

Lyndon C. Martin
University of British Columbia

The dynamical theory for the growth of mathematical understanding, developed by Pirie and Kieren (1992, 1994), offers a language and way of observing and accounting for developing mathematical understanding. The Pirie and Kieren theory differs from other views on the nature of understanding in that it characterizes growth as a 'whole, dynamic, levelled but non-linear, transcendently recursive process.' (Pirie & Kieren, 1991, p.1). The theory provides a way of considering understanding that recognizes and emphasizes the interdependence of all the participants in an environment. It shares the enactivist view of learning and understanding as an interactive process, and rejects the boundaries between mind and body, self and other, individual and social, known and knower (Davis, 1996). Although understanding is still the creation of the learner, the classroom, the curriculum, other students and the actions of the teacher act to occasion such understanding, which can be seen as co-emergent with the space in which it was created (Varela, Thompson & Rosch, 1991).

The theory contains eight potential levels for understanding by a specific person and for a specified topic. These are represented by a 2-dimensional model consisting of eight nested circles. (Figure 8) this set of unfolding layers suggests that any more formal or abstract layer of understanding action enfolds, unfolds from, and is connected to inner less formal, less sophisticated, less abstract and more local ways of

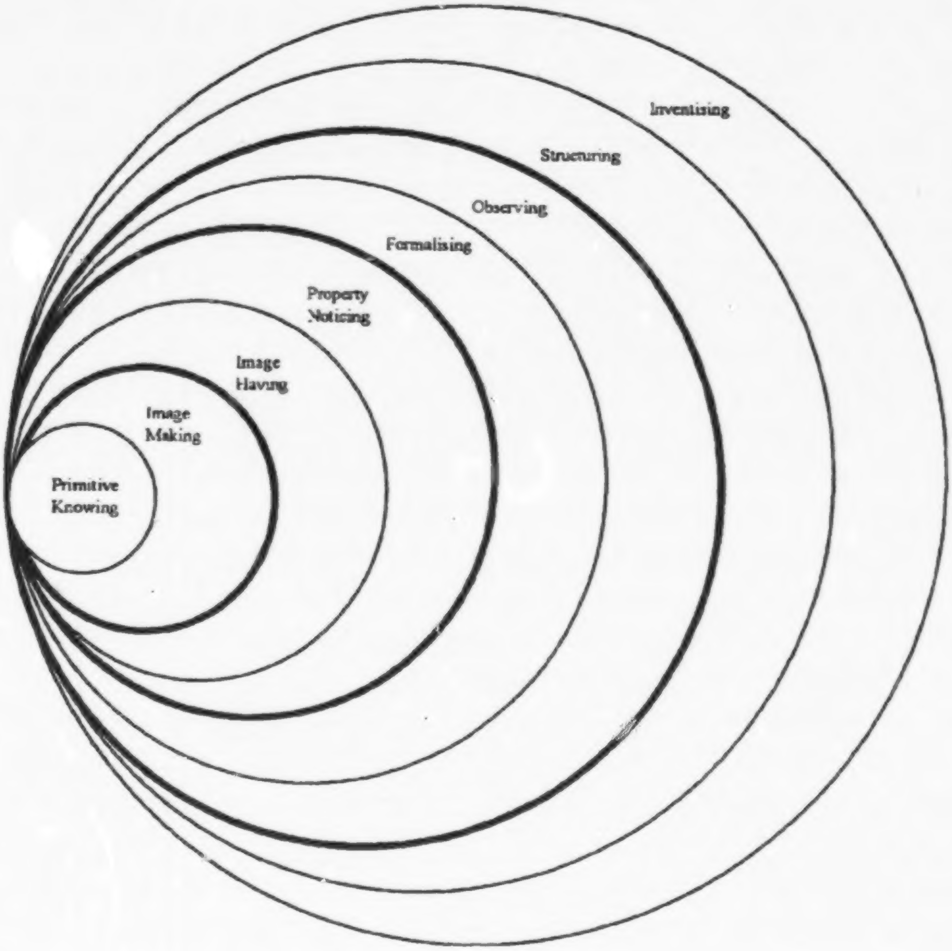

Inventising
Structuring
Observing
Formalising
Property Noticing
Image Having
Image Making
Primitive Knowing

Figure 8. The Pirie-Kieren Model for the Dynamical Growth of Mathematical Understanding.

acting. Although the rings of the model grow outward toward the more abstract and general, the non-concentric nesting implies that growth in understanding is not seen to happen that way. Instead, growth occurs through a continual movement back and forth through the levels of knowing, as the individual reflects on and reconstructs their current and previous understanding. The growing understanding of a learner, or a group of learners, can be represented by a drawn line on the model diagram. This is a technique known as mapping and may be used to represent or visually 'trace' the thinking actions of the children working mathematically, and the growth of their mathematical understanding.

The inner circle of Primitive Knowing contains all a pupil's prior knowledge, in particular understandings that may influence the growth of the new topic being considered, but not including existing knowledge of the topic. At Image Making the learners are engaging in specific activities aimed at helping them to develop particular images. This may involve reviewing their previous abilities and pre-existing understanding and using them in new circumstances. By "images" we mean any ideas the learner may have about the topic, any "mental" representations, not just visual or pictorial ones. By the Image Having stage the learners are no longer tied to image making activities, they are now able to carry with them a *mental* plan for these activities and use it accordingly. This frees the mathematical activity of the learner from the need for particular actions or examples. In Property Noticing, these images the learner now has are examined for properties, connections or distinctions which the learner can also articulate. Formalizing involves making "for all" statements: generalizing but not necessarily with algebra. Observing, Structuring and Inventizing are not defined here as they do not concern the analysis in this paper.

However, as stated above, a path of growing understanding for a learner working on a particular topic is not necessarily a mono-directional one, outwards, through these layers. Indeed the theory states that when faced with a problem at any level that is not immediately solvable, an individual will need to return to an inner layer of understanding. This shift to working at an inner layer of understanding actions is termed "folding back" and enables the learner to make use of current outer layer knowing to inform inner understanding acts, which in turn enable further outer layer understanding. The result of this folding back is that the individuals are able to extend their current inadequate and incomplete understanding by reflecting on and then reorganizing their earlier constructs for the concept, or even by generating and creating new images, should their existing constructs be insufficient to solve the problem. Each person now possesses a degree of self-awareness about his or her understanding, informed by the operations at the outer level. Thus, the inner layer activity will not be identical to that originally performed, and each person is effectively building a 'thicker' understanding at the inner layer to support and extend their understanding at the outer layer to which they subsequently return. The theory suggests that for understanding to grow and develop, folding back is an intrinsic and necessary part of the process.

Considering the Case of Veronique, William and Allan

The Pirie-Kieren theory provides a powerful theoretical lens through which to consider the mathematical actions and inter-actions of the three pupils in the video clip. There is not space here to fully 'trace' the growing understanding of the group, instead a few key aspects of this will be highlighted and discussed to provide a flavour of the theory in use.

As mentioned above, the theory accounts for growing understanding for a specific topic. Here, the topic will be taken to be that of "Patterns and Relations" – in line with the British Columbia Integrated Resource Package. However, this choice of focus is a decision made by the observer, and there exists the possibility of choosing a different topic frame, depending on the focus of interest of the observer.

From the start of the clip it is clear that Veronique and William do not have the same understanding of what the task is about. Veronique sees the task as a "number problem" and is therefore looking for a numerical result. She does not see it as a task concerned with the actual making of pyramids, nor even of exploring how they grow in any physical sense. William however, is thinking about a sequence of genuine pyramids that he needs to build to find out how many extra cubes are needed for each subsequent model. He sees the required result as being one that he will get by building and then counting. These different interpretations greatly influence the actions the pupils subsequently engage in. We immediately see Veronique clearly offering an image of a pyramid growing by the adding a layer on the bottom, and she is able to easily state how many extra cubes are required for the next pyramid. Interestingly, we do not see her working at Image Making in the clip, she seems to have very quickly made her image – and is now able to use it without the need to work with the cubes at all. She has a strong image for pattern as a number sequence. She sees it as an "add to" pattern, but does not relate this in any way to either the height of the pyramid or its position in the sequence. In contrast we see William explaining that he sees the sequence of pyramids growing by building and covering around the one before. He sees pyramids as if they are rooted in the ground, and he is building over them. Although he will obtain a numerical answer for each pyramid that he builds, he is not given a chance by the other students to use these numbers to find a pattern or rule. In this sense he is Image Making for the concept, his activity could allow him to spot a pattern, for example counting the cubes he has added to one side, multiplying by four and adding one, and so have an image for a number pattern but it would be different from that of Veronique. However, his possible numerical rule would probably be more complex to recognise and to state because of the way he has chosen to build his pyramids.

Interestingly, we see Allan working first with Veronique's image of adding a layer on the bottom, but then "seeing" what William was doing. He actually says "Oh yeah. I see, I see what, I see what he's doing. Put three more on top. There." and when Veronique offers rods for the base he says "No we already figured. I think we already ... we already got it", although William has not in fact quite finished building it on one side. Allan realises that the same pyramid will be produced by the two different methods but sees them as distinct and different approaches to the problem. If William had articulated his image, Allan would then have been in a position to explore the common nature of the two apparently different images. This mathematical activity we would call Property Noticing.

Later in the video we see a situation where the pupils are trying to calculate how many extra cubes will be needed to build the pyramid with a base of fifteen. The instructions to the task tell them to think that they have to guess answers to how many cubes are added to subsequent pyramids, and led them to think that using a calculator would therefore be cheating. However, performing the multiplication with pencil and paper is seen as acceptable by Veronique who, when Allan says “We do a wild guess. It says just guess.”, replies “So? I want it to be right!”. This decision to tackle the calculation manually causes some problems for the pupils as they cannot do this. Two digit written multiplication forms part of their Primitive Knowing, as they admit when Allan says “Okay. Just wait now. I can’t remember how you do this anymore” and Veronique replies “I did this last year”. It is, however, only their reading of the task that has made this relevant and applicable. They are not able to instantly recall how to do this multiplication and we see them folding back to try to “collect”[5]from their Primitive Knowing as it is not complete enough for them to merely remember and to apply in the context of the task. In the act of folding back here, the pupils actually move out of the topic of “number patterns and relations” and start working in the topic frame of “multiplication” – without direct reference to the original task. They have to figure out how to multiply fifteen by fifteen. In doing this they are trying to work on their Primitive Knowing, so that it is useful and applicable to the problem they are working on.

Some Concluding Comments

The Pirie-Kieren theory allows the observer to see and make sense of the different approaches to the task employed by the three pupils. In particular it accounts for the differing images being made and held by the pupils, and characterizes how these images are distinct yet, as Allan is able to recognise, they also possess similarities or common properties. The nature of the images made by Veronique and William also illustrate that as defined in the Pirie-Kieren theory, an image need not be visual. Although William is thinking about cubes and physical objects, the image of Veronique is that of a pattern that comes from these. She does not visualize cubes, but has a number sequence as her image.

The place and use of their Primitive Knowing is important for these pupils, they recognise what elements of this they need to draw on in their current working, and the incompleteness of this. They know that they need to perform a two digit multiplication sum, and also realize that they are not easily able to do this (although of course they were when using a calculator). The way in which they fold back and access their Primitive Knowing is an appropriate strategy and one that is fundamental to the continued growth of their understanding. However, in this clip their inability to successfully perform the calculation causes them to abandon “guessing” and turn instead to the calculator. In the given circumstances, with no teacher present, this was appropriate in order for them to be able to continue working on the topic of patterns.

References for All the Papers

Anghileri, J., & Johnson, D. (1988). Arithmetic operations on whole numbers: Multiplication and division. In T. R. Post (Ed.), *Teaching mathematics in grades k-8* (pp. 146-189). Toronto: Allyn and Bacon.

Bottorff, J. (1994). Using videotaped recordings in qualitative research. In J. M. Morse (Ed.), *Critical issues in qualitative research methods* (pp. 244-261). Thousand Oaks, CA: Sage.

Brousseau, G. (1986). *Fondements et méthodes de la didactique des mathématiques*. Thèse d'Etat, Université de Bordeaux, France.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Kluwer.

Byers, P. (1966). Cameras don't take pictures. *Columbia University Forum*, *2*, 27-31.

Davis, B. (1996). *Teaching mathematics: Toward a sound alternative*. New York: Garland.

Davis, R. B., & Vinner, S. (1986). The Notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior, 5,* 281-303.

Erickson, F. (1992). Ethnographic microanalysis of interaction. In M.D. LeCompte, W. L. Millroy & J. Preissle (Eds.), *The handbook of qualitative research in education* (pp. 201-225). San Diego: Academic Press.

Johnson, C. J. (2000). What you see is what you get: The importance of transcription for interpreting children's morphosyntactic development. In N. Bernstein Ratner & L. Menn (Eds.), *Methods for studying language production* (pp. 181 - 204). Mahwah, NJ: Erlbaum.

Martin, L. C. (1999). *The nature of the folding back phenomenon within the Pirie-Kieren Theory for the growth of mathematical understanding and the associated implications for teachers and learners of mathematics*. Unpublished doctoral dissertation, Oxford University.

Martin, L. C., Towers, J., & Pirie, S. E. B. (2000). Growing mathematical understanding: Layered observations. In M.L. Fernandez (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* (Vol.1, pp. 225-230). Tucson, AZ.

Mercer, N. (1995). Researching common knowledge: Studying the content and context of educational discourse. In G. Walford (Ed.), *Doing educational research* (pp. 41-58). London: Routledge.

Murray, H., & Olivier, A. (1989). A model of understanding two-digit numeration and computation. *Proceedings of The Thirteenth Annual Meeting of the International Group for the Psychology of Mathematics Education,* Paris, France, Laboratoire PSYDEE.

Mulligan, J., & Mitchelmore, M. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education, 28* (3), 309-330.

Nantais, N., & Herscovics, N. (1989). Epistemological evidence of early multiplication. *Proceedings of the Thirteenth Annual Meeting of the International Group for the Psychology of Mathematics Education,* Paris, France, Laboratoire PSYDEE.

Nickerson, R. S. (1985). Understanding understanding. *American Journal of Education, 93(2)*, 201-239.

Pirie, S. E. B. (1996). What are the data? An exploration of the use of video recording as a data gathering tool in the mathematics classroom. *Proceedings of the 18th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education Panama City, FL.* (Vol. II, pp. 553-560).

Pirie, S. E. B., & Kieren, T. E. (1991). A dynamic theory of mathematical understanding - Some features and implications. *A background paper for the Annual AERA Conference.* Chicago: AERA.

Pirie, S. E. B., & Kieren, T. (1992). The answer determines the question - Interventions and the growth of mathematical understanding. In W. Geeslin & K. Graham (Eds.), *Proceedings of The Sixteenth Annual Meeting of the International Group for the Psychology of Mathematics Education,* (Vol. 2, pp. 1-8). New Hampshire.

Pirie, S. E. B., & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26* (2-3), 165-190.

Schoenfeld, A. (1989). Exploring the process problem space: Notes on the description and analysis of mathematical processes. *Proceedings of the Eleventh Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Dekalb, IL: Northern Illinois University.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification--The case of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (MAA Notes 25, pp. 59-84). Mathematics Association of America.

Sfard, A. (1997). Commentary: On metaphorical roots of conceptual growth. In L. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 339-372). Mahwah, NJ: Erlbaum.

Sfard, A. (1998) On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27* (2), 4-13.

Sfard, A. (in press). Steering (dis)course between metaphors and rigor: Using focal analysis to investigate an emergence of mathematical objects. *Journal for Research in Mathematics Education.*

Sfard, A., & Kieran, C. (2000). *Cognition as communication: Rethinking learning-by-talking through multifaceted analysis of students' mathematical interactions.* Unpublished manuscript.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification—The case of algebra. *Educational Studies in Mathematics, 26*, 191-228.

Sierpinska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics, 18,* 371-397.

Sierpinska, A. (1990). Some remarks on understanding in mathematics. *For the Learning of Mathematics, 10(3)*, 24-37.

Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function, aspects of epistemology and pedagogy* (Vol. 25, pp. 23-5*8)*: MAA Notes.

Sierpinska, A. (1994). *Understanding in mathematics.* London: Falmer Press.

Sierpinska, A. (2000). *Master in the teaching of mathematics: Theory of didactic situations*. Concordia University.

Towers, J. (1998). *Teachers' interventions and the growth of students' mathematical understanding*. Unpublished doctoral dissertation, University of British Columbia.

Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge, MA: MIT Press.

Notes

[1] A timed activity trace always makes it very easy to tell a "neutral" account of their working.

[2] A multi-link cube is a 2 cm x 2 cm x 2 cm plastic cube that can be affixed to other cubes on any one of its six faces.

[3] See Pirie, 1996, and Martin, 1999 for the pros and cons of using video-taping as a means of data gathering, and considerations of the value of video-stimulated recall

[4] I shall use the word "lesson" although sometimes what we are capturing is groups of children working outside the classroom, and sometimes our participants are adults and/or graduate students, or possible in the future pre-schoolers.

[5] See Martin (1999) for a full description of the various forms of folding back.

INCREASING THE COMPREHENSION OF FUNCTION NOTION FROM VIRTUAL EXPERIENCE USING *CABRI-II*

Verónica Hoyos
Universidad Pedagógica Nacional, México
vhoyosa@correo.ajusco.upn.mx

Two recent teaching experiments (T1 & T2) were accomplished with two different groups of tenth graders, about two different scholarly mathematical contents by designing and setting up some learning scenarios using the software Cabri-II. During the first one (T1, achieved on 1998), the mathematical content approached was the resolution of rational algebraic inequations. On the second one (T2, achieved on 2000), we approached isometric transformations. Both setting up were designed to study the establishment of connections between different settings, systems of notation, meanings (Confrey, 1993; Noss & Hoyles, 1996), or even between different domains (Presmeg, 1997).

The students action in the T1 scenario was organized first around some problems of construction using the dynamic properties of the software, in order to ask pupils about an algebraic representation of a geometric configuration of variation (Hoyos & Capponi, 2000). Then, we set up two basic algebraic activities: (a) abutting the equation for a Cartesian hyperbola (Descartes, 1637; Hoyos, Capponi & Geneves, 1998b); and (b) the graphical representation of algebraic expressions (Hoyos & Capponi, 1999). We believe that the actions executed by the students by means of Cabri-II become fundamental building blocks for an understanding of the variation and functional dependence of values x and f(x) (Hoyos, 1999). The kind of tasks through which our results suggest that the establishment of connections occurred between the algebraic and graphic representations were of the following type:

> "Solve graphically $(1\div x)=(3+(x+1))$". Or also " $|5x+2| = |-2x+1|$"; and "Solve graphically $(1\div x) < (3\div(x+1))$, and $|5x+2| < |-2x+1|$." Note that in these tasks an operational perspective will not be sufficient for a satisfactory resolution.

In reference to T2, after the realization of geometric constructions using Cabri-II, we achieved an exploration scenario within the software around the study of the properties of isometric transformations. For example, in the case of the translations, the scenario had the following aspect:

> "1. Construct an AB segment. Then, construct the segment A'B' the translation of AB, using the command "translation" of Cabri (open the help menu from F1). Which geometric object are necessary to make the construction?" Note that if you select the command "translation", the help menu tells

you: "construct the image of a translated object to a vector given. The object is indicated and the vector afterwards" then the objects that you need to have in the screen are a vector and a geometric object (you must already have the segment AB in the case on time). After having created an OP vector, do the required translation."

"2. Which geometric objects can you move/vary dragging them with the mouse? Why?"

In fact, from both T1 & T2 experiments we obtained some signs that talked about pupils experiences of variability, dependence and composition; by the connections that the pupils established among different types of functional configurations (in the case of T1); or between different domains (in the case of T2). These connections probably would help pupils to make sense of the more general mathematical construct in play: the notion of function.

Note

The research presented here was partly supported by the National Council of Science and Technology (CONACyT) of Mexico. No. I29994S & No. 30430S.

NON-ANALOGOUS REASONING IN AN ANALOGOUS SITUATION OF REPEATING PATTERN

Peter Liljedahl
pgl@uniserve.com

Patterns are the heart of mathematics. Students' ability to recognize or develop patterns is related to their ability to reason mathematically in general and to develop reasoning by analogy in specific (White, Alexander & Daugherty, 1998). In this study I examine students' strategies as they engage in tasks that invite exploration of repeating patterns in familiar and unfamiliar situations.

According to English (1998) reasoning by analogy in problem solving involves mapping the relational structure of a known problem (that has been solved previously, referred to as "source") onto a new problem (referred to as "target") and using this known structure to help solve the new problem. Mapping the structure of the source problem to the target problem is the key aspect of reasoning by analogy. In order for a mapping to take place a relational structure of the source problem must be identified, as well as similar relational structure of the target problem. English (1997) suggests that failure to identify the relational structure of the source problem is a significant cause of children's impaired mathematical reasoning. Furthermore, successful solution of a target problem is not an immediate and natural consequence of a successful mapping (Novick, 1995). The solution procedure may need to be adapted or extended, especially in case of problems that are not completely isomorphic. Successful mapping is essential for recognizing the need for such adaptation. In summary, solution by analogy requires recognition/identification of a similar relational structure, mapping between source and target, and adaptation or extension.

Repeating patterns was used in this study to explore the relationships between the nature of the isomorphism between two problems and students' abilities to map solution strategies from one problem to the next. Written responses were collected from a group of 106 preservice elementary school teachers, enrolled in a core mathematics course. Participants were presented with five repeating pattern problems and the nature of their solution strategies were coded and analyzed in problem pairings as mappings.

Three distinct results came out of the data. The first is that simply varying the computational complexity of a problem caused a significant number of participants (27 out of 106) to change their solution strategy. As well, simply changing the situational setting (while keeping the problems mathematically isomorphic) was enough to change the problem solving approach of 21 of the participants. Finally, when presented with two problems that were only partially isomorphic, yet could be treated with the unifying solution strategy of *remainder in division*, 21 of the participants went from the *remainder in division* strategy in the "source" problem to a process to a *counting up/down from a multiple* strategy in the "target" problem; an inconsistency that can be viewed as a hierarchical regression in the development of students' schemes (Zazkis & Liljedahl, 2000).

Reasoning by analogy follows the sequence of recognition-mapping-adaptation. I suggest that isomorphic mapping of a problem does not necessarily imply isomorphic mapping of a solution. Solution depends on computational complexity and often patterns with small and compatible numbers are not easily generalized. Therefore, adaptation may be required not only for problems that are not completely isomorphic, as suggested by English (1998) and Novik (1995), but also for completely isomorphic problems of different degree of computational complexity.

Consideration of *remainder in division* presents a unifying scheme that enables a student to deal with a large class of problems, including all of the problems in our instrument. However, students' ability to generalize this strategy and apply it for various questions is not well developed. When faced with unfamiliar situations students often prefer to invoke *counting up from a multiple* strategy. I found in these data, as well as in-class interactions among students, that a *distance from a multiple* is a reasoning that is applied by participants in a more spontaneous and natural way. This slip back towards the familiarity of multiples is an indicator of the lack of recognition of the invariant multiplicative structure inherent in division with remainder statements.

References

English, L. D. (1998). Reasoning by analogy in solving comparison problems. In L. D. English (Ed.), *Mathematical cognition (special issue) - Mathematical reasoning: Nature, form, and development, 4* (2), 125-146. Hove, East Sussex: Psychology Press.

English, L. D. (1997). Children's reasoning processes in classifying and solving comparison word problems. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 191-220). Mahwah, NJ: Erlbaum.

Novick, L. R. (1995). Some determinants of successful analogical transfer in the solution of algebra word problems. *Thinking and Reasoning, 1* (1), 5-30.

White, C. S., Alexander, P.A., & Daugherty, M. (1998). Reasoning by analogy in solving comparison problems. In L. D. English (Ed.), *Mathematical cognition (special issue) - Mathematical Reasoning: Nature, Form, and Development, 4* (2), 103-123. Hove, East Sussex: Psychology Press.

Zazkis, R., & Liljedahl, P. (2000, September) *Arithmetic sequences as a bridge between conceptual fields.* Paper presented at the conference for Research in Undergraduate Mathematics Education. Chicago, IL.

THE DEVELOPMENT OF CHILDREN'S UNDERSTANDING OF EQUIVALENCE RELATIONSHIPS IN NUMERICAL SYMBOLIC CONTEXTS

Anne Morris
University of Delaware
abmorris@UDel.Edu

Children reason about equality relationships in many different contexts. Children with an appropriate understanding of an equivalence relationship involving symbolically represented numerical amounts, for example, can determine whether the following is true or false: 4+5 = 6+3. Children with an appropriate understanding of equivalence relationships involving collections of objects can infer that the addition of equal numbers of marbles to two unequal sets of marbles will result in unequal sets.

In the context of collections of objects, preschool children's reasoning about numerosity appears to include an equality relation and a numerical ordering relation (Gelman & Gallistel, 1978). There is also evidence that the majority of 6–7-year-olds have at least an implicit understanding of the following valid principles pertaining to the effects of additions and subtractions on numerical equivalence relationships:

1. $a + c - c = a - c + c = a$ ("inversion") (Brush, 1978; Cooper et al., 1978; Smedslund, 1966)
2. If $a = b$ and $c = d$, then $a+c = b+d$ ("compensation") (Cooper et al., 1978; Smedslund, 1966)
3. If $a = b$ and $c = d$, then $a - c = b - d$ ("compensation") (Cooper et al., 1978; Smedslund, 1966)
4. If $a = b$ and $c > d$, then $a+c > b+d$ ("complex addition") (Brush, 1978)
5. If $a = b$ and $c > d$, then $a-c < b-d$ ("complex subtraction") (Brush, 1978)

There is also a large body of literature, however, that suggests that in numerical symbolic contexts, the majority of 5–7-year-olds cannot interpret the equals sign as an indicator of an equivalence relationship and that an appropriate concept of an equality relationship is not ordinarily achieved in the absence of extensive and explicit instruction (e.g., Behr et al., 1980; Falkner et al., 1999; Sáenz-Ludlow & Walgamuth, 1998).

In the current research, experimental tasks were developed that were "symbolic versions" of the concrete tasks that have revealed young children's capacities, skills, and deficiencies at the concrete level (Brush, 1978; Cooper et al., 1983; Smedslund, 1966). These tasks were designed to investigate children's understanding of the notion of an equality relationship in a numerical symbolic context and their ability to apply the inversion, compensation, complex addition, and complex subtraction principles. The equals sign was not used in the tasks. "Traditional symbolic problems" that measured children's interpretation of the equals sign were also presented. The research examined (a) changes with age in children's understanding of the notion of an equality relationship in a numerical symbolic context and their recognition of principles pertaining to the properties of equivalence relationships in numerical symbolic contexts, (b) changes with age in children's interpretation of the symbol that represents an equality relationship (i.e., the equals sign), and (c) the relationship between (a) and (b). The tasks were presented to 6–9-year-olds in grades one, two, and four. Results will be presented at the conference.

References

Behr, M., Erlwanger, S., & Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching, 92,* 13-15

Brush, L. R. (1978). Preschool children's knowledge of addition and subtraction. *Journal for Research in Mathematics Education*, 44-54.

Cooper, R. G., Starkey, P., Blevins, B., Goth, P., & Leitner, E. (1978). *Number development: Addition and subtraction.* Paper presented at the meeting of the Jean Piaget Society, Philadelphia.

Falkner, K., Levi, L., & Carpenter, T. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics, 6* (4), 19-23.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.

Saenz-Ludlow, A., & Walgamuth, C. (1998). Third graders' interpretation of equality and the equal symbol. *Educational Studies in Mathematics, 35*, 153-187.

Smedslund, J. (1966). Microanalysis of concrete reasoning. *Scandinavian Journal of Psychology, 7*, 145-156.

THE INFLUENCE OF EXPLORING PATTERNS AND GENERALIZING ON THE FORMATION OF THE CONCEPTS OF SEQUENCES AND SERIES

Dirk Wessels
University of South Africa,
wessedcj@unisa.ac.za

Glenda Nixon
Kentron College,
nixonglenda@webonline.co.za

Cornelia Naude
University of South Africa
naudecg@unisa.ac.za

As a result of the traditional teaching approach, learners in the senior secondary phase in South Africa are inclined to rely on short cuts when trying to quickly master the formulae, important variables and necessary solution steps in sequences and series. This, however, is also true for a big percentage of our mathematics teachers. There is no broad view of seeing patterning in sequences and series as being related to visualisation, spatial thinking and reasoning. The processes of patterning and generalising are weakly developed and students experience difficulties in working with spatial patterns. Their problem solving skills and strategies are unarticulated and they have limited success in solving word problems based on sequences and series.

The purpose of this study is to determine the effects of visualization, exploring patterns, generalizing, proof and also the utilization of the relevant implications of the learning theories of Piaget, Van Hiele and Freudenthal on matriculation students' learning of sequences and series as well as proof by mathematical induction.

The investigation was done with six grade twelve students, including three boys and three girls. They were medium and top performers in their classes and were students of one of the English secondary schools in Centurion. They were exposed to fourteen 12 hour sessions, working in two small groups of three each. The material was prepared on worksheets and the students had to complete seven intensive questionnaires on the processes completed and the evaluation of their progress. The course was designed to favour exploring patterns and generalising through social interaction. It also adhered to the developmental phases identified by Van Hiele and Freudenthal, aiming to take the learners through the various thought levels so that, as a result of being provided with visual representations and patterns, they were able to reinvent for themselves the relevant principles regarding sequences, series and mathematical induction. Practical and realistic problems were also included and some historical investigations were incorporated.

The processes of "pattern formation' and "generalizing" as well as their development towards strategies (Hargreaves, Threfall, Frobisher & Sharrocks-Taylor in Orton (Ed.) 1998, p. 71) form part of the investigation ot solving sequences and series problems. One aim of the investigation is to evaluate the implications of the thinking levels of Van Hiele (supported by Freudenthal in this regard), as well as to evaluate the level and process description of working with sequences by Hargreaves et al., (1998, pp. 66-83) who also describes different levels of generalising (p. 70). At the beginning of the sessions with the grade 12 students, some learners experienced difficulties in finding and recognizing the numerical base of the spatial patterns as well as establishing formulae for spatial/visual representations. Although initially half the group did not believe that visualization played a role in normal algebra, this improves when dealing with series, limits and proof by mathematical induction. Computer values, graphs and simple illustrations were more effective than complicated block diagrams especially designed to illustrate concepts.

Students showed attainments of the different levels of thought in each new topic, finding patterns and establishing rules. The provision of exceptions to apparent rules made them aware of the need to prove their findings. The working together to reinvent results and solve problems contribute to their authentic experiences. The real progress will be known by the completion of this study in October, 2001.

References

Biggs, E., & Shaw, K. (1985). *Maths alive!* London: Cassell.

Freudenthal, H. (1991). *Revisting mathematics education: China lectures*. Dordrecht: Kluwer.

Hargreaves, M., Threfall J., Frobisher, L., & Sharrocks-Taylor, D. (1998). Children's strategies with linear and quadratic equations, In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics*. Oxford: Blackwell.

Mottershead, L. (1985). *Investigations in mathematics.* Oxford: Blackwell.

Orton, A. (1992). *Learning mathematics: Issues, theory and classroom practice* (2nd ed.). London: Cassell.

Van Hiele, P. M. (1986). *Structure and insight.* Orlando: Academic Press.

EQUIVALENCE: CONCEPT BUILDING IN A FIFTH GRADE CLASSROOM

Ann Baker
Washington State University Tri-Cities
almondbaker@hotmail.com

Amy McDuffie
Washington State University Tri-Cities
mcduffie@tricity.wsu.edu

Purpose

The purpose of this action research project was to examine the first author's practices to determine how she could help fifth graders improve their conceptual understanding of equivalence.

Background and Theory

Until recently algebra was only taught in upper grades, the updated National Council of Teachers of Mathematics (NCTM, 2000) Standards recommend teaching algebra in all grades. Teachers now need to develop and implement methods to teach algebra, in developmentally appropriate ways, to younger students. Students often have difficulty moving from an arithmetic view of the equal sign to an algebraic view. Behr, Erlwanger, and Nichols (1976) found that students view the equal sign as a "do something signal" not a relational symbol (p. 10). In algebra the equal sign is often used to compare two expressions and the concept of equivalence is fundamental to understanding algebra.

Hiebert and Carpenter (1992) recommend using multiple representations to help students build a better conceptual understanding of equivalence. Van Dyke and Craine (1999) describe four ways of representing information: graphs, equations, verbal statements, and tables. The Principles and Standards for School Mathematics (NCTM, 2000) have also acknowledged the importance of representation by giving it more prominence than it was given in the 1989 Curriculum and Evaluation Standards.

Methods

This action research study was conducted using qualitative analysis and descriptive statistics from the following data sources: (a) the first author's lesson plan book, (b) pre- and post-tests, (c) a student check sheet, (d) pre- and post-interviews, (e) student work, (f) a supplemental post-intervention assessment, (g) videotapes, tape recordings, and photographs. After learning that the students would be studying fractions during the first author's internship, we incorporated our research of teaching and learning the concept of equivalence into a unit on fractions. The first author spent more than half of her instructional math time using fraction concepts to teach the compare meaning of equivalence. The two methods used to teach the concept of equivalence were the use of multiple representations and the use of the equal sign as a comparison symbol. One representation used was the tangram. Students found fraction relationships using this model and explored equivalence relationships between the pieces.

Conclusions

Student work with tangrams indicated some understanding of the "compare" meaning of the equal sign. Equivalence had a concrete meaning with tangrams; it meant the pieces take up the same amount of area. Tangrams provided a rich way to explore fractions and algebraic thinking. By using models for equivalence with fractions, students described relationships that showed algebraic thinking.

This method of instruction requires the teacher to teach the students two concepts at once. This type of instruction requires that the students apply what they learned using fractions and generalize it to non-fraction problems. Two students demonstrated they had made this connection. It may be that the students who didn't understand the compare meaning of equivalence were the students who were the least equipped to draw connections between the concepts taught and their application to other problems. Equivalence, like other big math concepts, needs to be taught over a long period of time before student thinking can be changed (Falkner, Levi, & Carpenter, 1999).

Relationship to PME-NA Goals

It is apparent that teachers need to help students develop their understandings of equivalence. Symbols can be misinterpreted and teachers must use models and discussions to insure that their students have the same meanings for symbols as they have. This study serves to extend our understanding of students' interpretations of the equal sign symbol and of notions of equivalence and thereby serves to inform teachers' approaches to instruction.

References

Behr, M., Erlwanger, S., & Nichols, E. (1976). *How children view equality sentences* (PMDC Technical Report No. 3). Tallahassee: Florida State University.

Falkner, K., Levi, L., & Carpenter, T. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics, 6* (4), 232-236.

Hiebert, J., & Carpenter, T. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Van Dyke, F., & Craine, T. (1999). Equivalent representations in the learning of algebra. In Barbara Moses (Ed.), *Algebraic thinking: Grades k-12* (pp. 215-219). Reston, VA: National Council of Teachers of Mathematics.

MAPPING PEDAGOGICAL GROWTH ACROSS LAYERS OF THE PIRIE-KIEREN MODEL WITHIN A GROUP LESSON-PLANNING ACTIVITY

Matthew R. Clark
North Carolina State University
mrclark@unity.ncsu.edu

The Pirie-Kieren model is a theory for the growth of understanding that describes growth by an individual, for a particular topic, and over a range of time (Pirie & Kieren, 2001). The use of this model has traditionally been for the growth of mathematical understanding, but Berenson, Cavey, Clark, and Staley (2001) have proposed using this model for the growth of understanding when the individual is a pre-service mathematics teacher and the topic is what to teach and how to teach it. In another paper (Cavey, Berenson, Clark, & Staley, in press), we describe the application of this adaptation of the Pirie-Kieren model to the growth of four undergraduate math-education majors' understanding during and after a group lesson-planning task. In this study, these pre-service teachers had developed individual lesson plans and met to describe and compare their individual lessons and to develop a new lesson on the topic as a group. Student interaction in a context such as this can facilitate an individual's movement across layers of the Pirie-Kieren model, and this movement can be illustrated by paths drawn over the nested circles that represent the layers of the model (Towers, Martin, & Pirie, 2000). The purpose of this paper is to use and extend the visual representation of the Pirie-Kieren model to illustrate pedagogical growth, accommodate multiple learners in a small-group activity, and capture their interaction in a way that enables researchers to visualize individual growth and the group dynamics and to analyze how communication within the group contributes to the individuals' growth. At the poster session, I will display a chart for a group lesson-planning task.

The chart includes visual representations for the following variables: participant (coded by color), the layer of understanding (coded by Pirie and Kieren's terms within the nested circles), the line of the transcript from the group lesson-planning task, and the communication mode. The communication mode is coded by the first letter of one of the following categories: *chaining*, when an individual builds upon another's idea; *shifting*, when an individual changes the discussion topic; or *pulling*, when an individual tries to convince the other group members to accept a proposal. The path on the chart represents the discussion over time, and the line numbers correspond to the transcript of the discussion. For example, if a participant were coded by blue, the beginning of a blue segment on the path and the notation C(208) would indicate that at line 208 of the transcript that participant builds upon what the previous speaker had said. Additional text on the chart can indicate the sub-layers of the Pirie-Kieren model, such as image doing and image reviewing, which compose the layer of image making, and can mark segments when the participants shift from a pedagogical discussion to a mathematical discussion.

As we continue to analyze the transcripts from the pre-service teachers' group lesson-planning activities and evaluate the effectiveness of these activities for undergraduates in their first methods course, I hope to develop this visualization tool further and to determine whether it assists us with interpreting the data and communicating the results.

References

Berenson, S. B., Cavey, L. O., Clark, M. R., & Staley, K. (2001). *Adapting Pirie and Kieren's model of mathematical understanding to teacher preparation*. Paper presented at the Psychology of Mathematics Education, Utrecht, Netherlands.

Cavey, L. O., Berenson, S. B., Clark, M. R., & Staley, K. (in press). *Enhancing prospective teachers' understanding of what and how to teach through small-group collaboration*. Paper to be presented at the Twenty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Snowbird, UT.

Pirie, S., & Kieren, T. (2001). *From theory to practice: Susan Pirie and Tom Kieren look at how children's mathematical understanding grows*. Paper presented at the 18th Annual Rutgers Invitational Symposium on Education, May 2001, New Brunswick, NJ.

Towers, J., Martin, L., & Pirie, S. (2000). Growing mathematical understanding: Layered observations. In M. L. Fernandez (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (Vol. 1, pp. 225-230).

PRESERVICE ELEMENTARY TEACHERS' SOLUTION PROCESSES AND THEIR INTERPRETATIONS OF SOLUTIONS TO PROBLEMATIC WORD PROBLEMS INVOLVING DIVISION OF FRACTIONS

José N. Contreras
The University of Southern Mississippi
Jose.Contreras@usm.edu

Word problems provide students with opportunities and experiences to connect mathematical concepts and procedures to real-world situational contexts. They also provide simple examples of the process of mathematization or mathematical modeling. Mathematization can be defined as the representation of aspects of reality by means of mathematical procedures. The purpose of the present paper is threefold. First, we examine prospective elementary teachers' solution processes and their use (or lack of) of realistic considerations to solve problematic story problems involving division of fractions. Second, we examine the extent to which prospective elementary teachers explicitly interpret the solution to mathematical procedures. Third, we also examine the extent to which the participants' solutions support or refute a referential-and-semantic-processing model proposed by previous research (Silver, Shapiro, & Deutsch, 1993).

Sixty eight prospective elementary teachers participated in the study. The participants were enrolled in two sections of a mathematics course for elementary education majors at a southeastern state university in the USA. A paper-and-pencil test was constructed consisting of five items involving division of fractions and decimals. The test was given to students during class and they were told orally that they would have enough time to complete it. Students were not allowed to use calculators. Written instructions asked students to explain their solutions and to write down any questions, comments, or concerns that they might have had about each problem. This paper reports the results of the first problem which was stated as follows: Lida is making muffins that require 3/8 of a cup of flour each. If she has 10 cups of flour, how many muffins can Lida make?

Students' written responses to the problem were examined to detect four aspects of their solution processes: (a) mathematical model or procedures used, (b) execution of the procedures, (c) solution to the problem, and (d) explicit interpretation of the solution to the mathematical model. In this abstract we summarize findings about the participants' solution processes and their interpretations of solutions to the problematic story problem. Students' mathematical procedures, their execution, and evidence in support of Silver et al's (1993) model will be discussed during the poster presentation. A solution was coded as realistic (RS) if it was 26, 27 or any other solution with an appropriate justification. Nearly 28% of the solutions were categorized as RSs. A solution was categorized as reasonable unrealistic solution (RUS) if it was the result of an appropriate procedure executed correctly and it was different from 26. About 37% of the solutions were categorized as RUSs. A solution was categorized as incorrect reasonable unrealistic solution (nearly 6%) if it was the result of an appropriate procedure executed incorrectly. Finally, underestimated solutions close to 3 (3 and 3 3/4) or 23 (23.1 and 23) and overestimated solutions (28, 35, 375) were classified as unreasonable unrealistic responses (about 28%). An interpretation was defined as the written explanations provided by the students to justify the solution to the problem based on the execution of the procedure. Only two students provided appropriate interpretations, two gave inappropriate interpretations, and 63 (about 93%) students did not provide an interpretation to the solution produced by the mathematical model. We recognize that the muffins problem is problematic since it potentially admits multiple solutions or interpretations depending on the context but that was the whole point of the problem. The same division expression (in this case 10 ÷ 3/8) can represent different problem situations. We asked students to solve the muffins problem because we wanted to confront them with a problematic situation involving division of fractions for which the solution could be either the integer part of the quotient (26) or one more than the integer part of the quotient (27) depending on aspects of the situational context. We suspect that a contributing factor to students' failure to react realistically to the problem was students' impoverished experiences with standard problems in which the solution can be obtained by the straightforward application of one or more simple arithmetic operations with the given numbers.

References

Silver, E. A., Shapiro, L. J., & Deutsch, A. (1993). Sense making and the solution of division problems involving remainders: An examination of middle school students' solution processes and their interpretations of solutions. *Journal for Research in Mathematics Education*, *24* (2), 117-135.

MOTION OR TRANSACTIONS: DEVELOPING TEACHERS' UNDERSTANDING OF RATES OF CHANGE BY CONNECTING IDEAS FROM DIFFERENT CONTEXTS

José Aires de Castro Filho
Universidade Federal do Ceará
j.castro@ufc.br

Jere Confrey
University of Texas at Austin
jere@mail.utexas.edu

This study describes teachers' understanding of rate of change and accumulation during the implementation of a replacement unit. These ideas are not emphasized in most high school curricula, and are usually delayed until their formal introduction in calculus courses. Current research proposes that rates of change and accumulation can be introduced earlier in the curriculum (Kaput, 1994) and that they can provide a foundation for developing ideas about function and calculus (Confrey & Smith, 1994, Wilhelm, Confrey, Castro-Filho, & Maloney, 1999).

The study was conducted at an urban high school in Austin, Texas (Castro-Filho, 2000). Eight math teachers were implementing a replacement unit in the Algebra I course. The unit emphasized the use of rate of change to introduce the ideas of slope and linear function through a sequence of ten lessons. Four of these teachers were investigated in-depth while teaching two lessons called respectively the qualitative graphing and the bank account lessons. In the qualitative graphing lesson, motion detectors were used to introduce the idea of slope as a constant rate of change. In this activity, students were asked to draw graphs of position versus time and velocity versus time for a variety of situations involving walking at a steady pace. In the bank account lesson, a Java applet called Bank Account Interactive Diagram (Confrey & Maloney, 1998) was used to investigate the ideas of rate of change and accumulation in the familiar context of banking transactions. Within that context, rate of change and accumulation are represented as the daily transactions (deposits or withdrawals) and total balance respectively. The Bank Account ID was used in the curriculum as one of the activities that bridge qualitative aspects of graphing and numerical analysis leading to slope.

The study investigated teachers' understanding of rate of change and accumulation ideas in the two lessons. Data sources included: individual interviews with teachers before and after the lessons were taught; Classroom observations; videotape analysis of classrooms and of teachers' meetings with researchers. All data was coded into categories using the constant comparison method (Strauss & Corbin, 1998).

The work with different activities and technologies such as motion detectors or the Bank Account ID helped teachers develop a flexibility in their thinking. Previous activities were constantly being brought up by teachers to illustrate or explain ideas. A common connection used was relating the motion detector and the bank account lesson. Teachers switched between interpreting a graph of rates as transactions versus time or velocity versus time, and an accumulation graph as balance versus time or position versus time. The results show how technological tools such as motion detectors and the bank account ID can be catalysts to develop important mathematics ideas among teachers and students. They are relatively simple to use but they lead to important connections between fundamental ideas in mathematics.

References

Castro-Filho, J. (2000). *Teachers, math and technology: A reform effort.* Ph.D. dissertation. University of Texas at Austin.

Confrey, J. (1995). A theory of intellectual development, part 3. *For the Learning of Mathematics, 15* (2) 36-45.

Confrey, J., & Maloney, A. (1998). *Interactive diagrams* [Computer Software] Quest Math & Science Multimedia. Austin, TX.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26*, 135-164.

Kaput, J. (1994). Democratizing access to calculus. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 77-156). Hillsdale, NJ: Erlbaum.

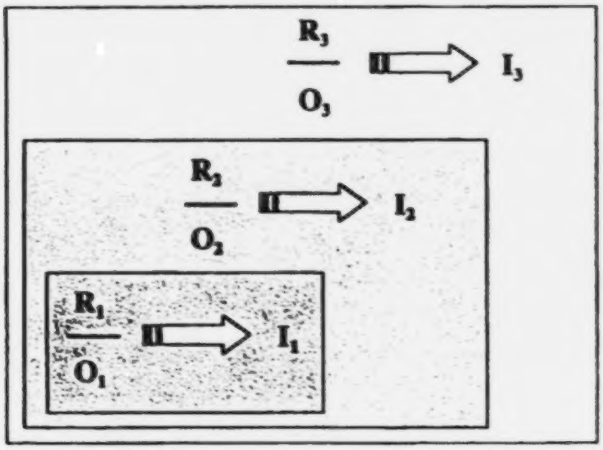
Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory.* Thousand Oaks, CA: Sage.

Wilhelm, J., Confrey, J., Castro-Filho, J., & Maloney, A. (1999). Interactive diagrams to address key student conceptions in mathematics. In D. Thomas (Ed.), *Proceedings of M/SET 99* [CD-ROM]. San Antonio, TX: Association for the Advancement of Computing in Education.

A PEIRCEAN NESTED MODEL OF SEMIOTIC CHAINING LINKING HOME AND SCHOOL MATHEMATICS

Norma C. Presmeg
Illinois State University

This model proposes to address some of the issues involved in teachers' use of their students' authentic everyday practices in the classroom teaching and learning of school mathematics at all levels. A theoretical framework, based on situated cognition and semiotic chaining, was developed (Presmeg 1997, 1998) and its viability and usefulness were tested with graduate students in courses on ethnomathematics (Presmeg, 1998), with practicing elementary school teachers (Hall, 2000), and with preservice K-8 teachers (Hunt, in progress). The original semiotic chaining model that we used was developed from the work of Walkerdine (1988) and Whitson (1994), based on Lacan and Saussure's dyadic model of semiosis. The need for a triadic model became apparent in our research. Hence I am developing a triadic nested model based on the semiotic triad used by Charles Sanders Peirce (1992). These triadic chains have the potential to address and provide insight into some of the complex issues met in our work using dyadic chains to link everyday activities and the mathematics of school classrooms.



Key

O = Object (signified)
R = Representamen (signifier)
I = Interpretant

These three components together constitute the Sign, thus the three nested rectangles represent Signs 1, 2, and 3 respectively

Figure 1. A Peircean representation of a nested chaining of three signifiers.

References

Hall, M. (2000). *Bridging the gap between everyday and classroom mathematics: An investigation of two teachers' intentional use of semiotic chains.* Unpublished Ph.D. dissertation. The Florida State University.

BEST COPY AVAILABLE

Hunt, C. A. (in progress). *The effect of knowledge of semiotic chaining on the beliefs, mathematical content knowledge, and practices of pre-service teachers.* Unpublished Ph.D. dissertation, Illinois State University.

Peirce, C. S. (1992). *The essential Peirce.* Volumes 1 & 2, edited by the Peirce Edition Project. Bloomington: Indiana University Press.

Presmeg, N. C. (1997). A semiotic framework for linking cultural practice and classroom mathematics. In J. Dossey, J. Swafford, M. Parmantie, & A. Dossey (Eds.), *Proceedings of the Nineteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 151-156). Columbus, Ohio: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Presmeg, N. C. (1998). A semiotic analysis of students' own cultural mathematics. Research Forum Report, in A. Olivier & K. Newstead (Eds.), *Proceedings of the Twenty-Second Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 136-151).

Walkerdine, V. (1988). *The mastery of reason: Cognitive developments and the production of rationality.* New York: Routledge.

Whitson, J. A. (1994). Elements of a semiotic framework for understanding situated and conceptual learning. In D. Kirshner (Ed.), *Proceedings of the Sixteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 35-50). Baton Rouge, Nov. 5-8, 1994.

UNDERGRADUATES' CONCEPTIONS OF CONGRUENCE OF INTEGERS

Jennifer Christian Smith
University of Arizona
jsmith@math.arizona.edu

Congruence of integers in modular arithmetic is an extremely important idea on which a good deal of mathematics is based. Currently no research exists examining students' understanding of this topic. The purpose of this study is to uncover undergraduates' understanding of this concept, in order to better understand their ways of thinking and misconceptions, with the eventual goal of improving instruction.

This poster represents a small piece of the researcher's dissertation project focusing on the rather broad question of determining the nature of undergraduates' understanding of congruence of integers. This poster will focus on one aspect of this understanding: How do students interpret the relationships between integers given by congruences of the form A = B (mod N)?

In elementary mathematical thinking (EMT), one begins with a focus on concrete objects, such as numbers, which are studied in order to generalize to related processes and concepts. Properties of the objects are induced from observation and comparison with other objects. Symbols are used to represent the objects, and one is gradually able to work with the symbols completely abstracted from the original objects. However, the transition from elementary to advanced mathematical thinking (AMT) requires a major shift in one's thinking processes. Instead of starting with the objects, one must start with a formal definition and from it construct a mental image of the object in question.

Since the topic is virtually unstudied, it was determined that an exploratory case study design was best suited for revealing the nature of students' understanding of congruence. Six above-average-performing students enrolled in an introductory number theory course at a large state university in the southwest completed questionnaires and were interviewed about their conceptions of statements of congruence. These data were triangulated with written responses and field notes from observations in class

The students held extremely varied conceptions of the meanings of statements of congruence. Some students interpreted the statements primarily as meaning that B is the remainder of A upon division by N, whereas others had independently constructed an informal notion of equivalence classes. Only one student seemed comfortable using the formal definition of congruence. In general, the students' level of flexibility in working with statements of congruence seemed to be related to their having an EMT or AMT perspective on the material.

In addition, many mathematical misconceptions appeared among these high-performing students, such as confusing the modular systems with base N notation, interpreting the integer B as the remainder in all cases, and assigning meaning to the position of a number in the congruence statement. That is, A = B (mod N) was generally interpreted differently from B = A (mod N) for fixed integers A and B. Several students appeared to have difficulty working with the notation used for congruence and viewed congruence as an arithmetic process to be performed on the given integers, rather than as a statement about the relationship between the integers.

Congruence of integers is a concept that seems to lie in the borderlands of EMT and AMT. It is a topic that can indeed be approached from an elementary perspective. However, the student must make the shift to thinking about congruence from an advanced perspective in order to develop a rich understanding of congruence as an equivalence relation. It may be the case that experience with AMT is necessary in order for the student to make sense of even simple statements of congruence.

References

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Dordrecht, The Netherlands: Kluwer.

Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 3-21). Dordrecht, The Netherlands: Kluwer.

Probability and Statistics

PROBABILITY MODELING: CLARIFYING THE CATEGORIES

Carol T. Benson
Illinois State University
benson@ilstu.edu

Edward S. Mooney
Illinois State University
mooney@ilstu.edu

Abstract: Benson and Jones (1999) identified probability modeling strategies that fell under the broad headings of idiosyncratic, correspondence, procedural, and conceptual probabilistic reasoning in their observations of seven students in grades two through university as they used probability generators (e.g., spinners, dice, colored bears, two-colored chips) to model probabilistic situations. The purpose of this paper is to illuminate the first two of these levels of student thinking based upon analysis of student responses to probabilistic situations obtained during a teaching experiment. Six students, three each in grades 3 and 4, participated in 7 sessions. Results of the analysis indicated 3 distinct categories of idiosyncratic and 4 distinct categories of correspondence probabilistic modeling strategies. The researchers also hypothesized that a transitional level of probabilistic modeling exists between the idiosyncratic and correspondence levels.

School mathematics reform movements worldwide have advocated the study of probability at all levels of the curriculum (e.g., Australian Education Council, 1994; National Council of Teachers of Mathematics [NCTM], 2000). In particular, proponents of these reform movements document the view that even young children will develop greater understanding of probability through experiences involving modeling and experimentation (NCTM, 2000). These experiences include the use of dice, spinners, and other probability generators in the experiments and simulations.

There has been substantial research on students' probabilistic thinking (Fischbein, Nello & Marino, 1991; Green, 1991; Jones, Langrall, Thornton & Mogill, 1997; Shaughnessy, 1992). However, little of this research has focused on probability modeling, the use of probability generators to determine probabilities, or students' ability to model contextual problems. There is evidence that such research-based knowledge of students' thinking can assist teachers in providing meaningful instruction (Fennema & Franke, 1992). With this in mind, Benson and Jones (1999) investigated how students used probability generators to model probability tasks set in three contexts. From the study, they developed a framework, consisting of four levels, to describe student thinking in modeling probability situations. Benson (2000) used the Benson and Jones framework to conduct a teaching experiment with six students, three each in grades 3 and 4. The purpose of this study is to illuminate the first two levels of the Benson and Jones framework based on the analysis of data for the assessment and instructional sessions of the teaching experiment.

Theoretical Perspective and Background

The Benson and Jones (1999) framework is based on the view that students' reasoning in probabilistic modeling develops over time and can be described across four levels. The notion of levels of thinking within a content domain coincides with cognitive research, which recognizes developmental stages (Piaget & Inhelder, 1975). The Biggs and Collis (1991) general developmental model incorporates five modes of functioning: sensormotor (from birth), ikonic (from around 18 months), concrete symbolic (from around 6 years), formal (from around 14 years), and post formal (from around 20 years). Within each mode, three cognitive levels (unistructural, multistructural, and relational) recycle and represent shifts in the complexity of students' reasoning. According to Biggs and Collis, each of the five modes of functioning emerges and develops in a way that incorporates the continuing development of earlier modes. Thus, they also recognize two other cognitive levels: the prestructural, which is related to the previous mode and the extended abstract, which is related to the next mode. We consider the ikonic and concrete symbolic modes to be most applicable to elementary school students.

Researchers have found that children demonstrate varying levels of thinking within probability. Piaget and Inhelder (1975) identified levels of probabilistic reasoning. Jones, Langrall, Thornton, and Mogill (1997) identified four levels of probabilistic reasoning in the categories of sample space, probability of an event, probability comparisons, and conditional probability. The Benson and Jones (1999) framework consisted of four levels of student thinking in modeling probability situations: idiosyncratic, correspondence, procedural, and conceptual probabilistic reasoning. A student's reasoning was classified as idiosyncratic if the student used subjective reasoning or incomplete correspondence reasoning to model a task. At the correspondence level, a student used one-to-one or other correspondences to pair sample space elements of the task and model. A student at the procedural probabilistic reasoning level used a formula, but did not indicate any understanding of why the formula was appropriate. A student at the conceptual probabilistic reasoning level provided justification for any formula or process used in modeling.

In conducting the teaching experiment, Benson (2000) observed that students who were classified at the same level often exhibited varied strategies. Moreover, students who predominately used strategies from one level occasionally used strategies from a different level. This study sought to provide detailed descriptions of students' strategies within the first two levels of the probabilistic modeling framework of Benson and Jones (1999), that is, the idiosyncratic and correspondence levels.

Methodology

Participants

Students in grades three and four at a Midwestern school formed the population for this study. Six students, three from each grade level, were selected for case-study analysis based on levels of performance in mathematics: one high, one middle, and one low. Teachers described these students as fluent in communicating their thinking and reasoning, with little or no prior school experiences in studying probability.

Procedure

Benson (2000) conducted a teaching experiment with assistance of a witness (Steffe & Thompson, 2000) to investigate the impact of instruction on the modeling strategies that children in grades 3 and 4 used for probability tasks. Based upon the framework of Benson and Jones (1999) and building upon the research literature (Fischbein, Nello & Marino, 1991; Fischbein & Schnarch, 1997; Green, 1991; Konold, 1989; Lecoutre, 1992), an assessment was developed to measure students' levels of modeling before instruction, and at one and four weeks after instruction. Tasks for instructional sessions were created, based upon ongoing assessment of modeling strategies used by students and in light of research literature (Cobb, 2000; Simon, 1997; Steffe & Thompson, 2000). Students were given standard dice, two-colored chips, an equally divided 6-section spinner, an equally divided 2-section spinner and plastic bears in six different colors to be used in modeling tasks. All interviews and teaching sessions from this study were transcribed and printed for coding. Researcher and witness field notes were also taken and students' written artifacts collected.

Data Sources and Analysis

Data sources consisted of the transcribed interviews, students' written work, researcher and witness field notes, and summaries generated during the analysis. All students participated in individual sessions with the exception that two student worked together during two instructional sessions. Thus, transcripts, field notes, and student work from 40 student sessions were analyzed for this study. A double-coding procedure (Miles & Huberman, 1994) was used to analyze students' responses. The researchers independently coded each student's response as indicating idiosyncratic or correspondence levels of probabilistic modeling. After all student transcripts from assessment and instructional sessions were read, the researchers compared the coded responses within a corresponding level to discern patterns of thinking. These patterns were used to categorize students' thinking in modeling probabilistic situations and the students' responses were then recoded using the revised categories. Throughout this process, differences in categorization and coding were discussed and agreement was negotiated.

Results

In this study, the researchers operationally defined modeling as the process of selecting a probability generator, relating corresponding sample spaces of the generator and the probabilistic situation, and demonstrating or describing the execution of a result. From analysis of the data, the researchers were able to distinguish eight categories of thinking in modeling probabilistic situations: three of the categories pertain to the idiosyncratic level, four pertain to the correspondence level, and one category, incomplete correspondence, pertains to neither level. A discussion about the patterns of thinking discerned from student responses is presented in the following paragraphs.

Idiosyncratic Level

At the idiosyncratic level, a student's probabilistic modeling is characterized by minimal to no understanding of the basic aspects of representing probabilistic situations. Examination of student interviews revealed three categories of idiosyncratic thinking: accommodation, authority, and competition. With *accommodation*, a student creates a method of "modeling" a situation that allows all perceived outcomes to occur. For example, in one task, six children each wanted to watch a different video. The students were each asked to model selecting which movie the children would watch. As shown in Figure 1, one student stated that the children could watch one movie one day and the remaining five movies another day.

In an *authoritative* model, a student creates a method of modeling in which the event or outcome(s) are controlled by the student or by the character in the probabilistic situation. For example, students were told that a child, Terry, had six shirts of different colors and two pairs of pants of different colors. Students were asked to model, which shirt/pants outfit Terry would wear if the room were dark and Terry had an equal chance of picking each shirt and each pair of pants. One student stated that Terry could go ask her mother for a flashlight and then go back and pick a red shirt and blue pants (see Figure 1). In another example (see Figure 1), the student made use of a probability generator, colored plastic bears, to make a correspondence between the sample space of the generator and the selection of ice cream desserts. However, the student did not execute a random selection of a dessert. Instead, the student stated that each person in the story was allowed to choose the dessert he or she wanted.

With *competition*, the selection of an outcome is based on skill or the winning of a contest. One student modeled the selection of one of six videos by having students act out the movie he or she wanted to see (see Figure 1). The video drawn by the person who acted the best would be selected. Another student modeled the selection of one of six different lollipops by assigning a die to each lollipop, rolling all six dice, and selecting the lollipop with the highest score (see Figure 1). The execution of this model highlighted two problems. First, the dice were not all different in color. Thus, after the student rolled all six dice, the student could not discern which color lollipop should be selected. Also, the outcome was difficult to determine when there was a tie among the dice rolled.

Correspondence Level

At the correspondence level, a student successfully models probability situations by establishing a correspondence between elements of the sample space(s) of the

Categories of Idiosyncratic Modeling

Category	Sample Responses
Accommodation	"[O]ne kid says we'll watch Aladdin and …the next time we come over we can watch the other five movies." "She could wear one pair of pants one time and then the other pair of pants the other day…She [could] wear one shirt one day, and then the other shirt the other day…"
Authority	"She could have her door open and she could ask her mom for a flashlight. And she would go back in her room and turn on the flashlight and she would open the top dresser and say, 'Oh, here's my red shirt'. And then she goes to the bottom drawer and says, 'Oh, here's my blue pants'. Then she might wear the red shirt and the blue pants." "The yellow [bear] could be chocolate in a cone, purple could be chocolate in a dish, orange could be vanilla in a cone, blue could be vanilla in a dish, red could be strawberry in a cone, and green could be strawberry in a dish. And, each person could pick their ice cream…whichever one they wanted."
Competition	"[The children] might play Charades…and each movie would be in the bucket and then they would draw a movie and they would act it out…then they would decide who did the best acting…and they would, say, watch Aladdin [since] that was the best act." "I could roll each [die] and whatever one is higher I could pick that lollipop."

Figure 1. Categories of idiosyncratic modeling and sample responses.

model and elements of the situation's sample space(s). Moreover, the student does so in a manner that maintains the respective probabilities of the elements. The researchers discerned four categories of correspondence when students modeled probability situations. All four categories of correspondence discussed in this section are of the form *x*-to-*y* in which *x* elements of the probability-generator sample space are related to *y* elements of the probability-situation sample space.

The first type of correspondence that children used in modeling was *1- to-1* correspondence. In this situation, each element of the sample space of the probability generator corresponds to an element of the sample space of the probability situation. For example, a student matched each section of a 6-section spinner to one of six desserts (see Figure 2).

The second type of correspondence consisted of two versions: *n-to-1* and *1-to-n*. The first version, *n-to-1*, is correspondence in which n ($n > 1$) elements of the sample space of the generator are matched to one element of the probability situation sample space. Figure 2 shows how one student assigned three numbers of the die to one person and three other numbers of the die to another person in order to model selecting one of the persons at random. The other version of correspondence, *1-to-n*, occurs when a student relates one element of the probability generator's sample space to n ($n > 1$) elements of the probability situation sample space. For example, a student assigned one section of a 2-section spinner to three girls and the other section of the spinner to three boys to model the selection of a boy or a girl (see Figure 2).

With the third type of correspondence, *n-to-n*, n ($n > 1$) elements of the sample space of the generator correspond to n elements of the situation's sample space. A student, for example, used a die by assigning three numbers to three red shirts, two numbers to two blue shirts and one number to one green shirt in order to model the selection of a shirt (see Figure 2).

The last correspondence, *n-to-m*, occurred when a student assigned n ($n > 1$) elements of the probability generator's sample space to m ($m > 1$; $m \neq n$) elements of the sample space of the probability situation. In the selection of a boy or a girl from three boys and three girls, a student used four purple bears and four red bears to represent the boys and girls, respectively (see Figure 2). The student went on to say that any number of bears could be used as long as the numbers of each are the same.

Incomplete Correspondence

A final category of modeling was identified as *incomplete correspondence*. In this category, students attempt to correspond elements of the probability-generator sample space to elements of the sample space of the probability situation. However, they do not attend to all the aspects of correspondence. Four patterns of incomplete correspondence were seen in this study. The first way was that students restricted the sample space of the probability situation. For example, students were asked to model the chance of selecting a Disney video from a group of three Disney movies and three non-Disney movies. Some students assigned correspondence to the three Disney videos and ignored the three non-Disney videos. For the second pattern, students left the correspondence of sample spaces undefined. One student, for example, modeled the selection of one of three sandwiches by placing three dice in a cup and selecting one at random. The student failed, however, to assign a particular sandwich to each die. After executing a trial, the student discovered the error and corrected the corre-

Categories of Correspondence Modeling

Category	Sample Responses
1-to-1	Task: Model selecting one of six desserts. Generator: 6-section spinner "…The smiles [section of the spinner] could be vanilla in a dish, the music could be vanilla in a cup, candles could be chocolate in a dish, the stars could be chocolate in a cup, and lightning could be strawberry in a cup and the hearts could be strawberry
n-to-1	Task: Model selecting one of two students. Generator: 1 die "Beth gets number one, two and three and Becky gets numbers three, I mean, four, five and six."
1-to-n	Task: Model selection of a boy or a girl from a group of three boys and three girls. Generator: 2-section spinner "Pink is girls and blue is boys."
n-to-n	Task: Model selecting one shirt from three red shirts, two blue shirts and one green shirt. Generator: 1 die "One is red, two is red, three is red, four and five are blue, and six is green."
n-to-m	Task: Model selection of a boy or a girl from a group of three boys and three girls. Generator: colored plastic bears I: So, you used one purple [bear to represent the boys] and one red [bear to represent the girls]. And, you used three purples and three reds, and now… S: Four [purple and four red bears].

Figure 2. Categories of correspondence modeling and sample responses.

spondence. A third way in which incomplete correspondence arose was when students failed to replace an element before executing another trial. For example, when asked to select a dessert for three people, a student assigned one of six colored bears respectively to one of six desserts. The bears were placed in a cup and one was randomly selected to determine the first dessert. In the subsequent trial, the bear was not returned to the cup. The fourth pattern occurred during probability situations with two dimensions. In these cases, students overlooked the second dimension when modeling the situation. For example, one student was asked to model the type of pizza topping and crust one could order for a pizza. The student appropriately assigned a 6-section spinner to the pizza toppings and three colored bears to the types of crust. However, when the student attempted to model the pizza selection, the student focused on the toppings and overlooked the selection of a crust type.

Discussion

The purpose of this study was to illuminate the first two levels of student thinking in probabilistic modeling described in the Benson and Jones (1999) framework. Based upon analysis of student responses to probabilistic situations obtained during a teaching experiment, eight categories of probabilistic modeling were identified. The researchers hypothesize that one of the categories, incomplete correspondence, may be a transitional level of probabilistic modeling between the idiosyncratic and correspondence levels of the Benson and Jones framework. Unlike students using authoritative, accommodative or competitive strategies of modeling, students demonstrating this type of modeling showed some understanding of the basic aspects of the probability situation, which we think takes them above the idiosyncratic level. However, these students did not consistently maintain the correspondence and probability between elements of the sample space of the probability generator and the elements of the sample space of the probability situation, which is necessary to be at the correspondence level. Further research is needed to determine if such refinement to the framework is warranted.

In the same light, further research is needed to refine the categories of modeling found in this study. Only six participants in grades 3 and 4 were used in this study. And although transcripts from 40 sessions were used in the analysis, the examination of more students across a wider band of grade levels may reveal other categories of modeling that lead to further refinement of the existing categories of probabilistic modeling. Results of such a study could be used to refine the Benson and Jones (1999) framework. Ultimately, this kind of research-based knowledge of students' thinking in probabilistic modeling can assist teachers in providing meaningful instruction (Fennema & Franke, 1992).

References

Australian Education Council. (1994). *Mathematics – A curriculum profile for Australian schools.* Carlton, VIC: Curriculum Corporation.

Benson, C. T. (2000). *Assessing students' thinking in modeling probability contexts.* Unpublished dissertation, Illinois State University, Normal.

Benson, C. T., & Jones, G. A. (1999). Assessing students' thinking in modeling probability contexts. *The Mathematics Educator, 4* (2), 1–21.

Biggs, J. B. & Collis, K. F. (1991). Multimodal learning and quality of intelligent behavior. In H. A. H. Rowe (Ed.), *Intelligence: Reconceptualization and measurement* (pp. 57-66). Hillsdale, NJ: Erlbaum.

Cobb, P. (2000). Working on the inside: Conducting teaching experiments in collaboration with teachers. In A. Kelly & R. Lesh (Eds.), *Handbook on research design in mathematics and science education* (pp. 307–334). Mahwah, NJ: Erlbaum.

Fennema, E., & Franke, M. L. (1992). Teacher's knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan.

Fischbein, E., Nello, M. S., & Marino, M. S. (1991). Factors affecting probabilistic judgments in children and adolescents. *Educational Studies in Mathematics, 22,* 523–549.

Fischbein, E., & Schnarch, D. (1997). The evolution with age of probabilistic intuitively based misconceptions. *Journal for Research in Mathematics Education, 28,* 96–105.

Green, D. R. (1991). A longitudinal study of pupils' probability concepts. In D. Vere-Jones (Ed.), *Proceedings of the Third International Conference on the Teaching of Statistics, Volume 1: School and general issues* (pp. 320–328). Voorburg: International Statistical Institute.

Jones, G. A., Langrall, C. W., Thornton, C. A., & Mogill, A. T. (1997). A framework for assessing and nurturing young children's thinking in probability. *Educational Studies in Mathematics, 32,* 101–125.

Konold, C. (1989). Informal conception of probability. *Cognition and Instruction, 6,* 59–98.

Lecoutre, M. P. (1992). Cognitive models and problem space in "purely random" situations. *Educational Studies in Mathematics, 23,* 557–568.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data.* Thousand Oaks, CA: Sage Publications.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Piaget, J., & Inhelder, B. (1975). *The origin of the idea of chance in children* (L. Leake, Jr., P. Burrell & H. D. Fischbein, Trans.). New York: Norton. (Original work published in 1951.)

Shaughnessy, J. M. (1992). Research in probability and statistics: Reflections and directions. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 465–494). New York: Macmillan.

Simon, M. A. (1997). Developing new models of mathematics teaching: An imperative for research on mathematics teacher development. In E. Fennema & B. S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 55–86). Mahwah, NJ: Erlbaum.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. Kelly & R. Lesh (Eds.), *Handbook on research design in mathematics and science education* (pp. 267–306). Mahwah, NJ: Erlbaum.

STUDENTS' DEVELOPMENTS IN SOLVING DATA-HANDLING ENDS-IN-VIEW PROBLEMS

Lyn D. English
Queensland University of Technology
Australia

Overview

This paper addresses components of a 3-year longitudinal study in which 9th and 10th grade students in Brisbane (Australia), Canada, and Zambia participated in data-handling programs through networked learning communities. Of interest here are the students' responses to a selection of "ends-in-view" problems, which formed the major part of the data-handling programs. The nature and role of these ends-in-view problems in promoting students' mathematical learning are addressed first. In the second part of the paper, the cognitive and social developments of groups of 9th and 10th grade students as they worked some of the ends-in-view problems are examined.

Data Handling

Data handling is recognized as an important topic within school mathematics and has gained an increasingly visible role in the K-12 curriculum (English, Charles, & Cudmore, 2000; Greer, 2000). As Shaughnessy, Garfield, and Greer (1996) noted, data handling is not just dealing with a body of statistical content, rather, it is "an approach to dealing with data, a frame of mind, an environment within which one can explore data" (p. 205). In addition to helping students cope with the increasing use of data and data-based arguments in society, data-handling activities provide realistic contexts for the application of basic mathematical ideas, and can assist students in developing an appreciation of mathematics as a way of interpreting their world (Hancock, Kaput, & Goldsmith, 1992).

Accompanying the calls for students' statistical development is a focus on student-driven classroom projects where students are given opportunities to engage fully in the practices and processes of meaningful statistics (Derry, Levin, Osana, & Jones, 1998; Lehrer & Romberg, 1996; Hancock et al., 1992; Moore, 1998). In the present study, these projects adopted the form of "ends-in-view" problems (English & Lesh, in press). Ends-in-view problems that involve data handling not only comprise meaningful contexts but also address the kinds of mathematical knowledge and processes that are fundamental to dealing with the increasingly sophisticated systems of our society. As indicated later, these problems take students beyond just computing with numbers to making sense of large volumes of data, quantifying qualitative information, identifying patterns and trends, producing convincing arguments supported by appropriate data, and assessing the products generated by their peers.

Ends-in-View Problems

Tasks that present students with particular criteria for generating purposeful, complex, and multifaceted products that extend well beyond the given information are referred to as "ends-in-view" problems (English & Lesh, in press). John Dewey (Archambault, 1964) initially coined the term when he highlighted the importance of inquiries that involve anticipating the consequences of any proposed courses of actions, as well as assessing the means by which these actions can be implemented to achieve a desired end product.

Ends-in-view problems differ in several ways from the classroom problems that students typically meet. With standard textbook tasks, the "givens," the "goals," and the "acceptable" solution steps are usually specified clearly such that the interpretation process for the learner is reduced or removed completely. In contrast, ends-in-view problems not only require students to work out how to get from the given state to the goal state but also require them to interpret both the goal and the given information, as well as the permissible solution steps. This interpretation process presents a challenge in itself: there could be incomplete, ambiguous or undefined information; there might be too much data, or there might be visual representations that are difficult to interpret. When presented with information of this nature, students might make unwarranted assumptions or might impose inappropriate constraints on the products they are to develop.

Another challenging feature of ends-in-view problems is that students are presented with criteria for generating various mathematical products, which could involve constructing a new problem, developing a persuasive case, or designing a data-gathering tool. Students do not know the nature of the products they are to develop; they only know the criteria that have to be satisfied. For example, in constructing new problems, students could be required to make their problems challenging but manageable, free from cultural bias, and interesting and appealing to the solver. There is more than one way of satisfying the criteria of ends-in-view problems, which means that multiple approaches and products are possible. Students at all achievement levels are thus able to tackle problems of this type.

Ends-in-view problems are also designed with group problem-solving processes in mind. That is, the problems require the combined abilities and experiences of a diverse group of students to develop products that can be shared with others (Zawojewski, Lesh, & English, in press). In contrast to other problems that might be worked in group situations, ends-in-view problems specifically require students to develop sharable, mathematical products. This means that the products must hold up under the scrutiny of others. When students don't have to produce something sharable, they can frequently "settle for second best."

Although ends-in-view problems share the above features, they do differ in the type of end products they request. These products may be classified under three broad categories: *tools*, *constructions*, and *problems*, each of which comprises many different examples (English & Lesh, in press). *Tools* as products include models, mathematical descriptions, explanations, designs, plans, and assessment instruments. In general terms, tools are products that fulfil a functional or operational role. A *construction* normally requires students to use given criteria to develop a mathematical item, which can take many forms including spatial constructions, complex artefacts, persuasive cases, and assessments (i.e., the products of applying an assessment instrument). *Problems* as products encompass not only criteria for their construction but also criteria for judging the appropriateness and effectiveness of one problem over another. Examples of these types of products are presented in the next section.

The Data-Handling Program

Participants

Students from the 9th and 10th grades (aged from 13 to 15 years), together with their teachers, participated in the present program. The Australian students were from a private, co-educational school in a middle socioeconomic area of Brisbane. Students from Canada and Zambia also participated in the programs, via the website we established. The Brisbane students commenced the program in their 9th grade and continued with it in the following grade (N=26). The Brisbane teacher remained with the class of students for both years.

Program

A program of data-handling experiences was implemented during the third and fourth terms of the students' 9th grade year and a second, related program was implemented during the first and second terms of their 10th grade year. We worked collaboratively with the classroom teacher in developing and implementing the programs. The students worked in small groups (3 to 4 members per group) to complete the activities. It was important that we established a classroom environment where the students could work as a community of learners (Cobb & Bowers, 1999). In doing so, we encouraged the students to work collaboratively and to express and justify their ideas and sentiments in an open and constructive manner. We also placed an emphasis on analytical, critical, and philosophical thinking.

Appropriate technological support was essential to the program. The students were provided with computer workstations that contained the required software (including Excel) and that were linked to the Internet. We established a website that housed the students' data, their responses to various questions, and their products. The website also provided a forum for the students to communicate with their international peers, and a forum for researcher/teacher collaboration. Each of the 9th and 10th grade programs comprised 16 learning sessions of approximately 70 minutes each.

The ends-in-view problems that were implemented during the programs included the following:

A Tool. In the 9th grade program, the students in Brisbane and Canada were to develop an international survey for completion by all the participating students. After preliminary discussion, the students worked in their groups to develop some questions for inclusion in their survey. This involved the students in extensive posing and refining of questions, taking into account the nature of the data that their questions would yield. The students were presented with the following directions (including criteria that had to be met):

> *Think of 20 questions that you would like to include in the international survey we are constructing. Include a mix of questions that will produce nominal data, ordinal data, and interval data. Write your questions on the back of this sheet as well as creating a Word document. Remember that your questions must be suitable for the students in the other countries.*

Problems. In both the 9th and 10th grade programs, the students were to generate their own statistical problems and related issues for investigation (drawing upon the data generated from their survey). The students had previously developed their own criteria for determining what constitutes a mathematical problem, a "good" mathematical problem, a challenging problem, and a problem that appeals to the solver (including their overseas peers). These criteria not only guided the students in initial problem creation but also in critical assessment of their completed problems. The latter included testing whether one problem creation was better than another (which helped them to refine their problems) and critically analyzing and assessing one another's problems.

Constructions. The students undertook several constructions, including the development of persuasive cases (e.g., using the data from the survey, students were to present a persuasive case that argued for a particular point of view, such as the need for more computers in their school). Another construction involved the writing of a newspaper article that reported on interesting and controversial findings from their data (including comparisons of data gathered across the three countries). The students were to support their constructions with appropriate statistical representations (e.g., tables, graphs). The students also completed critical assessments of their own and their peers' constructions (with provision of constructive feedback).

Data Sources and Analysis

Each session was videotaped and audiotaped, together with fieldnotes being made. We used two video cameras, one focusing on the teacher and the whole class, and the other on groups of students. We rotated the videotaping of the student groups across the sessions, but we audiotaped all groups during each session. Transcripts were made of all the audio and videotapes. We also collected copies of students' artefacts, which included the products they developed for each of the ends-in-view problems and printouts of their website entries.

"Iterative videotape analyses" (Lesh & Lehrer, 2000), along with analyses of the students' artefacts and the classroom fieldnotes, were used to produce a description of the students' cognitive and social progress during and across the sessions. Given that several complex, dynamic systems were involved in the study (e.g., the classroom community, the student group, the individual student, and the students' evolving products), iterative analyses were deemed essential. As is typical of such analyses, initial interpretations of the data were rather disjointed and barren. However, with each successive analysis, clearer patterns of the students' mathematical and social development emerged. Some of these developments are addressed in the remainder of this paper.

Some Findings

Interpreting the Problem

In the early stages of working the ends-in-view problems, the students' initial responses were quite unstable and disjointed. The students spent considerable time trying to interpret the nature of the end product they were to develop (i.e., interpreting the criteria that had to be met) and tended to focus on certain features of the problem and ignore others. For example, in constructing questions for their international survey, some 9th graders were content with just creating questions -- any questions (and frequently got off task in doing so). Other 9th graders were cognizant of creating questions that would yield different data types, irrespective of whether the questions were appropriate or not. Only a few students in the initial stages attempted to address all of the criteria simultaneously. However, over the course of the survey construction each student group developed the ability to coordinate all the criteria in their product generation. In the two excerpts below, Kate's group is primarily concerned with getting some questions recorded, although they do show some awareness of the appropriateness of their questions. In contrast, Laurel's group spends time interpreting the criteria first, recalling an earlier learning experience to help them.

Kate's group

Kate: What are we supposed to do?

Greer: I don't know. Make some questions up.

Kate: Are we supposed to ask questions like 'What's your name' and 'What school do you go to?'

Greer: These go at the beginning, so don't bother doing it.

Kate: OK. Let's ask a question about 'What do you think about wearing a school uniform?' But don't write that. We have to have 'yes' or 'no.' 'Do you think you should wear a school uniform, um, at your school?'

Greer: 'Do you think school uniforms should be compulsory?'

Kate: Yeah.
Greer: 'Do you think the legal age for driving should be…'
Kate: But it's not a legal age for everywhere.
Bill: Do you want another question? (Kate and Greer welcome his input.)
Bill: 'How old are you?'
Bill (After the girls explain this has been covered): What about like, 'How much TV do you watch?'
Kate: 'Do you think the school hours are too long?'
Greer: 'Do you go to church?'

Laurel's group

Laurel took the problem sheet and read the directions to the group.

Mary: Right. I think it says, like, you know how we did all that data on our three questions ---
Laurel: Yes. I know what it means.
Mary: --- and I think it says, like, what type of question we can ask about those data.
Cindy: That can be subjective or objective response? It can be, like, subjective---
Laurel: Tested. We did that in science: testable or untestable. Someone's opinion or someone's actual research.
Cindy: Yeah, whether it's opinion or whether it's fact.
Mary: It should be fact because we had data for them to use.
Laurel: Yeah.
Mary: Like, they're supposed to use that data.
Laurel: Yeah. An answer shouldn't be opinionated, because therefore you wouldn't get a proper answer… All right. So we have to think of what kind of data we expect in the things---

Reconciling Individual Interpretations with Those of Others

A noticeable feature of the students' interactions in working the ends-in-view problems was the reconciling of individual interpretations with those of others. That is, individual students brought their own understandings and experiences to the group situation, at times, ignoring the viewpoints of others. However, they gradually tapped into one another's ideas and subsequently reconciled their viewpoints and interpretations with those of their peers. This reconciliation process was observed to occur in a cyclic fashion during the working of the problems (not surprisingly, students who failed to reconcile their views faced considerable difficulties). In the following excerpt, Laurel, Mary, and Cindy (10th graders) are coming to a consensus on what they consider a problem to be (this was part of their discussion on the criteria they would use to help them create and assess their mathematical problems).

Mary: So, what is a mathematical problem?
Laurel: Something that has the answer in the question.
Cindy: A question involving numbers that has an answer.
Laurel: No, because you can have words ----
Mary: Involving calculations or....
Cindy: It has to involve numbers or formulae. Formulae!
Mary: It doesn't have to have formula.
Laurel: No, it doesn't have to.... you know those things in the maths book where you tick the different things?....Um..there's one down the edge. There's all words and not numbers.
Mary: What do you mean "all words"?
Laurel: You don't always have to have numbers.
Mary: Yeah, but you need some sort of calculation like even if you have just words you need to gather information.
Laurel: Something with a question and an answer has to have a solution, doesn't it?
Mary: That's just any problem. I could ask you "Laura, what is your name?" and the answer is "Laura".
Cindy: That's the most pointless question I've ever heard.
Laurel: Yes, one characteristic....no, but things like - philosophers ask questions that can never be answered. It's the same – and you have to go "I am what I am". It doesn't have to have an answer.
Mary: OK. So it's a problem. It's a question that needs calculation to get the solution. How's that?
Laurel: OK (as she records) A question......that.....involves....calculations... ....to reach an appropriate answer? To reach.....
Cindy: To reach an appropriate conclusion or solution.
Mary: Yes, that's good.

Repeated Cycling of Refinements to Intermediate Products

An important development in the students' responses, particularly when they constructed their own problems and cases, was their attention to refining their intermediate products. This was evident at both grade levels, but there was more commitment to this process in the 10th grade than in the 9th grade. As the students made repeated refinements to their products, they also demonstrated internalization of what was once external to the group (cf. Vygotsky, 1978). The students' cycles of refinement involved improving their contextual language, refining their mathematical language (the structure of the mathematical questions or issues posed), coordinating their contextual and mathematical language, and improving any open-ended questions they wished to include (the open-ended question asked the solver to think in a critical or philosophical manner). The students frequently became bogged down in contextual concerns (e.g., which characters they should include in their problem setting), until one of the group members alerted them to the need to return to the task directions. These refinement cycles were also evident when the students acted on feedback from their peer's critiques of their finished products. In the following discussion, Laurel's group is trying to improve the mathematical questions they wish to pose in their problem. Notice how Mary alerts the group to the inaccuracy of Laurel's suggested question. Mary had internalized the importance of using the appropriate statistical procedures for given data types (the students had explored this at the beginning of each program).

Laurel: All right. We have to find the mean, median, and mode. Or which one do you think? What do you think gives a more accurate answer to the most preferred musical artist: mean, median or mode? We could ask them that........ (No response from others.) That's a question we could ask.
Cindy: (to Laurel): I wouldn't want to be asked that.
Laurel: You don't want to be asked, but we're going to ask it. They're going to give us an answer because we don't have to think of it ourselves.
Cindy: But we do. We have to.
Mary: Well, we'd better get through this, you know.
Laurel: How about 'Find the mean of the most popular musical artist---'
Mary: You can't find the mean. (Laughs)
Laurel: Can we find the mode or....
Mary: Cardinal. Okay, for the cardinal question ---
Laurel: So it's cardinal data?
Cindy: You could ask them to find the mean.
Mary: Slash mode...median.
Laurel: ...and median.
Mary: And you might ask---
Laurel: And how about – can we ask them their opinion on which do you think give the most accurate----
Mary: Yes that's a good one.
Laurel: 'Which of these do you think give the most accurate....'
Mary: 'View of the general time spent on the---'
Cindy: 'Shows the trend in the data.'
Mary: Yeah, 'trend.'
Laurel: 'Which of these......(writing).....do you think...give a true reflection---'
Cindy: 'Gives the most accurate.'
Laurel: 'Gives the most accurate.....description of the trends?'

Following this, the group had considerable discussion on the wording of their problem context, and then returned to the nature of the questions they would pose in their problem. Laurel posed an open-ended question that asked the solver to suggest reasons for the findings: "We can ask, 'Is there . . . what factors do you think might influence the people's choice of listening and music.' The group then reverted to a mathematical question they had posed earlier and tried to incorporate it within their chosen story context, which they found somewhat difficult to do:

Laurel:	..look 'Is there any correlation between the people who are groovy and the people, what music they listen to.' ... 'Please help Groovy Greg discover....find out. . .'
Mary:	Kid's language....are you trying to insult us?
Laurel:	Find out whether . . . . 'Is there is a link between the people who are groovy and what music they listen to?'

They subsequently revisited their construction of an open-ended question to be included in their problem, and spent time refining it:

Laurel:	What was the question?
Mary:	Like, think about the reasons why there's this difference between the----
Laurel:	Why is this happening?
Mary:	Yes, why is this happening.
Cindy:	Why are there differences in the ....
Laurel:	Just say...I just have to say 'And why?' for that question and they can point out the reason why. Not much to that. And...now, why what?
Mary:	Why there are differences between the data of the different countries. Does that make sense?
Cindy:	'Why are there differences between each set of data?'
Mary:	Yes, each set of data. That's it.

Concluding Points

This paper has highlighted the importance of ends-in-view problems in the mathematics curriculum and has illustrated how such problems can facilitate learning in the domain of data handling. Ends-in-view problems bring many new features to classroom problem solving. In contrast to the usual tasks that students meet, these challenging authentic problems require students to interpret both the goal and the given information, as well as permissible solution steps. Criteria are presented for generating the end product (i.e., the solver develops an end-in-view of what is to be constructed) but information on the exact nature of the product is missing. However, even though this information is missing, students know when they have developed an appropriate product. This is because the criteria serve not only as a guide for product development but also as a means of product assessment. That is, the criteria enable learners to judge the suitability of their final products as well as enabling them to assess their intermediate products.

In the study addressed here, students displayed repeated cycling of refinement processes as they worked on perfecting their products. In constructing their own problems, for example, the students cycled through the processes of refining their contextual language, refining their mathematical language, and coordinating the two. The students had internalized the key features of effective problems, which they had developed earlier in the program. The fact that the students worked in groups and had to create a product that was to be shared with others strengthened their commitment to a high-quality product.

Students' school mathematical experiences could be made more meaningful, more powerful, and more enjoyable if some of their "standard" tasks were replaced by ends-in-view problems. Further research is needed, however, on students' learning through ends-in-view problems. We need to explore how students, both individually and as a group, develop the important mathematical understandings that these problems foster. At the same time, we need to analyze students' social developments as they work as team members in generating their products.

References

Archambault, R. D. (Ed.). (1964). *John Dewey on education: Selected writings*. Chicago: University of Chicago Press.

Cobb, P., & Bowers, J. (1999). Cognitive and situated learning perspectives in theory and practice. *Educational Researcher, 28* (2), 4-15.

Derry, S. J., Levin, J. R., Osana, H. P., & Jones, M. S. (1998). Developing middle school students' statistical reasoning abilities through simulation gaming. In S. P. Lajoie (Ed.), *Reflections on statistics: Learning, teaching, and assessment in grades K-12* (pp. 175-195). Mahwah, NJ: LEA.

Doerr, H. M. (1997). Experiment, simulation and analysis: An integrated instructional approach to the concept of force. *International Journal of Science Education, 19* (3), 265-282.

English, L. D., Charles, K. L., & Cudmore, D. H. (July, 2000). Students' statistical Twenty-Fourth *Conference of the International Group for the Psychology of Mathematics Education* (pp. 265-272). Hiroshima, Japan.

English, L. D., & Lesh, R. A. (in press). Ends-in-view problems. In R. A. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective on teaching, learning, and problem solving in mathematics education*. Mahwah, NJ: Erlbaum.

Greer, B. (2000). Statistical thinking and learning. *Mathematical Thinking and Learning, 2* (1&2), 1-10. Special issue on Statistical thinking and learning.

Hancock, C., Kaput, J. J., & Goldsmith, L. T. (1992). Authentic inquiry with data: Critical barriers to classroom implementation. *Educational Psychologist, 27* (3), 337-364.

Lesh, R. A., & Doerr, H. M. (in press). Foundations of a models and modeling perspective on mathematics teaching, learning and problem solving. In R. A. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: A models and modelling perspective on teaching, learning, and problem solving in mathematics education*. Mahwah, NJ: Erlbaum.

Lesh, R. A., & Lehrer, R. (2000). Iterative refinement cycles for videotape analyses of conceptual change. In A. E. Kelly & R.A. Lesh, (Eds.), *Research design in mathematics and science education* (pp. 665-708). Mahwah, NJ: Erlbaum.

Lehrer, R., & Romberg, T. (1996). Exploring children's data modeling. *Cognition and Instruction*, *14* (1), 69-108.

Moore, D. S. (1998). Statistics among the liberal arts. *Journal of the American Statistical Association, 93*, 1253-1259.

Shaughnessy, J. M., Garfield, J., & Greer, B. (1996). Data handling. In I. Gal & J. B. Garfield, (Eds.), *The assessment challenge in statistics education* (pp. 107-121). Amsterdam: IOS Press.

Zawojewski, J. S, Lesh, R., & English, L. D. (in press). A models and modelling perspective on the role of small group learning. In R. A. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: A models and modelling perspective on teaching, learning, and problem solving in mathematics education*. Mahwah, NJ: Lawrence Erlbaum.

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

Note

The present study was made possible by a large grant from the Australian Research Council. The assistance of Kathy Charles and Leonie Harris in data collection is gratefully acknowledged, as is the willing participation of the students and class teachers.

THE STABILITY OF PROBABILISTIC REASONING

Regina D. Kiczek
Rutgers University
rkiczek@westfieldnj.com

Carolyn A. Maher
Rutgers University
cmaher3@home.com

Abstract: In this paper, we report on the development of understanding for a group of eleventh graders who worked together on two classic probability investigations during four sessions over a period of seven months. Students eventually solved the problems and showed an isomorphism between the two tasks. In the process of problem solving, students folded back to and extended earlier ideas, using a variety of representations from prior investigations, and built arguments for their solutions.

Theoretical Framework

Considerable attention has been given to the study of students' misconceptions about probability ideas and students' lack of insight into fundamental notions about chance. See, for example, Fischbein (1975), Shaughnessy (1992), and Konold, Pollatsek, Well, Lohmeier and Lipson (1993). Our work differs in that it focuses on learning about particular aspects of probabilistic thinking prior to formal instruction. Specifically, we study: (1) how ideas about probability develop in learners in the course of working together on investigations that call for sense making; (2) how stable the ideas remain after introducing conflict; (3) how durable the ideas remain over time (days, weeks, months, and sometimes, years); and (4) how aware students are of the process of sense making. Our work is consistent with the social constructivist approach to teaching and learning described in the work of Cobb and Yackel (1996), the pursuit of meaning by Doerfler (2000), and the importance of trying to understand learners' thinking in order to help them develop their thinking further (Maher & Davis, 1990). For example, Maher and Davis write:

> Paying attention to the mathematical thinking of students engaged in active mathematical constructions, and trying to make sense of what students are doing and why they are doing it, is prerequisite, we believe, to gaining insight into the nature of the development of children's representation. Observation and analysis of children's constructions while working on sets of problem situations for particular mathematical concepts can provide teachers with children's verbal, pictorial and symbolic descriptions of them. This knowledge can provide a fundamental way to assess understanding of those concepts. Knowledge of children's thinking in this regard provides the basis for creation of appropriate activities that have the potential to encourage even further learning. (p. 89)

Our work is framed by the view that posing rich tasks and inviting students to work together provides the opportunity for collaborative building of ideas from very fundamental understandings (Maher & Speiser, 1999, 2001; Kiczek, Maher & Speiser, 2001). On the basis of observations of what the students do, write, say and build as they work on tasks, researchers make decisions about what might be appropriate follow-up investigations. The subsequent tasks are designed for a variety of purposes: to clarify, to validate, to challenge, or to extend student understanding. Inherent in all we do is the requirement that the students provide a convincing argument.

Pirie and Kieren (1998, 1994) provide a useful tool for looking at how understanding grows. According to their perspective, mathematical understanding is a dynamical non-monotonic process, in which the learner moves back and forth between their basic and more advanced mathematical understanding to retrieve previously built images, make new ones, and notice properties. This process can be helpful in tracing the movement to formal understanding and generalization. To make sense of a new situation, students "fold back" to an inner level of understanding, reflecting upon and reorganizing earlier ideas in light of new information and a wider scope of knowledge and experience.

Social interaction is essential to the building of ideas. As students discuss how they are thinking about a problem, individual ideas might be accepted, rejected or modified. We emphasize the importance of providing students the time and freedom to work together, think about their ideas, and consider new understandings and the ideas of others. Reflecting on earlier ideas and how they came about is also an essential component of a deep understanding (Maher, 1999, 1998a; Cobb, Jaworski, & Presmeg, 1996).

Modes of Inquiry

This research is a component of a longitudinal study[1] that made possible a large collection of data in the form of videotapes of the students working together on carefully chosen problems, verified transcripts of those videotapes, researchers' field notes and copies of student work. Analysis of the videotaped sessions includes the coding of critical events, identification of pivotal strands (collections of distinct episodes, connected by lines of investigation and resulting in the growth of understanding), and the tracing of the development of ideas (Powell, Francisco & Maher, 2001; Kiczek, 2000). The data for this paper were collected in four after-school small group sessions of several hours' duration each and in student interviews, during and following the students' junior year in high school. Two video cameras recorded each group session; one camera captured students' actions, while the other was focused on their work. One or two cameras were used to videotape interviews. Task design was critical to the study. Tasks were designed to invite students to explore collaboratively fundamental ideas as they invented strategies, built representations, recognized patterns, and justified results. The study design provided for opportunities to revisit tasks, sometimes in different contexts or with slight variations, to see if students noticed and adjusted for subtle differences. This approach also provided an opportunity for the researchers to study the durability of student ideas.

Background

The focus group of eleventh-grade students worked together since grade one on mathematics investigations in which they were encouraged to make conjectures, develop theories about their ideas and provide justifications for their reasoning. They explored probability and combinatorics through investigations that were designed to introduce new ideas, build on earlier ideas, and extend understanding. There was no formal instruction. The students worked in pairs or small groups, often arguing about methods and strategies along the way. As they justified their solutions, their ideas were refined, modified and clarified.

We report here on the students' work on two tasks: (1) The World Series Problem, a task that invites students to find the probability that a World Series will be won in four, five, six or seven games; and (2) The Problem of Points, a task that asks students to determine an equitable distribution of stakes when a game ends before a winner is declared.

> *The World Series Problem*: In a World Series two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assuming that the teams are equally matched, what is the probability that a World Series will be won: (a) in four games? (b) in five games? (c) in six games? (d) in seven games?

> *The Problem of Points*: Pascal and Fermat are sitting in a café in Paris and decide to play a game of flipping a coin. If the coin comes up heads, Fermat gets a point. If it comes up tails, Pascal gets a point. The first to get ten points wins. They each ante up fifty francs, making the total pot worth one hundred francs. They are, of course, playing "winner takes all." But then a strange thing happens. Fermat is winning, 8 points to 7, when he receives an urgent message that his child is sick and he must rush to his home in Toulouse. The carriage man who delivered the message offers to take him, but only if they leave immediately. Of course, Pascal understands, but later, in correspondence, the problem arises: how should the 100 Francs be divided?

Combinatorics tasks involving pizza, towers, and dice, introduced in elementary school, then revisited and explored in depth in tenth grade, were especially relevant to the ideas expressed by the students in the sessions reported here. Coding used to represent winning outcomes were derived from earlier investigations of pizza and tower problems[2] (Maher, 1998b; Maher, Muter & Kiczek, in press; Kiczek & Maher, 1998; Muter & Maher, 1998). Student explorations led to the generalization that there are 2^n combinations when building towers n tall, selecting from two colors, or when n pizza toppings are available (Muter, 1999), or when *n* games are played in a world series (Kiczek, 2000).

Two different models for a sample space were first proposed by the students in the dice games of sixth grade (3/21/94). Under consideration was the list of possible outcomes when two six-sided dice were rolled. Some students supported a thirty-six element sample space, while others argued for one containing twenty-one elements. In the seventh grade pyramidal dice games (10/20/94, 10/25/94, 10/26/94), the students again argued about the number of ways a particular triple could occur. There were some conversations within groups about certain doubles being twice as likely as others or that particular triples were three or six times as likely as others. When the students revisited a pyramidal dice game[3] in tenth grade (3/20/98), they made the game fair in the same way, reasoning from two different sample spaces.

Results

The World Series problem was presented, solved, revisited and extended in four separate eleventh grade sessions spanning a seven-month period from January 22 to August 31, 1999. It was, on the surface, a new problem for the students, not explicitly related to their previous investigations. Their prior knowledge of the World Series of baseball, a familiar real-life situation, emerged in some of the ideas they expressed as they worked on the problem. The Problem of Points was introduced in the third session, revisited and solved in the fourth. The students recognized the isomorphism between the two tasks.

Session 1 (January 22, 1999)

Each student began the World Series Problem by writing strings to represent winning series. Four used the letters A and B to represent a win for Team A or Team B. The fifth student, Michael, began in the same way, but soon switched to a code of ones and zeros. The students considered each case – winning a world series in four, five, six or seven games – separately and calculated each probability before moving on to the next. They employed three different strategies. First, they used multiplication to find the probability that the series would be won in four games. They tried to extend that idea to the other cases and reported their initial result as being counter-intuitive. They indicated that the resulting fractions were getting smaller, which suggested that the probability was getting smaller. This conflicted with their intuitive idea that it should be easier for a team to win a world series when more games are played and resulted in their search for another way to solve the problem. For five and six-game world series, they switched to a procedure of listing favorable outcomes, then finding the ratio of the number of favorable outcomes to the total number of possibilities. Finally, to find the probability of a world series continuing for seven games, they subtracted the other probabilities from one. The strategy of listing was used to check this last result.

The students expressed an awareness that the four probabilities must sum to one. This led them to discover a flaw in their initial numerical results, which they corrected. The fact that the revised probabilities did indeed total one gave them confidence in their solution. This confidence was strengthened as a result of a subsequent exploration of the relationship of their results to entries in Pascal's Triangle. When Michael pointed out the pattern he noticed, the other students worked with him to explain why there was a connection to the solution of the World Series Problem. This exploration actually began with Michael's explanation of connections he had investigated the month before (12/14/98), when he expressed how he saw the entries in each row of the triangle in terms of pizza toppings and then in terms of binary code (Kiczek, Maher & Speiser, 2001). Noticing the similarity between these situations and games won or lost, Michael suggested a connection also might exist in this case.

In their discourse, we observed the students using terms such as "slots" or "spaces," invoking images that originated in prior work with tasks involving building towers or making pizzas. One student introduced the idea of holding the last game fixed when listing the winning five-game world series, an approach similar to a strategy that he had used in a tenth-grade session (6/12/98) to count particular towers built by selecting from red and blue cubes. In the current situation, he realized the last game in the series had to be a win, so what actually needed to be determined was the number of ways that a team might win three of the first four games.

Session 2 (January 29, 1999)

The next week, the students were asked to explain their work on the World Series Problem. They again expressed confidence in their solution and reported that it made sense to them. Although they were sure they should multiply to find the probability of a particular series, the students were initially unable to explain why that method worked. Eventually, students built an area model to represent the possibilities when two games were played by equally matched teams. An understanding of this scenario allowed them to generalize their solution for a longer series of games. Later, they had an opportunity to test and extend their ideas when they considered the general case, a game played by two teams that were not equally matched.

Towards the end of this session, the students were given the opportunity to reconsider their solution to the World Series Problem. The researchers introduced an alternate, incorrect solution that was proposed by some graduate students. The group tried to make sense of the conflicting answers. This triggered a folding back to the ideas on which their original solution was based. Those ideas were re-examined in light of new understandings, built in the course of explaining their work and justifying their reasoning. Several students concluded that their solution was valid and identified the flaw in the graduate students' reasoning. One student, Michael, indicated that he was not yet convinced of either solution.

Session 3 (February 5, 1999)

During the third session one week later, Michael reported that he was still thinking about the World Series Problem and the discrepancy of the solutions. From the beginning of their work on the problem, the students agreed that winning a series in seven games would be easier than winning it in four. They grappled with the concept of equally likely outcomes. One student explained: "The chance of you winning the first way has to be the same as you winning the seventieth way." Another student focused on a four-game world series to point out the flaw in the reasoning behind the incorrect solution, explaining that there were only sixteen possible outcomes, not seventy. Two of those were winning series. Therefore, the probability of a world series ending in four games was 2/16, not 2/70.

It is interesting to note that in several of the earlier dice game investigations (grades six, seven and ten) students had proposed and argued for two different sample spaces, one consisting of equally likely outcomes and one comprised of elements that were not equiprobable. There had been no resolution of the two models. When the students first worked on the World Series Problem, they developed one model, consisting of seventy outcomes to which they had assigned correct weights. The graduate students' solution was based on the same seventy outcomes, but they were treated as equiprobable. The group concluded that this solution was incorrect because the seventy ways of winning the world series were not equally likely. In this session, the students were given the opportunity to create a new model, to fit a world series in which all seven games are played, and found they were able to reconcile it with their original one.

They were able to test their ideas in a new situation when they worked on the Problem of Points. After developing a solution based on the 128-element sample space, the students were introduced to this follow-up task, which was isomorphic to the original problem. Two different solutions were proposed, one based on the faulty reasoning behind the incorrect solution to the World Series Problem. The students saw a relationship between this problem and their investigation of the World Series Problem. They developed two models for the Problem of Points, one in which the game stopped when a player reached ten points and the other in which the play continued for four more flips, regardless of when a winner emerged. For the World Series Problem the two models yielded the same solution. In this case, they did not. It was suggested that the discrepancy might be because the outcomes are not equally likely or because Fermat had a one-point lead. One student remarked that it appeared that Fermat's chances got better when they continued to flip, but another pointed: "Chances can't get better." This discrepancy was problematic for the students.

Session 4 (August 31, 1999)

Seven months later, Michael revisited the World Series problem, still unresolved for him, with a classmate, Robert, who had not yet worked on it. In this session, Michael worked out the difficulty and explained it to Robert. Michael began writing strings of zeros and ones to represent winning series. Robert adopted his notation, using an organization based on his earlier work with towers. Their strategy was to count first the number of ways a world series might be won in four, five, six or seven games, and then calculate the probability of each case. They began by listing strings which represented winning world series, but Michael soon began to look for another way to determine each count. He recalled an expression for combinations, which he referred to as "choose." Once Michael was able to recreate the formula for nCr, he and Robert developed a procedure to determine the number of ways a world series might be won in seven games. They determined there were 70 ways to win a world series and calculated the probabilities using the same faulty reasoning as the graduate students. Challenged by the teacher/researcher to explain why they were treating the outcomes as equally likely, Michael reviewed for Robert how the students had originally determined the probabilities. A deeper examination of the two solutions and the reasoning behind them resulted in Michael and Robert's supporting with conviction the solution proposed by the group of five students seven months earlier.

Michael then had the opportunity to revisit the Problem of Points with Robert. Both students quickly and successfully solved it. Michael noted a relationship to the World Series Problem: "It's kind of like the World Series Problem. I think it's just exactly like it, so you're probably gonna do the same thing." They determined that Fermat and Pascal would have to flip the coin two, three or four more times and wrote strings of zeros and ones to represent winning outcomes. Robert's first idea was that the money should be split according to the number of ways each player could win. This would have led to the incorrect "sixty-forty" solution. Michael pointed out that they had to consider the "chances" of each possibility.

Robert: It's going to be sixty-forty, right? Because there's six chances for Fermat to win and four for Pascal. I think that's all of them.

Michael: I want to write it down. ... It's just kind of, like, coming back to the World Series one. Like, all the different chances of coming up. ...

Robert: No, you just can't take ... the chance of that –

Michael: No, you can't do that ... that's what we did in the World Series one.

Robert accepted this reasoning and took the lead, finding the probability of each winning outcome by multiplying individual probabilities. After working on the problem for six minutes, they concluded that the probability of Fermat winning was 11/16 and the probability of Pascal winning was 5/16. Michael remarked: "That's what we came up with before – another way, a different way of doing it, though. I remember eleven and five."

Robert presented their solution to the teacher/researchers. Michael, impressed with how quickly they solved the problem, indicated: "I think that, you know, we spent like an hour on this problem, or something, and we came up with that. We spent ten minutes and, you know, I think that it's correct." The researcher asked why it only took ten minutes to solve the problem this time, to which Michael responded: "Um, 'cause I guess I saw it like the World Series Problem, like, kind of the same, so I went that way. We did, as you can see, we did it the same way."

Conclusions

As the students worked on the World Series Problem and the Problem of Points, they moved among representations of winning outcomes that were isomorphic. In earlier combinatorics tasks, the students used codes of letters or numerals to represent particular outcomes, while Michael developed a binary code. The other students followed and referenced Michael's representation in subsequent investigations, yet they continued to use letters to represent games won or lost or flips of a coin. Robert adopted Michael's notation when they worked together in the final session.

To determine the number of winning outcomes, listing all possibilities was initially the strategy of choice. As the number of possible outcomes increased, other methods were sought. In the first session, a strategy for finding the probability of the last outcome was based on an awareness that the sum of the probabilities was one. In the fourth session, Michael and Robert developed a strategy based on recent work with combinations. Making sure the probabilities totaled one served as a check of their results.

The students shared a common language developed in the course of solving combinatorics problems involving towers, pizza and binary code. They relied on this language to explain to each other how they were counting certain combinations and what happened when another choice became available. One student invoked an image of Unifix cube towers built of "two colors and five things" to explain how he visualized five-game world series. The metaphor of adding another "slot" or "place" when increasing the number of choices was used by another student to explain why the total number of series doubled when another game was played. The generalization of 2^n to find the number of n-game world series also derived from earlier work with towers and pizza.

The students spontaneously employed the operation of multiplication to find the probability of independent events. Pressed to explain why it worked, the students responded by building a justification for a two-game series, in which the teams were equally matched, and generalized for n games. They extended that reasoning to explain further why multiplication also worked when the teams were not equally matched.

The relationship between Pascal's Triangle and the World Series Problem was identified by the students and led to the development of other ideas. Michael articulated how he saw the entries in each row in terms of pizza and then in terms of binary code. He indicated that those situations were similar to that of games won and lost, and suggested a connection that the other students were able to investigate with him. When Michael talked about the addition rule of Pascal's Triangle in terms of pizzas, he said that when another topping is added, the number of possibilities doubles, since each pizza becomes two new ones, one with the new topping and one without. He was unsure of that doubling pattern in the context of the World Series Problem. He indicated that there was one way to win three out of the first three games and pointed out that there was a "fifty-fifty chance of winning the next one – which might be why we double it." Another student followed the ideas Michael introduced and explained the addition rule of Pascal's Triangle in terms of the World Series Problem. The idea of doubling resurfaced in the fourth session and enabled Michael to make sense of the probabilities they found for a world series ending in six and seven games.

Intuitive feelings that particular events were "more likely" or "less likely" were expressed throughout the sessions. From the beginning, the students agreed that it was more difficult to win a world series in four games than in seven. This idea was key to their decisions about whether probabilities they were finding made sense or not. There was concern when they obtained the same result for the probabilities that a series would end in six or seven games. What appealed to them about the graduate student's solution was that those probabilities were different. Michael and Robert confronted the same issue in the fourth session, rejecting the group's original solution until they were able to make sense of why those numbers were the same.

Ideas related to likelihood of events had surfaced in sixth and seventh grade investigations with dice and fair games. Different models for a particular task were proposed and vigorously defended, as the students argued about whether re-orderings of number pairs or triples counted as the same or different outcomes. Two different models were proposed in these investigations. The discrepancies between the models were noted but unresolved, and there was no discussion about whether the sample points had the same likelihood of occurrence. The concept of equally likely outcomes was introduced by the teacher/ researcher in the second session. The students responded affirmatively when asked if the probabilities shown in their area model were equally likely. Later in that session, one student stated that the solution proposed by some of the graduate students was incorrect because the events from the seventy-point sample space were not equally likely. A subsequent explanation was based on the idea that for outcomes to be equally likely, their probabilities must be the same.

Throughout the four sessions, we were able to observe students folding back to earlier ideas and images to make sense of new information and situations. Our observations support Pirie and Kieren's (1998, 1994) theory for the growth of mathematical understanding.

Implications

Over a seven-month period, the students were able to make appropriate judgments about probability ideas. This seemed to be facilitated by their work on problems through which they could confront their intuitions and biases, build their own representations and strategies, and form strong conceptual foundations. They were invited to think about these ideas over several months and they noted the importance of revisiting earlier problems and reflecting on their reasoning. We find that an important condition of this study was leaving open for later exploration questions that had not yet been resolved. The students talked to each other about their ideas and tried to make sense of each other's reasoning. At times they worked individually, and other times, collaboratively, listening and questioning in their sense-making efforts. This seemed to help them move forward, not always together, but with a willingness to talk about and explain their thinking. In individual interviews conducted during this time period, students expressed an awareness of their thinking and reasoning about these problems. We contend that the growth of mathematical understanding can be fostered in students by teachers who are aware of and attend to students' thinking, and offer investigations to extend that thinking. Posing interesting, challenging problems early and revisiting them on more sophisticated levels allows for the building of ideas over time.

References

Cobb, P., Jaworski, B., & Presmeg, N. (1996). Emergent sociocultural perspectives on mathematical activity. In P. Nesher, L. P. Steffe, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 3-19). Mahwah, NJ: Erlbaum.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, 31, 75-190.

Doerfler, W. (2000). Means for meaning. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms* (pp. 99-131). Mahwah, NJ: Erlbaum.

Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children.* Boston: D. Reidel.

Kiczek, R. D. (2000). *Tracing the development of probabilistic thinking: Profiles from a longitudinal study*. Doctoral dissertation, Rutgers University, NJ.

Kiczek, R. D., & Maher, C. A. (1998). Tracing the origins & extensions of mathematical ideas. In *Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Kiczek, R. D., Maher, C. A., & Speiser, R. (2001). Tracing the origins of Michael's representation. In *NCTM 2001 yearbook: The roles of representation in school mathematics*. Reston, VA: NCTM.

Konold, C., Pollatsek, A., Well, A., Lohmeier, J., & Lipson, A. (1993). Inconsistencies in students' reasoning about probability. *Journal for Research in Mathematics Education, 24* (5), 392-414.

Maher, C. A. (1999). *How can listening to students promote the development of mathematical ideas?* Invited Plenary Address, International Conference on Mathematics and Science Education. Taipei, Taiwan.

Maher, C. A. (1998a). Constructivism and constructivist teaching – can they co-exist? In O. Bjorkqvist (Ed.), *Mathematics teaching from a constructivist point of view* (pp. 29-42). Finland: Abo Akademi, Pedagogiska fakulteten.

Maher, C. A. (1998b). Learning to reason probabilistically. In *Proceedings of the Twentieth Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 82-87). Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Maher, C. A., & Davis, R. B. (1990). Building representations of children's meanings. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 79-90). Reston, VA: National Council of Teachers of Mathematics.

Maher, C. A., Muter, E. M., & Kiczek, R. D. (in press). The development of proof making in students. In P. Boero (Ed.), *Theorems in school.*

Maher, C. A., & Speiser, R. (2001). *Polynomials, Pascal's triangle, and the building of isomorphisms.* Paper presented at the annual meeting of the American Educational Research Association. Seattle, WA.

Maher, C. A., & Speiser, R. (1999). The complexity of learning to reason probabilistically. In *Proceedings of the Twenty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Muter, E. M. (1999). *The development of student ideas in combinatorics and proof: A six-year study.* Doctoral dissertation, Rutgers University, NJ.

Muter, E. M., & Maher, C. A. (1998). Recognizing isomorphism and building proof: Revisiting earlier ideas. In *Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Pirie, S., & Kieren, T. (1998). *Pathways to formalising.* Paper presented at the Annual Meeting of the American Educational Research Association. San Diego, CA.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: how can we characterise it and how can we represent it? *Educational Studies in Mathematics,* 26, 165-190.

Powell, A. B., Francisco, J. M., & Maher, C. A. (2001). *An evolving analytical model for understanding the development of mathematical thinking using videotape data.* Paper presented at the annual meeting of the American Educational Research Association. Seattle, WA.

Shaughnessy, J. M. (1992). Research in probability and statistics: reflections and directions. In D. A. Grouws (Ed.), *Handbook of mathematics teaching and learning* (pp. 465-494). New York: Macmillan.

Notes

[1] This work was supported in part by National Science Foundation grants #MDR-9053597 (directed by R. B. Davis & C. A. Maher) and REC-9814846 (directed by C. A. Maher) to Rutgers, the State University of New Jersey. Any opinion, findings, conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation.

[2] *The Towers Problem*: Build all possible towers four [five, three, *n*] cubes tall when two colors of Unifix cubes are available. Provide a convincing argument that all possible arrangements have been found.

The Pizza Problem: A pizza shop offers a basic cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms & pepperoni. How many choices for pizza does a customer have? List all the possible different selections. Find a way to convince each other that you have accounted for all possibilities.

[3] *Another Pyramidal Dice Game*: Roll three pyramidal dice. If the sum of the three dice is 3, 4, 7, 8, or 12, Player A gets one point; if the sum is 5, 6, 9, 10 or 11, Player B gets one point. Continue rolling the dice. The first player to get ten points is the winner. (a) Is this a fair game? Why or why not? (b) Play the game with a partner. Do the results of playing the game support your answer? Explain. (c) If you think the game is unfair, how could you change it so that it would be fair?

REFINING A FRAMEWORK ON MIDDLE SCHOOL STUDENTS' STATISTICAL THINKING

Edward S. Mooney
Illinois State University
mooney@ilstu.edu

Cynthia W. Langrall
Illinois State University
langrall@ilstu.edu

Pamela S. Hofbauer
Illinois State University
pshofba@ilstu.edu

Yolanda A. Johnson
Illinois State University
yajohns@ilstu.edu

Abstract: This purpose of this study was to refine an existing cognitive framework (Mooney, in press) designed to characterize middle school students' statistical thinking. A case-study analysis was used to focus on two subprocesses of statistical thinking that were not adequately represented in the framework: students' use of multiplicative reasoning in analyzing data, and categorizing and grouping data. Twelve students, 4 from each of grades 6 - 8, were interviewed using a protocol comprised of 4 tasks designed to assess students' thinking across 4 levels: idiosyncratic, transitional, quantitative, and analytical. Based on an analysis of the interview data, descriptors were developed for each of the 4 levels of statistical thinking for both of the sub-processes. These sub-processes will be merged with Mooney's framework and the refined framework will be validated in a future study.

It is widely acknowledged that proficiency in statistical skills enables people to become productive, participating citizens in an information society and to develop scientific and social inquiry skills (National Council for the Social Studies, 1994; National Council of Teachers of Mathematics [NCTM], 2000; Secretary's Commission on Achieving Necessary Skills [SCANS], 1991). Moreover, in preparation for the workforce, SCANS (1991) recommended that benchmarks be established to inform education at the secondary level. Thus, calls for reform in mathematics education have advocated a more pervasive approach to statistics instruction at all levels (NCTM, 2000). This focus on the development and implementation of statistical application highlights the need to examine the development of students' statistical understanding, especially at the middle school level.

Researchers like Cobb et al., (1991) and Resnick (1983) have identified the need for cognitive models of students' thinking to guide the planning and development of mathematics curriculum and instruction. There is evidence that research-based knowledge of students' thinking can assist teachers in providing meaningful instruction (Fennema & Franke, 1992). With this in mind, Mooney (in press) developed and validated the Middle School Students Statistical Thinking (M3ST) framework based on a synthesis of the literature and observations and analyses of students' thinking in interview settings. The M3ST framework incorporated four statistical processes: describing data, organizing and reducing data, representing data, and analyzing and interpreting data. For each of these processes, the framework included descriptors that characterize four levels of students' statistical thinking ranging from idiosyncratic to analytical reasoning (Biggs & Collis, 1991).

In validating the framework, Mooney (in press) found that two key aspects of students' statistical thinking were not adequately assessed by the interview tasks used in his study. For the process organizing and reducing data, tasks did not elicit students' thinking about ways to categorize and group data. And for the process analyzing and interpreting data, no task was effective in evaluating students' use of multiplicative reasoning; that is, reasoning about parts of the data set as proportions of the whole to describe the distribution of data or to compare data sets. These two subprocesses of statistical reasoning are considered important to the overall development of students' statistical thinking (Cobb, 1999; Curcio, 1987; NCTM, 2000). Addressing these gaps in the framework is especially important if the framework is to be used by teachers to inform statistics instruction.

Theoretical Perspective

In this research, statistical thinking is taken to mean the cognitive actions that students engage in during the data-handling processes of describing, organizing and reducing, representing, and analyzing and interpreting data (Reber, 1995; Shaughnessy, Garfield & Greer, 1996). Descriptions of these cognitive actions are based on the general developmental model of Biggs and Collis (1991). Their model incorporates five modes of functioning: sensormotor (from birth), ikonic (from around 18 months), concrete symbolic (from around 6 years), formal (from around 14 years), and post formal (from around 20 years). Within each mode, three cognitive levels (unistructural, multistructural, and relational) recycle and represent shifts in the complexity of students' reasoning. According to Biggs and Collis, each of the five modes of functioning emerges and develops in a way that incorporates the continuing development of earlier modes. Thus, they also recognize two other cognitive levels: the prestructural which is related to the previous mode and the extended abstract which is related to the next mode. We consider the ikonic, concrete symbolic, and formal modes to be most applicable to middle school students.

Following the Biggs and Collis (1991) model, Mooney (in press) hypothesized that students could exhibit five levels of statistical thinking: *idiosyncratic*, (associated with the prestuctural level and representing thinking in the ikonic mode), *transitional*, *quantitative* and *analytical* (associated respectively with the unistructural, multistructural and relational levels; representing thinking in the concrete symbolic mode) and *extended analytical* (associated with the extended abstract level; representing thinking in the formal mode). However, students in his study only demonstrated the first four levels of statistical thinking and, thus, his M3ST framework characterized students' thinking for the idiosyncratic through analytical levels. Since the purpose of this study was to refine the M3ST framework, four levels of descriptors were developed to describe students' thinking when using multiplicative reasoning and in categorizing and grouping data.

Method

Participants

Students in grades six through eight at a Midwestern school formed the population for this study. Twelve students, four from each grade level, were selected for case-study analysis based on levels of performance in mathematics: one high, two middle, and one low.

Procedure

Based on the M3ST framework (Mooney, in press) and drawing from the research literature (e.g., Cobb, 1999; Bright & Friel, 1998; Lehrer & Schauble, 2000), four levels of descriptors were initially developed for the subprocesses, categorizing and grouping data and multiplicative reasoning. An interview protocol was designed to assess the middle school students' thinking within these subprocesses. Using the protocol, each student was individually interviewed during a 60-minute session. All interviews were audio taped and all student-generated work was collected. The interviews were transcribed for analysis.

Instrument

The researcher-developed interview protocol comprised four tasks (see Figure 1); each with a series of follow-up questions designed to assess students' thinking across the four levels, idiosyncratic through analytical. Questions were designed so students could respond orally or by generating tables or data displays. Although aspects of various statistical processes may be involved in completing each of the tasks, tasks 1 and 2 were designed to focus on the ways students grouped and categorized data. Tasks 3 and 4 focused on students' use of multiplicative reasoning.

Data Sources and Analysis

Data sources consisted of the transcribed interviews, students' written work and data displays, researcher field notes, and summaries generated during the analysis. Following the methodology used by Mooney (in press) to generate the M3ST framework, a double-coding procedure (Miles & Huberman, 1994) was used to analyze students' responses. The first two authors independently coded each student's response for each of the four tasks. Responses were coded by levels based on: (a) the initial descriptors for each level of the two statistical subprocesses and (b) descriptors generated from the data analysis that characterized students' responses, yet were not present in the initial descriptors. This occurred in the following manner. After all students'

Task 1: Shoe Sizes

50 eighth-grade students were surveyed about their shoe size. This list shows the data collected. The same information is on these cards. Your job is to arrange the data to be presented in the school newspaper.

Task 2: Pet Store

The teachers at your school were asked what kinds pets they have at home. In all, the teachers had 39 pets. A list of these pets is shown on this page. The same information is on these cards. Your job is to arrange the data to be presented in the school newspaper.

Task 3: Oscar Winners

This table shows the ages of the last 30 winners for the Best Actor and Best Actress in a movie. The same information is on these cards. The editor of the school newspaper wants you to arrange the data to be presented in the school newspaper and to determine which of these 3 headlines should go with the data.

Task 4: Study Habits

Mrs. Jones talked to the students in her mathematics classes one day about an article she read. It said that children who listened to classical music while studying performed better on tests than children who did not listen to classical music while studying. Some of her students planned to listen to classical music while studying for the next math exam. The results of the 80-point test are listed on this table. The students who listened to classical music have an "X" marked next to their name. The cards have the same information as the table.

Your job is to arrange the data to see if students who listened to classical music while studying performed better on the math test than students who did not listen to classical music while studying. The editor of the school newspaper wants you to present the data along with a headline about the comparison.

Figure 1. Interview protocol tasks.

responses to a question were read, the first two authors compared the responses to the corresponding descriptors to describe the levels of students' statistical thinking. If descriptors did not adequately characterize students' responses, the responses were examined as a whole to discern patterns of thinking. These patterns were used to revise the corresponding descriptors and the students' responses were then recoded using the revised descriptors to characterize students' levels of statistical thinking. If few or no students demonstrated thinking at a particular level of a subprocess, we interpolated the descriptor for that level based on students' thinking at other levels. Throughout this process, differences in coding were discussed and agreement was negotiated.

Results

Based on students' responses to the four interview tasks, we were able to revise descriptors for the subprocesses, categorizing and grouping data and multiplicative reasoning (see Figures 2 and 3). Overall, minor revisions were made in the wording of descriptors for purposes of clarification. A discussion about the construction of initial descriptors, patterns of thinking discerned from students' responses, and the revisions of descriptors based on the patterns of thinking for each subprocess are presented in the following paragraphs.

Categorizing and Grouping

In the M3ST framework, the statistical process *Organizing and Reducing Data* involves arranging, categorizing, or consolidating data into a summative form. The components of this process are considered critical for analyzing and interpreting data. Mooney (in press) identified three subprocesses: (a) categorizing and grouping data, (b) describing data using measures of center, and (c) describing the spread of the data. In his study, students were reluctant to organize data by arranging them into categories and they frequently displayed data in no particular order or arrangement. Thus, descriptors for this subprocess warranted further examination.

The initial descriptors for the categorizing and grouping subprocess (see Figure 2) were a modification of the descriptors in the M3ST framework and focused on students' ability to group or order data in a summative form. We hypothesized that students displaying idiosyncratic thinking would make no attempt to group or order the data in a summative form while students displaying analytical thinking would be able to group or order the data in a representative, summative form including a non-summative characteristic of the data such as the mean or a percentage. The grouping or ordering would be considered representative if information obtained in the arrangement coincided with information obtained from the original data source.

In analyzing students' responses to tasks 1 and 2, two patterns emerged. First, some students created arrangements (e.g., lists, tables, or graphs) by simply shifting data values around. In these arrangements, no new information was presented. For example, the student response at the transitional level shown in Figure 2 displays a listing of shoe sizes in numerical order. Nowhere in this listing is the frequency of each shoe size presented, even though this information could be determined from the arrangement. The second pattern pertained to students' creation of new categories to organize or consolidate the data. As shown by the student response at the analytical level in Figure 2, the categories of "fish" and "birds" were created to display the number of teacher's pets. In the data presented in the task, the particular types of pets were listed (e.g., goldfish, canary, cocker spaniel, garter snake), but no categories such as birds or fish were mentioned.

Initial descriptors were revised (see Figure 2) to reflect these patterns. In interpreting the first pattern, we recognized that students could arrange data in a non-summative form yet in a way that was reasonable or meaningful. Therefore, the descriptor at the idiosyncratic level was changed to characterize students who do not attempt to group or order the data. The descriptor at the transitional level maintained that students group or order data but not in a summative form. The second pattern in students' responses indicated that the development of new categories, rather than the inclusion of non-summative information, distinguished thinking at the quantitative and analytical levels. The descriptor at the analytical level was modified to reflect this distinction. In examining students' responses, we realized that it was not necessary for an arrangement to be representative of the data to assess a students' thinking about grouping and categorizing data. Given that the issue of representativeness is addressed in the descriptors of other subprocesses in the M3ST framework, it seemed appropriate to delete it from the descriptors for categorizing and grouping data.

Multiplicative Reasoning

The *Analyzing and Interpreting Data* process in the M3ST framework does not include descriptors pertaining to multiplicative reasoning because Mooney (in press) concluded that the tasks used in his study did not adequately distinguish levels of students' thinking for this subprocess. Based on the data handling processes that typically occur at the middle school level, we operationally defined multiplicative reasoning as reasoning about parts of the data set as proportions of the whole to describe the distribution of data or to compare data sets. We also considered multiplicative reasoning in terms of relative thinking as opposed to additive thinking. Thus, our initial descriptors (see Figure 3) focused on how students used relative thinking to describe or compare data. We hypothesized that a student exhibiting idiosyncratic thinking would not use relative thinking to describe or compare data while a student exhibiting analytical thinking would describe or compare data quantitatively and draw or justify conclusions using relative thinking.

Two patterns emerged in the analysis of students' responses to tasks 3 and 4. First, all students drew conclusions about the data regardless of how they used relative thinking to describe or compare data. Second, when comparing data, some students used relative thinking with only part of the data. For example, in task 3, students were asked to compare the ages of actresses and actors to determine whether there was evidence

Categorizing and Grouping

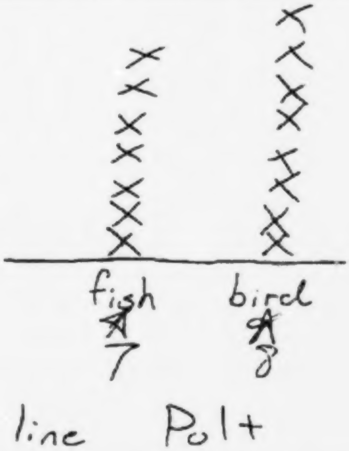
Levels	Descriptors	Sample Responses (Part of student work shown.)
Idiosyncratic	*Initial* Does not attempt to group or order the data in a summative form. *Revised* Does not attempt to group or order the data.	No responses at this level.
Transitional	*Initial* Groups or orders data but not in a summative form nor representative of the data. *Revised* Groups or orders data but not in a summative form.	5½ 6 6 6½ 7 8½ 8½ 8½ 8½ 11 11 12 14
Quantitative	*Initial* Groups or orders data in a summative form, representative of the data. *Revised* Groups or orders data in a summative form.	C.S 1 G. Sna. 1 Goldfish 5 B. cats 2
Analytical	*Initial* Groups or orders data in a representative summative from that includes at least one non-summative characteristic of the data. *Revised* Groups or orders data in a summative form by creating new categories or including categories not represented by data points.	

Figure 2. Categorizing and grouping descriptors (initial and revised) and sample responses.

BEST COPY AVAILABLE

of an age bias in Oscar winners. The student response in Figure 3 is indicative of thinking at the quantitative level. Here the student used multiplicative reasoning to conclude that for one-third of the years the actresses are older than the actors but made no explicit reference to the age comparisons in the other two-thirds of the years. In contrast, the student response for the analytical level indicates consideration all of the data when making the comparison. This student determined how many of the actresses were older than 40 (10/30) and how many of the actors were older than 40 (19/30) and concluded that the number of actors was greater than the number of actresses.

Our interpretation of these patterns led to changes in the descriptors for the quantitative and analytical levels. We concluded that it was students' use of only part of the data to make comparisons that distinguished thinking at these levels. Changes to the descriptor at the quantitative level were made to indicate that students at this level use relative thinking to make comparisons using only part of the data. At the analytical level, the descriptor was changed to indicate students' consideration of all of the data when making comparisons. Also, the reference to drawing and justifying conclusions was removed, based on the finding that all students made conclusions about the data.

Discussion

We sought to refine the Middle School Student Statistical Thinking framework (Mooney, in press) by developing descriptors for two subprocesses of statistical thinking that were not adequately addressed in the framework — categorizing and grouping data and multiplicative reasoning. The process of developing descriptors involved (a) creating initial descriptors, based on the M3ST framework and drawing from related research; and (b) refining these descriptors through the analysis of students' responses to an interview protocol designed to assess students' thinking across four levels — idiosyncratic, transitional, quantitative and analytical. For categorizing and grouping data, descriptors across levels characterized the summative forms of students' data arrangements and their use of categories to group data. However, further refinements to the descriptors for this subprocess might be necessary given that the tasks designed to assess thinking for this subprocess used only categorical data. The use of numerical data might reveal different patterns of thinking with regard to categorizing and grouping data. For multiplicative reasoning, descriptors characterized students' use of relative thinking and their consideration of the data set as a whole.

This study was one component of an extended research program that includes merging the descriptors developed in this study with the M3ST framework; validating the revised M3ST framework; and using the resulting framework with middle school teachers to guide instruction in statistics. As with Mooney's (in press) study, the descriptors we developed were based on the responses of only 12 students. Therefore, there is a need to validate the revised framework with a larger sample of students using a more comprehensive interview protocol. It is anticipated that the validation process will result in further refinement of the M3ST framework before it is used with teachers.

Categorizing and Grouping

Levels	Descriptors	Sample Responses
Idiosyncratic	*Initial* Does not use relative thinking to describe data or make comparisons. *Revised* Does not use relative thinking in situations that warrant it.	No responses at this level.
Transitional	*Initial* Uses relative thinking but does not use it to make quantitative descriptions of the data or comparisons. *Revised* Uses relative thinking but does not numerically describe data or make comparisons.	"Most of the male actors are middle age and most of the female actors are still young."
Quantitative	*Initial* Uses numerical relative thinking to describe or compare an event within data sets, but not across data sets. *Revised* Uses relative thinking to describe data or make comparisons, but considers only part of the data set.	"About in one-third of the years, the best actress is older than the best actor."
Analytical	*Initial* Describes or compares data quantitatively and draws or justifies conclusions using relative thinking. *Revised* Uses relative thinking, with the data set as a whole to describe data or make comparisons.	"Make a fraction out of it . . . 10/30 or 1/3 of the actresses are older [than 40] and 19/30 [of the actors] is . . . more than 1/3, that's older than 40."

Figure 3. Multiplicative reasoning descriptors (initial and revised) and student responses.

References

Berg, C. A., & Phillips, D. G. (1994). An investigation of the relationship between logical thinking structures and the ability to construct and interpret line graphs. *Journal of Research in Science Teaching, 31*, 323-344.

Biggs, J. B., & Collis, K. F. (1991). Multimodal learning and quality of intelligent behavior. In H. A. H. Rowe (Ed.), *Intelligence: Reconceptualization and measurement* (pp. 57-66). Hillsdale, NJ: Erlbaum.

Cobb, P. (1999). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical Thinking and Learning, 1*, 5-43.

Bright G. W., & Friel, S. N. (1998). Graphical representations: Helping students interpret data. In S. P. Lajoie (Ed.), *Reflections on statistics: Learning, teaching and assessment in grades k –12* (pp. 63-88). Mahwah, NJ: Erlbaum.

Cobb, P., Wood, T., Yackel, E., Nicholls, J., Wheatley, G., Trigatti, B., & Perlwitz, M. (1991). Assessment of a problem-centered second-grade mathematics project. *Journal for Research in Mathematics Education, 22*, 3-29.

Curcio, F. R. (1987). Comprehension of mathematical relationships expressed in graphs. *Journal for Research in Mathematics Education, 18*, 382-393.

Fennema, E., & Franke, M. L. (1992). Teacher's knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan.

Lehrer, R., & Schauble, L. (2000). Inventing data structures for representational purposes: Elementary grade students' classification models. *Mathematical Thinking and Learning, 2*, 51-74.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data*. Thousand Oaks, CA: Sage.

Mooney, E. S. (in press). A framework for characterizing middle school students' statistical thinking. *Mathematical Thinking and Learning*.

National Council for the Social Studies. (1994). *Expectation of excellence: Curriculum standards for social studies*. Washington, DC: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Reber, A. S. (1995). *Dictionary of psychology*. London: Penguin Books.

Resnick, L. (1983). Toward a cognitive theory of instruction. In S. G. Paris, G. M. Olson, & H. W. Stevenson (Eds.), *Learning and motivation in the classroom* (pp. 5-38). Hillsdale, NJ: Erlbaum.

Secretary's Commission on Achieving Necessary Skills. (1991). *What work requires of schools: A SCANS report for America 2000*. Washington, DC: Department of Labor. (ERIC Document Reproduction Service No. ED 332 054)

Shaughnessy, J. M., Garfield, J., & Greer, B. (1996). Data handling. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (Part 1, pp. 205–237). Drodrecht, The Netherlands: Kluwer.

Wainer, H. (1992). Understanding graphs and tables. *Educational Researcher, 21* (1), 14-23.

STUDENTS' REASONING ABOUT SAMPLING DISTRIBUTIONS AND STATISTICAL INFERENCE

Luis A. Saldanha
Vanderbilt University
luis.a.saldanha@vanderbilt.edu

Patrick W. Thompson
Vanderbilt University
pat.thompson@vanderbilt.edu

Abstract: Reasoning proportionally about collections of samples is central to developing a coherent understanding of statistical inference. This paper discusses results of a classroom teaching experiment designed to support the development of such understanding. Instruction engaged students in activities that focused their attention on the variability among randomly drawn samples. There occurred a critical development in students' thinking away from the composition of *individual* samples and toward the distribution of *collections* of samples. This critical shift supported further developments in students' thinking on how to compare entire distributions of samples as a basis for conceptualizing a notion of statistical unusualness. We characterize aspects of these developments in relation to students' classroom engagement.

Background

Statistics and probability have been relatively minor topics in school mathematics, both at elementary and secondary levels. Even when they were the explicit topic of study, the focus was on calculating mean, median, and mode for sets of numbers, making bar charts from data sets, or calculating "number of successful outcomes divided by number of possible outcomes".

In like fashion, the preponderance of studies through the early 1990's on statistical and probabilistic reasoning, though relatively small, focused on students' understanding of center of a data set or determining numbers that somehow represented a data set (Strauss & Bichler, 1988; Mokros & Russel, 1995; Shaughnessy, 1992) and on representing and interpreting statistical graphs (Wainer, 1992). Connections with inferential statistics were examined only indirectly (Konold, 1993), such as examining peoples' intuitive reasoning strategies related to population proportions and judgements one might make about them (Nisbett, Krantz, Jepson, & Kunda 1983). Research on learning and teaching statistics as data analysis or its connections with probabilistic reasoning had not focused on conceptual operations and imagery ("ways of thinking and understanding") that might support coherent, sound stochastic reasoning.

Recent research has begun to address the question of what ideas and ways of thinking might be central to students' developing mature stochastic reasoning. For example, Watson and Moritz (2000) used cross-sectional data to investigate students' developing understandings of sample and sampling. Shaughnessy and colleagues (Shaughnessy, Watson, Moritz, & Reading, 1999) investigated conditions under which students would interpret situations with the understanding that outcomes will probably be different if the same situation were examined with a different set of data.

Purpose and Method

This study investigated students' abilities to conceive the ideas of variability, samples, and sampling distributions as an interrelated scheme. The constructivist teaching experiment (TE; Steffe & Thompson, 2000) was the method of inquiry: we investigated the development of students' thinking as they engaged in classroom instructional activities designed to support their ability to reason proportionally about collections of samples, and to thus conceive of them as distributions. Our aim was to produce epistemological analyses of these ideas—ways of thinking about them that are schematic, imagistic, and dynamic—and hypotheses about their development in relation to students' classroom engagement (von Glasersfeld, 1995; Thompson & Saldanha, 2000).

Eight 10th- and 11th- grade students enrolled in a year-long non-AP statistics course at a suburban high school in Nashville, Tennessee participated in a 20-session TE during fall 1999. Students' understandings and emerging conceptions were investigated in three ways: (1) by tracing their participation in classroom discussions (all instruction was videotaped); (2) by examining their written work; (3) by conducting individual clinical interviews.

Three research team members were present in the classroom during all lessons: the second author designed and conducted the instruction; the first author observed the instructional sessions and took field notes; a third member operated the video camera(s). Although an a priori outline of the intended teaching/learning outcomes guided the TE's progress, instructional activities and lessons were revised daily according to what the research team perceived as important issues that arose for students in each instructional session.

The experiment began with sampling activities intended to focus students' attention on the variability among randomly drawn samples. Students then used computer simulations to investigate the question of what it means to think that an outcome of a stochastic experiment is unusual. This paper discusses several fundamental ideas that emerged in class activities.

Results

In the initial sampling activities, groups of students hand-drew random samples from different populations of objects whose compositions were unknown to them. They recorded the samples' results and investigated patterns in them. Discussions centered around what those patterns suggested about the sampled populations, what individual samples might look like if the experiment were repeated, and how to decide whether two sets of outcomes were alike or not. In these discussions, we first focused students' attention on the variability among samples, and there soon emerged a consensus that this variability made it difficult to make claims about a population's composition based on any individual sample's result. This led to the idea of looking at collections of samples, instead. Each group drew 10 samples of equal size from a population of objects and the class investigated how these collections, as a whole, were distributed. At this point, students' discourse began to shift away from *individual* samples to *collections* of samples. The ensuing discussions focused on how to look at such collections in order to claim something about the composition of the underlying population. One group of students, José and Keisha, had selected 10 samples of 5 candies each, with replacement, from a sack of red and white candies whose composition was unknown to everyone. After some discussion of their results and noticing that they had many samples having more white than red (see Figure 1.), students concluded that there were more white candies than red in the sack. They based this conclusion on their observation that 80% of the samples were "heavy on the white", containing three or more white candies.

No. Reds	0	1	2	3	4	5
No. Samples	0	5	3	1	0	1

Figure 1. Ten samples of 5 candies each drawn by two students from a population having 50% red and 50% white candies.

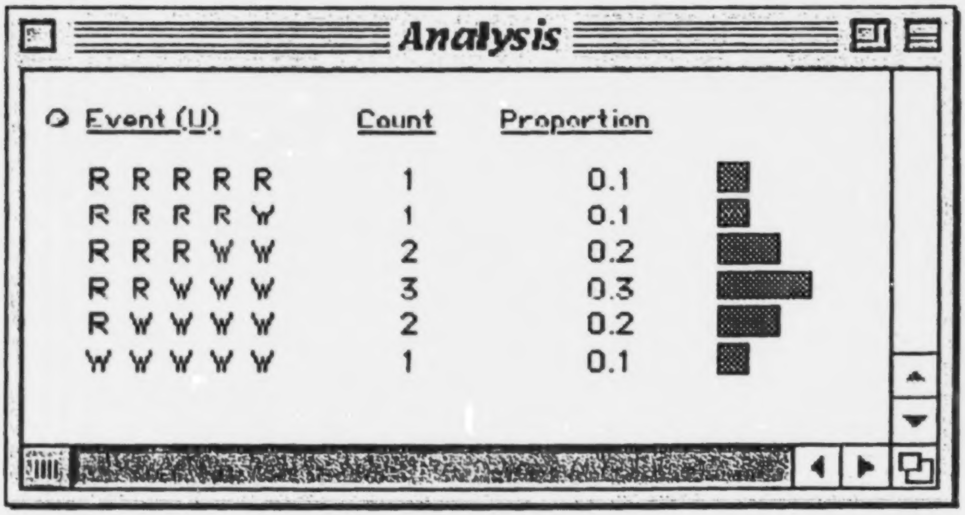
Thus, students seemed to reason that if a majority of the collection of samples each contained a majority of white candies, then this was sufficiently strong evidence to infer that the population also contained a majority of white candies. This line of reasoning suggests that they were able to coordinate two levels of thinking: one level involves individual samples and their composition; another level involves partitioning the collection of sample compositions in order to ascertain what proportion of them are composed mostly of white candies. This was the first instance in which students displayed an ability to operationalize the criterion for deciding what to infer about the population. It is worth noting that at this point, their inference seemed to be quasi-quantitative — it still lacked a precise measure. They claimed only that the population, like a majority of the samples, had a majority of white candies.

Students were surprised to learn that there were actually equal numbers of red and white candies in the sack. A sustained discussion then ensued about two questions: "might José and Keisha's results be unusual?", and "how might we investigate this question?". It was in this context that a student suggested repeating the process of selecting 10 samples of 5 candies several times with the intent of comparing this sequence of results to José and Keisha's distribution: "test it over and over", she said, without specifying the nature of the proposed comparison.

We used Prob Sim (Konold & Miller, 1994), a sampling simulation program, to efficiently simulate the experiment of drawing 10 samples of 5 candies from a population of 50% red and 50% white candies. Each time that we collected 10 samples we displayed a histogram of the number of samples having various numbers of white candies (Figure 2). For each collection of samples a class vote was taken to decide whether that distribution was similar to José and Keisha's. At first, students were uncertain about how to decide; some expressed opinions that they would not justify further. We suggested that they think of a distribution in terms of the relative weight of parts of its histogram. It emerged through the discussion that "heavy toward the white" was a criterion for saying that a simulated result resembled José and Keisha's. The class applied this criterion over 5 simulations, and the decisions were recorded and displayed in a table as shown in Figure 3. Students dismissed the possibility that José and Keisha's results were unusual when they saw that three of the first five simulations produced results "similar" to what José and Keisha had gotten.

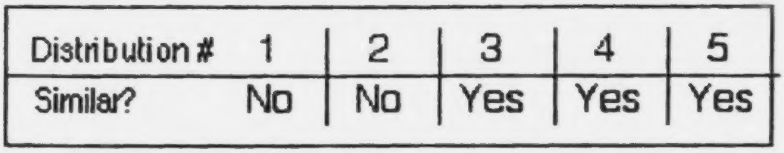
Discussion

From our perspective, a first critical development that occurred for students was their realization that the variability among samples necessitated a consideration of how



Event (U)	Count	Proportion
R R R R R	1	0.1
R R R R W	1	0.1
R R R W W	2	0.2
R R W W W	3	0.3
R W W W W	2	0.2
W W W W W	1	0.1

Figure 2. Simulated results of selecting 10 samples of 5 candies each from a collection having equal numbers of red and white candies.



Distribution #	1	2	3	4	5
Similar?	No	No	Yes	Yes	Yes

Figure 3. Record of decisions of whether simulated process produced results similar to José and Keisha's.

collections of samples were distributed in order to make an inference to the underlying population. Students seemed to reason proportionally about these collections; their inferences were based on the relative number of samples in a collection having a majority of white candies. We hypothesize that this reasoning entails conceptualizing a collection of samples as having a two-tiered structure: on a first level one focuses on individual sample compositions and develops a sense of their accumulation; on a second level one objectifies the entire collection and partitions it to determine a part's weight relative to the entire collection. These conceptual operations and their coordination can be taken to characterize critical aspects of imagining a collection of samples as comprising a distribution.

In retrospect, this line of reasoning was crucial to students' continued productive engagement in the instructional activities: it supported their eventual ability to determine a criterion for comparing entire distributions of samples (of candies) and for deciding when two such distributions are alike. These developments arose initially out of students' desire to resolve their feeling of surprise at results and they were heavily scaffolded by instruction that capitalized on this surprise. In the end, when investigating whether José and Keisha's results could be considered unusual, students were essentially engaging in informal hypothesis testing. They arrived at a conclusion by considering what proportion of a collection of 5 distributions of 10 samples were "similar" to José and Keisha's distribution. Their reasoning can thus be described as entailing the same conceptual operations described above, but applied to a more complex object: *a collection of distributions of samples*.

In sum, as students participated in directed activities and discussions, their thinking seemed to progress in complexity by moving from thinking about single samples, to reasoning proportionally about collections of samples, and finally to reasoning proportionally about collections of such collections.

References

Glasersfeld, E. v. (1995). *Radical constructivism: A way of knowing and learning*. London: Falmer Press.

Konold, C., & Miller, C. (1994). *Prob Sim*. Computer Program . Amherst, MA.

Konold, C., Pollatsek, A., Well, A., Lohmeier, J., & Lipson, A. (1993). Inconsistencies in students' reasoning about probability. *Journal for Research in Mathematics Education*, *24* (5), 392–414.

Mokros, J., & Russell, S. J. (1995). Children's concepts of average and representativeness. *Journal for Research in Mathematics Education*, *26* (1), 20-39.

Nisbett, R. E., Krantz, D. H., Jepson, C., & Kunda, Z. (1983). The use of statistical heuristics in everyday inductive reasoning. *Psychological Review*, *90*, 339–363.

Shaughnessy, J. M. (1992). Research in probability and statistics: Reflections and directions. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 465–494). New York: Macmillan.

Shaughnessy, J. M., Watson, J., Moritz, J., & Reading, C. (1999, April). School mathematics students' acknowledgement of statistical variation. Paper presented at the Research presession of the annual National Council of Teachers of Mathematics meeting, San Francisco.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-306). Dordrecht, The Netherlands: Kluwer.

Strauss, S., & Bichler, E. (1988). The development of children's concepts of the arithmetic average. *Journal for Research in Mathematics Education, 19* (1), 64-80.

Thompson, P. W., & Saldanha, L. A. (2000). Epistemological analyses of mathematical ideas: A research methodology. In M. L. Fernandez (Ed.), *Proceedings of the Twenty Second Annual Meeting of the International Group for the Psychology of Mathematics Education - North American Chapter* (Vol. 1, pp. 332-332). Tucson, AZ: ERIC Clearinghouse for Science, Mathematics, and Environmental Education. Columbus, OH.

Wainer, H. (1992). Understanding graphs and tables. *Educational Researcher*, 21 (1), 14–23.

Watson, J. M., & Moritz, J. B. (2000). Developing concepts of sampling. *Journal for Research in Mathematics Education, 31*(1), 44-70.

Note

Research reported in this paper was supported by National Science Foundation Grant No. REC-9811879. Any conclusions or recommendations stated here are those of the authors and do not necessarily reflect official positions of NSF.

THE ROLE OF PROBALISTIC SIMULATIONS AND TREE DIAGRAMS IN DETERMINING AND NURTURING STUDENTS' PRIMITIVE MATHEMATICAL INTUITIONS

Leslie Aspinwall
Florida State University
aspinwall@coe.fsu.edu

Kenneth L. Shaw
Florida State University

World-wide curriculum reforms in school mathematics (e. g., National Council of Teachers of Mathematics [NCTM], 2000; Australian Curriculum Corporation, 1994) have called for middle school students to carry out simulations of random phenomena and, through this process, to: (1) understand what it means for a game or process to be *fair* [mathematically]; (2) compute simple theoretical probabilities using tree diagrams; and (3) compare the results of simulations to the theoretical probability of an event.

This study sought to extend Fischbein's (1975) seminal body of work by presenting probabilistic situations to 8th grade students to investigate their mathematical intuitions as they relate to fairness of a game or process. Using multiple sources of data – audio-taped interview tasks and assessments, students' journals, worksheets of case study students, and researchers' journals – case study analyses were undertaken to identify patterns and changes in students' thinking as they were guided through a process to investigate their primary intuitions. Students were: (1) given a probabilistic task (or game) and asked if they thought the game was fair, (2) provided with simulations to conduct and compare to their primary intuitions, and (3) shown tree diagrams to provide a visual and enrich their intuitions. This analysis of case study students' learning attempted to isolate problem tasks centered on the determination of mathematical fairness and focused on how probability simulations and tree diagrams served as a catalyst for growth in students' intuitions and understandings.

Middle school students often develop games for competitions such as deciding "who goes first," and many seem to have good intuitive abilities in determining which games are fair. For example, students feel comfortable in playing "Odd It Out" to determine the winner. In "Out It Out," players count aloud, "one, two, three!", shaking their fists down at each count. On the count of "three," each player holds out either one or two fingers. There are two results, either the total number of fingers is even or the total number of fingers is odd. Most middle school students intuitively understand that "Odd It Out" is a fair game, though none in our study was able to explain in detail why this game is fair, other than to say that the outcome "could be odd or even." For one of the simulations, three blue cubes and one red cube are placed in a bag, and two players take turns reaching in (without looking), removing two cubes, examining and recording whether the cubes are the same color (one-color) or different colors (two-color), and then replacing them. One player is the "One-Color" player, and the other is the "Two-Color" player. Before playing the game, we asked students whether they thought the game is fair. Students' primary intuitions tended to obstruct understanding, and they incorrectly decide that the game is not fair, although they were unable to reach consensus whether "One-Color" or "Two-Color" would win. After performing the simulation, consisting of 50 trials, the result was 24 "One-Color" draws and 26 "Two-Color" draws, and this left them puzzled. They thought their results were anomalous. Students were shown the tree diagram – with the same number of equally-likely outcomes for "One-Color" or "Two-Color" and began debating among themselves; their discussions were directed toward the concept of tree diagrams, not their primary intuitions. It was clear that their mathematical intuitions were being stretched, and, in fact, they tended to believe their intuitions despite multiple simulations and the visual of the tree diagram.

Notwithstanding the positive influence of students' mathematical intuitions in understanding the fairness of such games as "Odd It Out", the researchers' study suggests middle school students need a substantial assortment of experiences with probability tasks to assist them in overcoming primitive intuitions. The tree diagrams assisted the students in visually understanding the outcomes and probabilities. In this study, students persisted in their primary intuitions despite contrary evidence from simulations and diagrams. As teachers, we need to understand that students' intuitions are powerful constructs and may adversely affect teaching and learning regardless of the clarity and logic used.

References

Australian Curriculum Corporation. (1994). *Mathematics – A curriculum profile for Australian schools.* Carlton, VIC: Curriculum Corporation.

Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children.* Dordrecth, The Netherlands: D. Reidel.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

Problem Solving

STUDENTS' USE AND UNDERSTANDING OF DIFFERENT MATHEMATICAL REPRESENTATIONS OF TASKS IN PROBLEM SOLVING INSTRUCTION

Fernando Barrera
ESFM-IPN
barrera@esfm.ipn.mx

Manuel Santos
Cinvestav, IPN, Mexico
msantos@mail.cinvestav.mx

Abstract: What kind of tasks should instructors design and implement in order to engage students in mathematical practices? This is a crucial question to discuss while implementing problem solving activities in the classroom. This study documents what first year university students showed when were asked to work on a set of learning activities that involve dealing with tasks that can be examined via the use of various representations. Although students experienced some difficulties in identifying and organizing key information of the tasks, in general, they recognized strengths and limitations associated with the use of each representation.

Introduction

Recent mathematics curriculum reforms (NCTM, 2000) have pointed out the importance for students to become engaged in the process of solving and formulating problems in which they need to identify and apply several strategies to find solutions in a variety of contexts. With this aim, the use of technology can become a powerful tool for students to achieve different representations of tasks and also as a means to formulate their own questions or problems that is a very important aspect in the learning of mathematics. We agree that the use of technology offers an important window for students to observe and analyze mathematical ideas and relationships. Students' life experiences are plenty of events that can be examined or analyzed via a mathematical formulation. These experiences can be a vehicle to promote and enhance students' mathematical habits that include the tendency to formulate conjectures, the use of diverse representations and several methods of solutions, and the use of a variety of arguments, including counterexamples, to support and communicate mathematical relationships or results. We argue that instructional tasks should provide students opportunities to engage them in mathematical thinking throughout their solution processes. Hence, to explore tasks mathematically, students need to identify and take into account information that goes beyond the superficial features of statement, and class environments should value students' active participation and contribution. What type of learning experiences do students need in order to develop ways and strategies that help them observe and examine fundamental mathematical features of tasks? This study documents the work shown by first year university students who were asked to work on tasks in which it was important to analyze the situation in terms of mathematical resources and strategies. In some tasks, students were encouraged to use a symbolic calculator to determine patterns and representations associated with the behavior of particular phenomena.

Conceptual Framework

There might be different learning trajectories for students to achieve mathematics competence; however, a common ingredient is that they all need to develop a clear disposition toward the study of the discipline. Such disposition includes a way of thinking in which they value: (a) the importance of searching for relationships among different elements (expressed via mathematical resources), (b) the need to use diverse representation to examine patterns and conjectures, and (c) the importance of providing and communicating different arguments. Thus, an important goal during the process of learning mathematics is that students develop an appreciation and disposition to be involved in genuine mathematical inquiry during their school learning experiences. Romberg and Kaput (1999) stated that a *genuine inquiry* is "the process of raising and evaluating questions, marshaling evidence from various resources, representing and presenting that information to a larger community, and debating the persuasive power of that information with respect to various claims (p.11)". Indeed, the idea that students should pose questions, search for diverse types of representations, and present different arguments during their interaction with mathematical tasks has become an important component in current curriculum proposals (NCTM, 2000). Here, the role of students goes further than viewing mathematics as a fixed, static body of knowledge; it includes that they need to conceptualize the study of mathematics as an activity in which they have to participate in order to identify, explore, and communicate ideas attached to mathematical situations.

> Students themselves become reflective about the activities they engage in while learning or solving problems. They develop relationships that may give meaning to a new idea, and they critically examine their existing knowledge by looking for new and more productive relationships. They come to view learning as problem solving in which the goal is to extend their knowledge (Carpenter & Lehrer, 1999, p. 23).

The design and implementation of tasks, which favor the use of these experiences, continue to be a great challenge in problem solving instruction (Santos, 1998). Thus, during the implementation of tasks in the classroom there is interest to discuss aspects that include the understanding of the statement of the task (the extent to which students were able to identify relevant information of the task); the use of diverse mathematical resources and strategies to deal with information embedded in the task; and the analysis of extensions proposed by students during their interaction with the task. Goldenberg (1995) states that "in current practice, the great bulk of mathematics teaching takes place within a single representational system. Much time and effort are spent in building students' skills in manipulating the formal symbolic language of traditional classroom mathematics, while relatively little time is devoted to other representations of the same ideas" (p.156).

Subjects, Tasks, Methods, and Procedures

Twenty-five first year university students who were taking a calculus course participated in the study. The instructor allotted two sessions per month (1.5 hour each session) during one semester to implement a series of tasks that demand the use of various representations and problem solving strategies. Students worked individually and in small groups. An important tool available to students was the use of a graphic and symbolic calculator. The information gathered for the analysis comes from the students' written report and students' interviews. Ten tasks were implemented throughout the development of the course and the work shown by students while working individually, in small groups and during the whole class interaction was used to support the results. In particular, the students' interviews were seen as a means to enhance their own learning (in addition to providing data for the analysis). Students worked initially on each task individually, then shared his/her ideas within a small group and eventually some of the small groups' approaches to the task were used as reference to engage students in a whole class discussion.

Features and Examples of Tasks

An important property shared by the tasks we have used during the study is that students do not need sophisticated resources to understand their statement and that it was easy to identify goals to pursue. Another component is that students' previous knowledge plays and important role in constructing other ways to approach the task. In this process, students get engaged in discussions where new and powerful mathematical ideas appear as a need to solve the task.

Two tasks given to students in the first and third sessions of the course are presented below as a way to illustrate features that guided their selection and formulation. The first one is within a mathematical context and involves dealing with quadratic equations of the form $x^2 + px + q = 0$; p and q are real numbers. Students were told that all equations of this form could be represented as points in the plane. For example, the quadratic equation $x^2 + 2x + 3 = 0$ can be represented as (2, 3), and a point in the plane as (0, 4) represents the equation $x^2 + 4 = 0$. Here, students were given a set of quadratic equations and were asked to identify their corresponding points in the plane and also they were asked to write the corresponding quadratic equations associated with a set of points. The second part of this task involved asking the students to solve both set of equations and organize them in the plane in accordance to the number of real solutions (one, two, or none solutions). The students were asked to identify points that seem to be located on the same curve and find the equation of this curve. They finally were inquired about the behavior of equations with one, two, or none real solutions in relation to their position on the curve. The main purpose of this task was to explore what type of ideas and resources students displayed in topics that they had previously studied.

The second task represents a situation in which a physician explains to his patient the way his prescribed drug will work during a period of time (treatment). The patient will take some tablets and receives the following information:

- Doses or amount of active substance of the tablet: 16 units
- When the patient takes the tablet, his organism begins to assimilate the active substance and 10 minutes after; his body will assimilate the total amount of 16 units. That is the patient organism will assimilate the doses 10 minutes after he took the tablet.
- When the patient organism assimilates the total doses of the tablet, his organism begins to eliminate the medication. The patient must take the next tablet when the previous doses are reduced to a half of that amount. The physician tells the patient that this reduction takes place every four hours. Thus, the patient will take the second tablet when the previous amount (16 units) assimilated from the first tablet is reduced to 8 units. This doses reduction will occur four hours later since the patient takes the first tablet. Thus when the patient registers 8 units, he will take the second tablet and his body will assimilate again the 16 units ten minutes later, here the amount of units stored in his body will be 8 + 16. Here, again his body commences the elimination period and when it reaches half of this amount (12 units after four hours), then the third supply takes place, and so on.

The patient will follow this treatment during a week. Based on the treatment description there is interest to analyze the amount of medication stored by the patient organism at different stages or moments during the treatment. In particular, what amount of medication does the patient store when he/she takes each tablet/ and 10 minutes later? Is it possible to discuss the evolution of the patient treatment in terms of mathematical resources?

Presentation of Results

The first task (quadratic equations) recalled students their experiences in solving equations. Although at the beginning they were surprised that quadratic equations could be represented as points in the plane, they, in general did not experience difficulties in identifying quadratics as points in the plane and conversely, points in the plane as quadratic equations. Few students used "completing the square" as a method to determine the solution of the equation. In general, they relied on the use of calculator or the general formula. Graphing the function associated with the quadratic equation was also a difficult task for some students and few students identified a relationship between points where the graph crosses the x-axis and roots of the equations. Indeed, these were the main issues that emerged during the whole class discussion that were clarified via the use of different examples.

While dealing with the part that involves grouping the roots of the equation, a small group utilized the computer to represent the points and observed that the points that seemed to appear on the same curve were actually points of a parabola. Here, the same software gave them the algebraic representation of this function, that is $y = (.5x)^2$. Based on this representation, it was easy to recognized that points which represent quadratic equation with only one real solution were located on the graph of $y = (.5x)^2$; while those points which represent quadratic equations with two solutions were located below the graph of that function. Finally, points that represent quadratic equations with no real solutions were located above the graph of $y = (.5x)^2$. Another small group detected a pattern among the points that represented equations with exactly one solution and proposed that those points had the form $(a,(a/2)^2)$. When those approaches were discussed within the entire class, students recognized that the algebraic function provided by the software was the same as that suggested by the students. During the whole group discussion, students were also asked to examine any relationship between values of discriminat involved in the general formula to find the roots of quadratic equations (Figure 1). Some students mentioned that the sign of the discriminat provides information regarding the number of real roots of quadratic equations.

It is important to notice that the visual representation of this task helped students observe relationships among the points (equations). Another observation that emerged from observing the graph representation was that "if a quadratic equation is given randomly then it is highly probably that this equation has two real solutions". This statement was based on observing the regions in which the corresponding points are located.

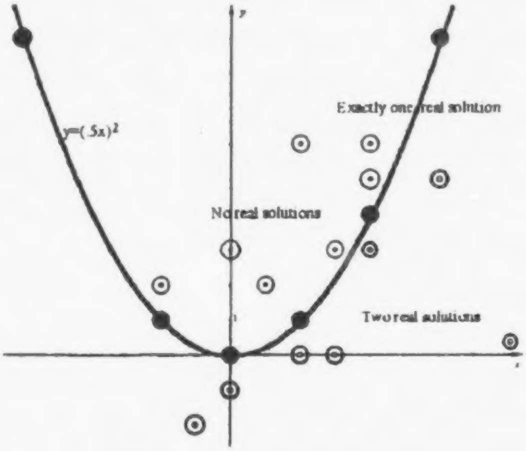


Figure 1.

While working on the second task, students initially realized the need to identify relevant information associated with the treatment. Data that they judged to be relevant included the amount of active substance of the tablet (16 units), the time in which the patient assimilates the active substance (10 minutes); frequency of dosage, amount of active eliminated by the patient during each drug supply. The identification of key information for this task came from students' discussion in which they tried to explain variation in the amount of drug during the treatment. This phase became crucial to propose ways to organize and present the information.

Use of a Table

Two ways were proposed to organize the information provided in the task, a systematic list and a table initially. Later, students decided to use a table to present the information. An important aspect to decide was to determine the entries that could help display the behavior of the relevant information during a period of time. Thus, the entries of the suggested table included frequency of dosage, total time (hours) elapsed from the beginning of the treatment at each take, amount of active substance at each supply and ten minutes after each supply. A table provided by one small group is shown next (this table shows fewer cases that those presented by the students)

Supply #	Elapsed time (hours)	Amount of substance at each supply	Amount of active substance 10 minutes after each supply
1	0	0	16
2	4	8	24
3	8	12	28

Students noticed that in both cases (at each supply and after 10 minutes) the amount of substance gets stable in the patient's body. Indeed, they mentioned that the amounts get close to 16 and 32 units respectively.

Graphic Representation

By using the data shown in the table, it was possible to present a visual approach or graphic representation of relevant information (Figure 2).

An Algebraic Approach

Is it possible to analyze regularities observed in the two above representations via the use of algebra? Here, students, who did not use their calculators to report results for the first table, were able to recognize and examine the corresponding pattern. For example, a student suggested the next approach to represent the amount of active substance (Figure 3).

Supply Number	Amount of active substance stored in the patient's body at each supply
1	0
2	$\frac{0+16}{2} = \frac{16}{2}$
3	$\frac{\frac{16}{2}+16}{2} = \frac{16+2\times 16}{2^2}$
4	$\frac{\frac{16+2\times 16}{2^2}+16}{2} = \frac{16+2\times 16+2^2\times 16}{2^3}$

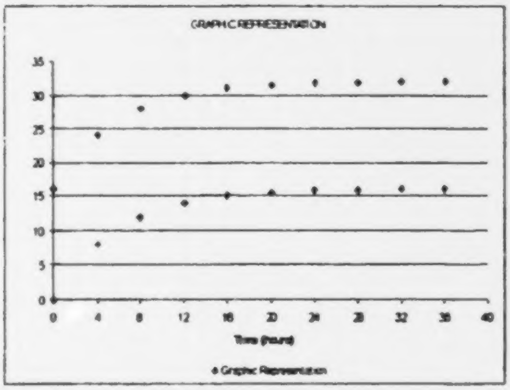
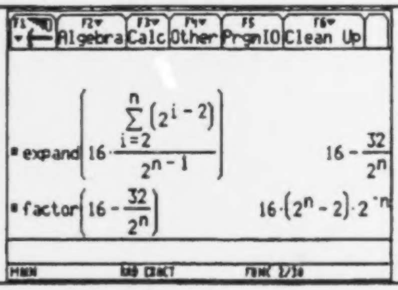


Figure 2.

A student proposed to use Excel to represent the data shown in the table into a graph.

What differences do you see between data represented in the table and data displayed in the graphic representation? Students again observed that both representations show that the amount of active substance at each taking and 10 minutes after converge, to 16 and 32 respectively.

Following this arrangement, the students in general were able to find a general expression to describe the general case. However, they showed difficulties in finding the corresponding formula. Here the use of the TI92 calculator became a powerful tool to determine a condensed expression.



■ expand$\left(16\cdot\frac{\sum_{i=2}^{n}(2^{i}-2)}{2^{n-1}}\right)$ $16-\frac{32}{2^n}$

■ factor$\left(16-\frac{32}{2^n}\right)$ $16\cdot(2^n-2)\cdot 2^{-n}$

Figure 3.

Concluding Remarks

There were different features of mathematical practice that appeared during the students' interaction with tasks. The representation of quadratic equations via points in the plane offered a visual approach that students exploited to understand the nature of the roots of this type of equation. Questions about existence of the roots addressed by some students led them to connect the discriminant with regions of points in the plane. In the second task, students observed mathematical qualities associated with the use of different representations to solve it. They noticed that the use of a table offered a discrete information to identify the stability of the behavior of the amount of substance stored in the patient' organism, while the graphic representation offered them a visual way to observe that behavior. Finding a general expression in the second task helped students to select a suitable representation (rational) to find the corresponding pattern. Students were aware of the need to constantly pose questions, discuss qualities of data representations, look for diverse ways of solutions, and the need to provide explanations. The tasks became basic ingredients to promote these students' approaches; but also the role of the instructor was a key component to direct and value students' participation.

References

Carpenter, P. T., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 19-32). Mahwah, NJ: Lawrence Erlbaum.

Goldenberg, P. (1995). Multiple representations: A vehicle for understanding. In D. Perkins, J. Schwartz, M. West, & M. Stone (Eds.), *Software goes to school. Teaching for understanding with new technologies* (pp. 155-171). Oxford: Oxford University Press.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Romberg, T. A., & Kaput, J. J. (1999). Mathematics worth teaching, mathematics worth understanding. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classroom that promote understanding* (pp. 3-17). Mahwah, NJ: Lawrence Erlbaum.

Santos, M. (1998). On the implementation of mathematical problem solving: Qualities of some learning activities. *Research in collegiate mathematics education.* (Vol. III, pp.71-80). Washington, DC: American Mathematical Society.

THE PROCESS INVOLVED IN FORMULATION AND REFORMULATION OF PROBLEMS OR QUESTIONS IN THE LEARNING OF MATHEMATICS

Juan Estrada
Facultad de Ingeniería UNAM
estrada@servidor.unam.mx

Abstract: What mathematics resources and strategies do students display in activities that involve processes of reformulation or design of tasks? This study documents the process shown by high school students while working on tasks in which they were asked to formulate questions or pose their own problems. Two tendencies appeared in students´ interaction: the operational and qualitative approaches.

Introduction

One of the factors that most influence the results of learning in students is the type of teaching implemented by the instructor. Here we may ask what type of teaching would promote mathematics learning, how to design the type of tasks, which would promote thinking in the students, which is compatible with mathematics practices. Rarely are students given the opportunity to practice formulation or reformulation of problems, which are an essential part of doing mathematics. Usually students are faced with a given problem, they are confronted with something, which has already been done and have not participated in the processes leading up to elaboration: they must only solve it. Where the problems come from or how they are created is left aside. The importance of this activity is acknowledged by NCTM (2000): "Teachers and parents can foster this inclination by helping students make mathematical problems from their worlds" (p.53). Therefore, aspects which should be investigated are the processes which students display when they are faced with situations or information and are asked to provide a description or analysis and based on this, to formulate problems and follow them up. This study documents the work carried out regularly by students participating in a teaching focused on formulation of problems and follow-ups.

Conceptual Framework

Brown and Walter (1990) suggest a strategy for problem formulation. This consists of analyzing a mathematical situation (e.g., an equation, formula or situation); then giving a description of the qualities; then changing or denying these qualities using the strategy "what if not"; and last analyzing the problem arising therefrom. Silver & Cai (1996) point out three phases in problem formulation: *before, during* and *after*. Based on this framework they analyzed the relationship between solving problems and problem formulation. This link could be inferred from the differences found in the problems posed by students who are successful in problem solving and those who are not, e.g., in the investigation carried with the first group of students they found that the number of mathematical questions created were much higher. On the other hand, the questions created by the second group showed a higher number of *statements*. English (1998) also analyzed these relations but with eight year old children. She arrived at similar conclusions; children who are strong problem solvers tend to pose more structurally complex problems, the weaker problem solvers tend to pose more operationally complex problems. From another viewpoint, Santos (1999) investigated problem formulations in tasks that involve the use of dynamic software and showed that such environments favor conjectures and then followed these up.

Thompson, *et al.*, (1994) assume that the activities developed by a teacher in a classroom are influenced by his conceptualization of mathematics and teaching. Two of the inclinations they found were the *operational* tendency and the *conceptual* tendency. Some of the characteristics they found in the first are the tendency to carry out numerical operations under any circumstances, without taking into consideration the context of numerical operations and not understanding the said numerical operation. Conceptually oriented teachers promote their students to acquire "mathematical disposition". They also ask their students questions which tend to stimulate the reasoning process –such as "what are you trying to find when you do this calculation (in the situation as you currently understand it?)." (p.86)

Lesh & Kelly (1997) identify the behaviors of students in problem solving:

1. The first interpretation by students are often very uncertain, superficial and prejudiced;
2. Students often ignore information which later on is important;
3. Superficial information catches their attention more instead of the profound information;

d) They don't realize inconsistencies in their reasoning.

Another characteristic of these models pass through cycles: examination, opening up, differentiation or are discarded.

Subjects, Tasks and Procedures

For one semester 26 high school students, 18-19 years, took part in learning activities centered on the formulation of problems and follow-ups. Participants had studied algebra, Euclidean geometry and analytic geometry in their previous courses. The tasks carried out had the following characteristics: a) we provided information, which showed the behaviors of a phenomena (e.g., epidemics, fluctuations in currency, position of an object). But no problem or question appeared; the students were requested to describe the given information, formulate problems and follow up. Another facet was that the prior information was presented in diverse contexts (tables, graphs, and verbal statements). This was done to help the students overcome the difficulties they have interpreting information or identifying mathematical ideas when these appear in different contexts, b) we presented a table with some information lacking and requested the students to provide the missing information, c) selected problems from a text book with the question or problem omitted with the remaining information the students were requested to formulate the "missing" problem and follow up, d) problems formulated by the students themselves. These were presented in front of the class for criticism. These activities also helped develop pre-calculus ideas (variation, approximation, limits, accumulated totals).

The work dynamics in the classroom were:

1. The class was organized into teams of three or four students. They were given written instructions with the activities mentioned above.
2. In the next class the teacher chose some of the teams to present their task in front of the classroom.
3. An important part of the teaching activity was to take the problems or questions posed by the students as a base for classroom discussions.

The gathered information came from written reports, small group discussions and open class discussions. We videotaped two students working on problem formulation. The teacher was the interviewer. We wished to observe in detail the processes displayed by the students when formulating problems and follow ups.

The analyses was divided as follows:

1. We took three phases during the semester [initial, middle and final phase].
2. We observed the same teams, the same pair or the same single student in each phase of the different activities.
3. We identified difficulties shown while working on the tasks.
4. We followed through on all of the above patterns in all later activities.

These procedures allowed as to observe the behaviors of the students during the three different phases.

Presentation and Discussion of Results

We present the outstanding results of each stage of the task carried out by the students.

Activity: Data on a table

In the initial phases information arising from the following type of activities was analyzed. The following table shows the number of infected people each week during twelve weeks. Based on this information to formulate questions or problems and solve them.

Table 1.

Week N°	1	2	3	4	5	6	7	8	9	10	11	12
N° of infected	100	244	356	436	484	500	461	392	266	160	80	40

Some of the objectives of the above activity were documented:

1. How students interact when exposed to information where there is no problem or question.
2. What recourses were elicited by the students. This allowed to know what type of knowledge was applied by the students.
3. What aspects of the information were taken into consideration and which not and the kinds of problems posed and how they were followed up.

The analysis for the above activities were based on written reports. In the initial stage, two tendencies emerged: the *operational* and the *qualitative* approach. These findings was observed by Thompson *et al.* (1994) in middle school teachers. In the operational approach the questions posed were of the calculation type. With information taken from the above table questions such as the following were posed: "How many patients per week represent the average?" Other types of questions which could be answered directly from the table: "Which week had the highest number of infected people?" The follow-up is begged from the question (calculate an average). In the second approach the students go much further with the given information—they use this information—to generate additional information. This was used to analyze the behaviors of the phenomena e.g., they create a table of differences based on the data in the original table ("first calculate the increase or decrease in the number of patients per week"). With this new information they were able to formulate the following questions: which showed highest increase of infected people?; and they were also able to make the following statements: "Week six showed the highest number of infected people and the least increase week (16)". Another characteristic was the students' willingness to use graphic representation to interpret the phenomena e.g., "the information was drawn on a graph to have a better a idea of how the epidemic developed" or "as seen on the graph the number of patients was decreasing".

The second type of approach attends the qualitative aspects of the phenomena therefore, the questions posed reflect this image. The students tried to arrive at a deeper understanding of the situation.

The computational approach appeared in some of the groups' work and during the class discussions. We noted a change from the operational to the qualitative approach e.g., in a graph which showed the distance traveled by a train, the question posed was

"Is there constant movement of the train and if so, from where?" It seems that the questions or problems formulated and the follow – up are closely tied in with the recourses of the students. Unfortunately in this stage we noted a tendency to change the formulation and follow- up of the problem activities into "rules" e.g., when the students were requested to design their own activities, the questions which arose were "Draw a table showing the behaviors of the phenomena"; "Draw a graph".

The question formulating tasks were influenced by an activity, which was discussed previously. That is to say, that they followed closely the activity presented by the teacher e.g., if they worked with a table representation of a variation phenomena, the resulting problem took on the same format. Not all students were influenced by the tasks seen previously. Another difficulty seen here was the reluctance showed by some students to participate in these activities. To sum up, this stage was to familiarize the students, to introduce precalculus ideas and, mainly, to make the students aware of the importance of this type of activity in the learning of mathematics.

Activity: Problems without questions

The intermediate stage showed an important turn in the type of activities carried out by the students e.g., the students faced situations involving ideas of continuous variations which may have induced them to formulate optimal problems. The tasks were selected from a textbook and the question or problem was deleted. With the remaining information the students were asked to formulate or complete the statement of the problem and follow up. One of the problems was:

1. There are 50 m.of wire to fence-in 3 sides of a rectangular field. The side is a wall which limits the field (on the dotted line write the problem which is missing and solve it).
2. The graphs of the straight lines **y =x** , **x=3** are shown in the first quadrant. Both straight lines and the axis **x** form a triangular region. Inside this there is a rectangle with one of its vertices (3,0) fixed (figure 1) (Write your question or the *missing* problem on the dotted line).

Characteristics of the tasks

By omitting the question or problem in both tasks, the reference to a problem of variation is left out. Therefore, the remaining information opened unlimited questions. We did not necessarily expect the students to pose problems of variation. The second task, in contrast to the first, showed a graph. This could cause difficulties in interpreting the information e.g., to realize that one vertice of the rectangle is on the straight line y = x and therefore, it is possible to draw an infinite number of rectangles which fulfill the condition of having a fixed vertice is not easy. If the students do realize this, they may recognize the problem of variation.

For this activity we selected two students (with operational tendency) to work on the two above tasks. The session was videotaped and the instructor participate as interviewer. The purpose of this was to analyze in detail the steps carried out by the two students when they were requested to formulate problems.

The most outstanding in this activity is that the formulation of a problem is shown to be a process. We were able to identify to following cycles in task two. This process was similar to that in task one.

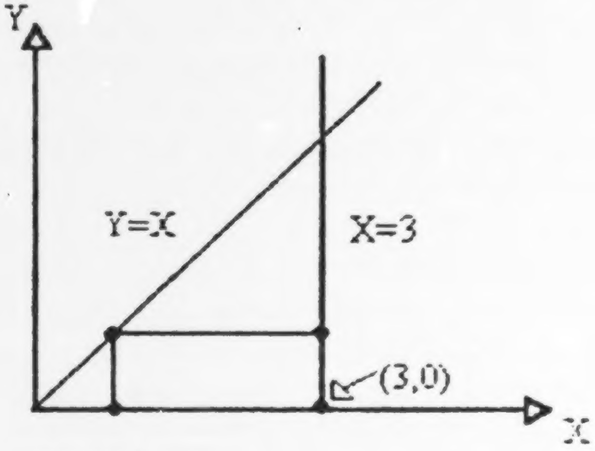


Figure 1.

Appropriation of information

At the same time the student is able to possess the information he formulates the first question and at this same time he builds his model of the situation. This model is built on partial elements ("two straight lines and the point were they intersect to form a triangle"). Other aspects are left aside or simply ignored (area, perimeter, a vertice on the straight line y = x). From the first sighting arises the first formulation of the problem ("find the measurements which will be the length an the width of this rectangle")

Recognition of new information

Here the teacher attempts to help the students overcome the prior level. His questions to the students oblige them to recognize new aspects of the situation e.g., the students recognize that there is an unlimited number of rectangles. This recognition causes them to have conflicts with the original problem. ("Here since there is nothing fixed, we were going to find too many rectangles and we wouldn't know their measurements because they are *moving,* it's not fixed and then we could find its measurement"). They weren't able to overcome this conflict by themselves at this stage. In order to overcome this they had to reexamine the original problem posed. But the recognition of new aspects does not bring about reformulation of the original problem.

Reformulation of the original problem

Since the students were not able to overcome the initial difficulties, the teacher tried to help them surpass their conflict asking them, "what happened to the areas of each one of the rectangles that your drew" (figure 2). This question causes a change; a pupil replies, "it reduces or it is going to increase". He notices both rectangles in figure 3 and states "this rectangle (without shading) is the same as that shaded.

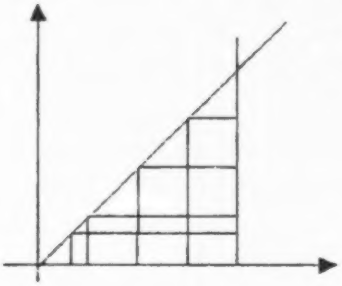


Figure 2

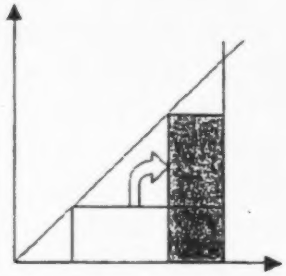


Figure 3

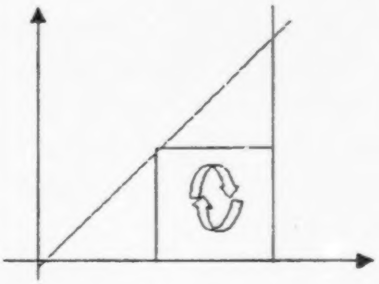


Figure 4

The only thing that will happen is that we have to turn the rectangle –the areas will increase and then decrease." Carrying out these actions, the students arrive at a new understanding, which takes them to the reformulation of the original problem: "The only thing we could find would be an equal sided rectangle, which would be the maximum area" (figure 4). The process of formulation is still not ended. It is still very general and imprecise and requires more development to be more precise in mathematical terms.

Activity: Problems posed by students

In the final phase of one the activities, we requested the students to design the activity for formulation and follow-up and then to present it in front of the class. This activity was also requested all through the course. This was videotaped, transcribed and analyzed. We selected a team. The task they designed and presented provided information on a table, which showed the price of coffee from the 15 - 30 September, 1998, but some data were lacking. The question they formulated: "Is it possible to find the missing data from the table?" "What is the price function?" "Is it possible to find data for days before September 15 – 30, etc. In the follow-ups, the students assumed the price always increased and the price variation (difference) was constant. Based on these assumptions, they calculated some prices for after the 15th, using the formula 15+.31 (day). However, when they tried to find prices for before the fifteenth, they had difficulties. The point here, and which took up the major part of the discussion, was the introduction of a negative sign in the variation (-.31) for the days before the fifteenth , which of course, contradicts their assumption –the increase in the coffee prices. Here, the group began to contradict themselves e.g., one of the team first said that to calculate values for days before the fifteenth, the days should be considered as negative, then he changed his mind and said that what should be considered as negative is the variation and not the days. This misunderstanding was cleared up in the open class criticism. The importance of this activity was to show the value of holding up a work to open criticism and being able to defend it.

Conclusions

At the beginning of the teaching course, we saw two tendencies –operational and qualitative. This was not ignored and we encouraged the student to reflect on the qualities and limitations of both approaches through questions, team discussions and open class discussions. However, when the pupils worked in teams they apparently surpassed this tendency, but when working individually the tendency came out.

In future investigations, we suggest that the teacher place more emphasis on students' problem posing and follow-up. Attention should also be given to some difficulties: verbal facility in expressing mathematical ideas, lack of recourses, inability to move among diverse representations, and emphasis on operations. Activities should be revised. The context of the course may have limited the type off questions or problems formulated by students. These questions or problems were very similar to the ones in the school. Even within these boundaries, the experience was fulfilling and the students were very receptive to this type of teaching.

References

Brown, S. I., & Walter, M. (1990). *The art of problem posing* (2nd ed.). Hillsdale, NJ: Lawrence Erlbaum.

English, L. (1998). Children's problem posing within formal and informal contexts. *Journal for Research in Mathematics Education*, *29*, 83-106.

Lesh, R., & Kelly, A. (1997). Teacher's evolving conceptions of one to one tutoring: A three-tiered teaching experiment. *Journal Research in Mathematics Education. 28*, 398–430.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.*

Santos, M. (1999). The use of technology as a means to explore mathematics qualities in proposed problems. In F. Hitt & M. Santos, (Eds.), *Proceedings of the Twenty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol 1, pp. 139-146). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmentl Education.

Silver, E., & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education, 27*, 521-539.

Thompson, A., Philipp, R., Thompson, P., & Boyd, B. (1994). In D. B. Aichele & A. F. Coxford (Eds.), *Professional development for teachers of mathematics* (pp. 72-92).Yearbook. Reston, VA: The Council.

STUDENTS CONNECTING MATHEMATICAL IDEAS: POSSIBILITIES IN A LIBERAL ARTS MATHEMATICS CLASS

Barbara H. Glass
Rutgers University

Abstract: Students in a community college mathematics class for liberal arts majors were studied as they met in small groups to work on various non-routine problems. Several problems from the area of combinatorics were given during these problem-solving sessions. These students were engaged in thoughtful mathematics. They found patterns, justified that their patterns were reasonable, and recognized isomorphisms between the different problems.

Introduction

Researchers have documented children's thinking as they investigate problems in the area of combinatorics to determine how they think about the problems and justify their solutions (Maher & Martino, 1996, 1997, 1998; Muter, 1999; Adleman, 1999; Maher, 1998; Maher & Speiser, 1997; Kiczek & Maher, 1998; Muter & Maher, 1998; English, 1988, 1993, 1996; Weir, 1999). One of these problems is the Towers Problem, which invites a student to determine how many different towers of a specified height can be built when selecting from two different color cubes, and to justify that all possibilities have been found. A second problem, the Pizza Problem, invites a student to determine how many different pizzas can be created from a given number of toppings and to justify that all possibilities have been found.

Objectives

The purpose of this study is to examine how a small group of community college students, enrolled in a liberal arts mathematics class, solve the towers and pizza problems as well as modifications and extensions of these problems. Some questions that were investigated for those students include: (1) How do they solve non-routine mathematical investigations; (2) What connections, if any, are made to isomorphic problems and previous mathematical knowledge; and (3) To what extent are justifications made.

Background

The study was conducted at a moderate size New Jersey community college in a mathematics course for liberal arts majors. The curriculum for the course includes algebra and problem solving. The sections of the course that are included in the study met for two seventy-five minute classes each week for fifteen weeks. They spent one class each week working on various non-routine problems in a small group setting.

This paper will focus on Rob who worked in one of these small groups with his partners, Steven and Samantha.

The Towers Problem was given during the ninth week of the semester. By this point the students had become accustomed to working on non-routine problems and justifying their solutions. The students began by working on towers that were four cubes tall. They then were asked to consider towers that were five cubes tall. Rob's group also worked with towers where three different color cubes were available.

The Pizza Problem was given during the fourteenth week of the semester. The students first worked on finding the number of pizzas that could be made if four toppings were available followed by the problem with five toppings. After solving the basic problem they were asked to consider the Pizza With Halves problem in which a topping could be placed on either a whole pizza or a half pizza.

Theoretical Framework

The growth of mathematical knowledge is the process whereby a student constructs internal representations and connects these representations to each other. Understanding is the process of making connections between different pieces of knowledge that are already internally represented or between existing internal connections and new knowledge (Hiebert & Carpenter, 1992). Students build their understanding of concepts by building upon previous experience, not by imitating the actions of a teacher or being told what to do (Maher, Davis & Alston, 1991). Learners who first learn procedures without attaching meaning to them are less likely to develop well-connected conceptual knowledge. When students encounter new problems they are more likely to retrieve prior knowledge that is well connected than to retrieve loosely connected information (Hiebert & Carpenter, 1992). In traditional mathematics classrooms, the answer key or the teacher is the source of authority on the correctness of answers. Quick right answers are often valued more than the thinking that leads to the answer. Teachers generally only question students when an error has been made. They do not ask students with correct answers to explain their approaches and results (Burns, 1985). As a result students develop the belief that all problems can be solved in a short amount of time and will often stop trying if a problem cannot be solved quickly. Moreover, since students view school mathematics as a process of mastering formal procedures that are removed from real life they learn to accept and memorize what their teachers tell them without making any attempt to make sense of it (Schoenfeld, 1987). In contrast the students in this study were encouraged to think about their solutions to the problems that they were given and justify their answers rather than relying on the instructor of the class as the sole authority.

Data Collection

Rob's group was videotaped as they worked on the Towers Problem and the Pizza Problem. Following the class sessions each student was required to submit a write-up of the problems. In addition a videotaped, tasked based, interview of Rob was conducted one week after the class session on the pizza problem. All videotapes from the three sessions have been transcribed, coded for problem solving strategies, methods of justification, and connections, and analyzed.

Results

Rob's group began the Towers Problem by organizing their towers by the number of red cubes and yellow cubes that were in each tower and used a modified "proof by cases" to justify their solution. They showed that they had produced all towers which contained only one color, all towers with a single red or a single yellow cube, all towers with two of each color in which the two yellow cubes were kept together, and then all towers in which the two yellow cubes were separated. After producing the towers five tall, they noticed that the number of towers had doubled from the towers of four to the towers of five. They extended the doubling pattern to make a prediction about how many towers they would get if the towers were three tall, two tall or one tall. They then built the towers one tall and two tall to test their theory. While justifying their number of towers five tall to the instructor, they refereed to both their doubling pattern and their proof by cases. The instructor asked the students to think of a reason why the number of towers was doubling. After a few minutes of thought Rob explained to Steve that it doubled because you could add either a red or a yellow cube to the bottom of each tower.

> Rob: Okay lets say the top of our tower is X, X (Rob writes an X on his paper) Then were putting one on the bottom. For every X we can have a Y down here, or for every X we can have a red down here. So for each block we have there is now two more things it could be. So before we just had X. This is X (Rob picks up the solid red tower. of four as an example) Now we have XR and XY derived from this. (Rob continues to talk as he looks for the corresponding towers of five) XY and XR (Rob holds up RRRRY and RRRRR)

Steve demonstrated that he understood Rob's explana-tion by starting with the towers that were one tall and showing how the towers that were two tall could be created.

Rob's group was next presented with the task of selecting from three colors. They solved it immediately by using the same inductive method that they had developed for the towers of two colors. Rob's written work for the two-color problem is shown in Figure 1. It is interesting to note that Rob's written justification refereed only to the proof by induction and made no reference at all to the original proof by cases.

Five weeks later, the group was given the Pizza Problem. They completed the problem with four toppings by listing the various pizzas by the number of toppings each contained. Before moving on to the problem with five toppings Steve commented: *"I think it's good we find a pattern...It's going to be the same as the last time*

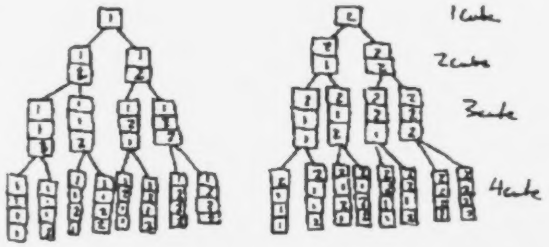


Figure 1. Rob's written work for the two-color Towers Problem.

we did this." After a brief period of time Rob noticed a row of Pascal's triangle in the number of pizzas with the various number of toppings, although he couldn't remember the name. He referred to it first as "*Pythagorean's triangle*" then as "*what's his face with the triangle*."

When the group told the instructor about their discovery, she challenged them to find a reason why the addition rule of Pascal's triangle should work for the pizza problem. After the group worked for a while without results the instructor's restatement of the question appeared to give Rob an insight. He listed the various pizzas on a copy of Pascal's triangle and then explained his thinking to the rest of the group.

> Rob: Two toppings. We take this list of two toppers, which is this one, this one, and this one. And then we take this list and add our sausage to the end of that. So put sausages there and add it to those. And that's this.
>
> Steve: All right. I got it. It's just. It's tough, because it's very crammed in there.

Rob then proceeded to make a list of the five topping pizzas by filling in the fifth row of Pascal's triangle as shown in Figure 2.

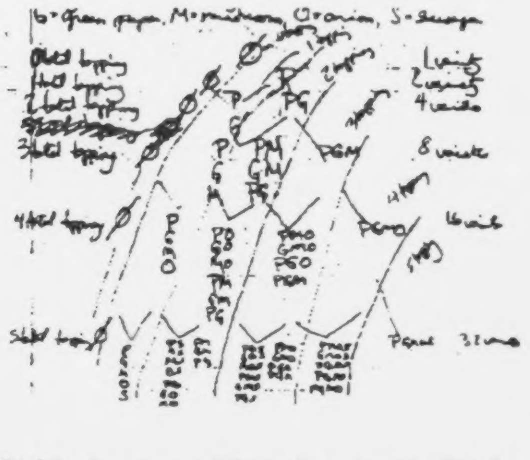


Figure 2. Rob's drawing of Pascal's triangle for the Pizza Problem.

After completing work on the basic Pizza Problem, the group was given the Pizza with Halves problem. This time the group decided to start by working on an easier problem. They started with three toppings and found twenty-seven pizzas. (There actually are thirty-six.) They then moved down to two topping and found nine pizzas. (There should be

BEST COPY AVAILABLE

ten) Convinced that they had found a pattern they checked the one topping case and found three pizzas. They now said that they were sure that the number of pizzas with halves is 3^n, where n is the number of toppings available. When asked why the answer should be three to a power Rob answered *"when we did it before it was binary. It's either on or off. Now it's off, on one or on two."*

The Instructor then asked the group about the relationship between the pizza problem and the towers problem. Steve and Samantha tried to relate the toppings to the different color blocks, but Rob stated that it is not the colors that represent the toppings, but rather the position of the blocks.

Rob: Your colors are. See, if you do it like the binary. It's not that your colors are the topping, it's the topping is the position. So if you have a tower 4 blocks high, your first block is plain or pepperoni, onion, sausage, mushroom. If it's orange, it's on. If it's yellow, it's off. (Rob writes as he talks.) So if this is all yellow, that means there's no toppings, and it's plain. OK. Now.

Steve: Got it.

Rob: If this is orange, that means it's one topping pizza with pepperoni on it.

Steve: Got it.

Samantha: Hold on.

Rob: Yeah.

Samantha: Are we doing the half pizzas?

Rob: Not yet, that would be three colors.

Steve: OK. You need three colors to do that.

Samantha: OK. Well, I thought she was asking on this problem that. That's why I didn't understand why we only had two colors.

Rob: So then it? Yeah.

Samantha: That's why it's two raised to the something power.

Rob: Right. So then?

Samantha: So it's three raised to something power.

Rob: If it's blue, it's half on.

After the group explained the relationship to the instructor, she drew a YBBO tower on Rob's paper and asked which pizza that this would represent. At this point Rob recognized that you could not tell whether the half toppings were on the same half or opposite halves of the pizza. Samantha and Steve described the pizza to the instructor. During the description Steve also stated that there was a problem with the two halves.

Steve:	I see a problem, though.
Rob:	The problem is, you can't tell.
Samantha:	No, because it's.
Steve:	Which if it's two on one side. If two have one side. Like pepperoni, onion, on the other side plain. So maybe we could make it a bigger tower.
Rob:	Or should we add another color? Blue is right side, red is left side.

The class ended with the instructor asking the students to think about the problem.

About a week later Rob was interviewed by the instructor about the problems. After Rob built the twenty-seven towers of three with three colors the instructor asked about the relationship between the towers problem with three colors and the pizza problem with halves. Rob related each tower to a pizza using white for on, blue for off, and red for half on. Rob arrived at the first tower that had two red cubes. The instructor pointed out that this was the case that Rob though might cause problems.

Rob:	Right, because we know we've got half green peppers, and we know we've got half mushrooms,
Instructor:	Uh, huh.
Rob:	but we don't know which half.

Rob then proceeded to find all possible pizzas for each tower that had more than one white cube to arrive at an answer of thirty-six pizzas.

Conclusions/Implications

The students in Rob's group were engaged in thoughtful mathematics. They found patterns and justified that their patterns were reasonable. The students not only related the Pizza Problem to Pascal's triangle by recognizing the number pattern, but they also explained how the addition rule of Pascal's triangle related to the pizza problem. In addition they explained the isomorphism between the Towers Problem and the Pizza Problem, and related the Towers Problem with three colors to the Pizza With Halves problem. The students' reasoning about the Pizza With Halves Problem resulted in an incorrect conclusion, however, the instructor intervened by asking them to describe a pizza with two half toppings. This enabled the students to realize the mistake that they had made without being told that they were wrong. Rob later used the towers with three colors to correctly find all pizza with halves. The three color Towers Problem also demonstrates how the students applied the methods they had discovered in one problem to another problem. The findings support the importance of introducing rich problems to students and giving them opportunities to work together toward a solution.

References

Adleman, K. A. (1999). *Problem-solving strategies of young children working on a combinatorics task in a computer environment*. Unpublished doctoral dissertation, Rutgers, The State University of New Jersey, New Brunswick, NJ.

Burns, M. (1985). The role of questioning. *Arithmetic Teacher, 32* (6) 14 - 17.

English, L. D. (1988). *Young children's competence in solving novel combinatorial problems.* Unpublished doctoral dissertation, University of Queensland, Brisbane, Australia.

English L. D. (1993). Children's strategies for solving two-and three-dimensional combinatorial problems. *Journal of Research on Computing in Education, 25* (3), 339 - 346.

English L. D. (1996). Children's construction of mathematical knowledge in solving novel isomorphic problems in concrete and written form. *Journal of Mathematical Behavior, 15*, 81 - 112.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65 - 97). New York: Macmillan.

Kiczek, R. D., & Maher, C. A. (1998). Tracing the origins and extensions of mathematical ideas. In S. Berenson et al., (Eds.), *Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 377-382). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Maher, C. A. (1998). Constructivism and constructivist teaching - can they coexist? In O. Bjorkquist (Ed.), *Mathematics teaching from a constructivist point of view* (pp. 29 - 42). Finland: Reports from the Faculty of Education, No. 3, Abo Akademi University.

Maher, C. A., Davis, R. B., & Alston, A. (1991) Implementing a thinking curriculum in mathematics. *Journal of Mathematical Behavior, 10*, 219 - 224.

Maher, C. A., & Martino, A. M. (1996). The development of the idea of proof: A five-year case study. *Journal for Research in Mathematics Education, 27* (2), 194 - 219.

Maher, C. A., & Martino, A. M. (1997). Conditions for conceptual change: From pattern recognition to theory posing. In H. Mansfield & N.H, Pateman (Eds.), *Young children and mathematics: Concepts and their representations.* Sydney, Australia: Australian Association of Mathematics Teachers.

Maher, C. A., & Martino, A. M. (1998). *Brandon's proof and isomorphism: Can teachers help students make convincing arguments?* (pp.77 - 101). Rio de Janeiro, Brazil: Universidade Santa Ursala.

Maher, C. A., & Speiser, R. (1997). How far can you go with block towers? *Journal of Mathematical Behavior, 16* (2), 125 - 132.

Muter, E. M. (1999). *The development of student ideas in combination and proof: A six year study.* Unpublished doctoral dissertation, Rutgers, the State University of New Jersey, New Brunswick, NJ.

Muter, E. M., & Maher, C. A. (1998). Recognizing isomorphism *Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 461-467). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Schoenfeld, A. H. (1987). What's all the fuss about metacognition. In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189 - 215). Hillsdale, NJ: Lawrence Earlbaum.

Wier, D. (1999). *The application of professional development in combinatorics and proof to classroom teaching*. Unpublished doctoral dissertation, Rutgers, The State University of New Jersey, New Brunswick,NJ.

HIGH SCHOOL CURRICULUM CONTENTS FROM AN INTERNATIONAL PERSPECTIVE

Lourdes Guerrero
Universidad Michoacana de San Nicolás de Hidalgo
gmagana@zeus.umich.mx

Antonio Rivera
CINVESTAV, México
arivera@mail.cinvestav.mx

Abstract: In this paper we report some results of the analysis of released items of the Third International Mathematics and Science Study (TIMSS). This analysis is part of a research project in process related to Mexican high school curricula from an international perspective and, the mathematics students' performance in this school level. The results obtained up to now are related to the mathematical curriculum content underlying the set of TIMSS assessment items. The findings suggest a great wealth of mathematical contents and some skills and abilities needful of answering to this set of items. However, in this paper we report only the results corresponding to mathematical content, highlighting some detected differences between the Mexican high school curricula and these contents.

Introduction

The need to know how young people are being prepared at school leads us to the evaluation of student performance as well as instruction, curricula and teacher practices. The Third International Mathematics and Science Study (TIMSS), conducted in 1992-1995, and its repetition (TIMSS-R), in 1999, was the most extensive and ambitious international mathematics and science evaluation of comparative education achievement ever undertaken (Michel, & Kelly, 1996; Gonzales et al., 2000). The main goal was to examine student achievement in some school grade levels to try to understand the nature and extent of student achievement and the context in which it occurs (Michel & Kelly, 1996).

For TIMSS implementation, analysis and data collection, different instruments were prepared. They included questionnaires and interviews for students, teachers and principals of their schools, videotapes, classroom activities and case studies of three countries, curriculum analyses of participant countries, and assessment item sets for students (Mullis & Michel, 1996).

One of the most important tasks carried out by TIMSS was the design and selection of the item sets in such a way that this would reflect the knowledge and some skills and abilities that students should acquire and develop according to their age and school grade level. For this purpose, the TIMSS team took into account the analysis of curriculum guides, textbooks and other material that allowed them to determine the common characteristics of most of the participant countries.

In this paper we report some results of a research project in process related to Mexican high school mathematics curricula from the international perspective, implicitly suggested by the TIMSS assessment instruments. This research project began with the analysis of the TIMSS instruments from different perspectives:

- Determining the underlying content in the set of items to identify a curriculum, which we can refer to as international, at the content level. This international curriculum can serve as a reference for the curriculum reform in Mexican high school mathematics education, since it will allow us to compare the Mexican mathematics curriculum with this broad opinion of specialists in both mathematics and mathematics education world-wide.
- Identifying particular skills and abilities, procedures and solution strategies for each item. Doing this we can have a broad picture of the things that can be assessed with this set of items. This information allows us to have a richer curriculum that includes not only content knowledge but also processes, abilities and skills.
- Designing instruments complementary to those of TIMSS to identify more clearly the knowledge, skills and abilities that students should develop and acquire. The enrichment of these items by means of adaptation and incorporation of new ones, will allow us to investigate more deeply into the knowledge, skills, and abilities identified in the original instruments, seeking to have a more profound diagnostic of Mexican high school mathematics education.

Under the assumption that the TIMSS set of items reflects a desirable curriculum for high school systems from an international perspective, we intended to determine the common contents and differences between the Mexican high school and this international curriculum. We suppose that Mexican high school curriculum contents cannot be very different from the one outlined by this international perspective, since it is not isolated from international expectations, mainly due to the current global tendency in which we are immersed.

Assessment of Knowledge, Skills and Abilities

One fundamental stage in the design and analysis of items to assess student performance is to know accurately what it is to be assessed. To be clear in this aspect, it is desirable to distinguish between knowledge, skills and abilities.

Knowledge acquisition is a process that involves both the memorization of facts, principles and laws and the appropriation of mathematical concepts (Haladyna, 1997). Although the purposes and goals of mathematics education in most of the Mexican educational systems are guided to promote the development of competencies and skills (SEP, 1999), the assessment has been focused mainly on content knowledge. In many countries, however, there is a growing interest in the way that we are assessing student performance. For example, *The Principles and Standards for School Math-*

ematics suggests that assessment "should focus on students' understanding as well as their procedural skills" (NCTM, 2000, p. 23).

Skills and abilities are cognitive processes that can be inferred by means of studying the students' behavior when they are carrying out a certain activity. Skills are specific actions that are characteristic of the activity, while abilities are the psychological characteristics of that person who carries out the activity (Krutetskii, 1976). These aspects have been attended to little in the assessment process, not because they are seen as less important, but because of the difficulty in studying them. It is important to mention, however, that currently this is a central part of some mathematics education research. It seems to be that a qualitative inquiry methodology applied to the assessment process is an appropriate way to investigate these issues (Romberg & Collins, 2000).

Knowledge, skills and abilities are not acquired and developed in an isolated way, they interact and develop in a combined form (Strenberg, 1999).

Taking into account these aspects, our analysis is focused on the identification of the mathematical contents reflected by the TIMSS items, which in turn give us a well-rounded picture of the knowledge involved.

Methods and Procedures

From the TIMSS perspective, curriculum was considered to have three manifestations:

- The intended curriculum, that is, the mathematics that one expects students to study and learn in school;
- The implemented curriculum refers to what the teacher actually teaches, in terms of their own interpretation of the intended curriculum; and,
- The attained curriculum, which refers to what the students are really learning (Michel & Kelly, 1996).

These manifestations tell us that official intentions can be different from what the students are really learning. Determining the underlying contents in the TIMSS items leads us to identify one part of a deliberate international curriculum that may not correspond to what the students are really achieving; this international curriculum, however, integrates the expectations for students at an international level.

The attained curriculum can be determined by assessing student achievement in those expectations marked by the intended curriculum. As we have mentioned, one of the purposes in determining the underlying curriculum in the TIMSS item set is to have a broad picture of the knowledge, skills and abilities that we can assess with this pool of items. Their implementation will allow us to have a diagnostic of the actual situation of the Mexican high school system, and a framework with which to compare expectations and students' performances in those contents that are common in both the Mexican and the international curricula.

In 1996-98 in Mexico a team of mathematics education researchers and mathematics researchers, conducted the study "*Propósitos y Contenidos de la Enseñanza de las Matemáticas en el Nivel Medio Superior en México*" (Rivera et al., 2000), which was focused on the intended curriculum across the country. This study shows the complexity and diversity of Mexican high school systems; it also highlights some schools systems that are representative of them.

For our analysis, we used the results of this study and the 57 released TIMSS mathematics items that correspond to the high school system in Mexico. The data gathered by this means is appropriate for a qualitative approach; so we used this method of inquiry.

Results

TIMSS classified the items in various ways. One of them is by the format, multiple choice and open-ended (short and extended response) items.

The use of multiple-choice items has some advantages, for example, the ease in assigning and applying scores, mainly when we want to test a large population. The main purpose to using them in assessing student performance is to determine the knowledge acquired as a product of the learn process (Haladyna, 1997). This kind of format, however, is not appropriate for assessing some skills and abilities, since these are aspects that can only be evaluated by analyzing the solution process of a problem or the students' behavior when facing an activity. Others that, besides measuring the students' knowledge base, also try to develop certain competencies are beginning to substitute multiple choice items, which is in widespread use in standardized tests. Examples of this are the performance assessment packages that consist of combined-solving problems and/or open-ended problems.

The item format gives us insight into the extent of the answer that students should give us, and this extent reflects the depth of the assessment process, so this is an aspect that we take into account in the analysis.

TIMSS classified the items in the Final Year of Secondary School test (final year of high school in Mexico) into mathematics literacy and advanced mathematics item sets. The former was to be applied to a sample of all students in the last year of secondary school, while the advanced mathematics test was to be applied to students who were taking advanced mathematics courses in their final year of secondary school. The items in the latter were classified in turn as numbers, equations and functions, geometry, calculus, probability and statistics, and validation and structure.

Although the mathematics literacy items were not explicitly classified in more detail, our analysis suggest the following more relevant contents:

- *Numbers and number sense*: Whole numbers and rational numbers, rational numbers representations in decimal and fractional forms and their relationships, basic operations with whole numbers and rational numbers, basic operation properties, unit conversion, reasonableness of results, order of magnitude, estimation, rounding, number patterns, and various types of percent problems.

- *Proportionality*: Direct and inverse proportionality, ratio, and scales in graphs.
- *Probability and statistics*: Data representation; interpreting graphs, plots and charts, prediction and inference from graphs and charts, notions of sample and population, randomness, measures of central tendency (mean), fitting lines, numerical probability; notions of mathematical expectation, use of notation and vocabulary.
- *Variables and functions*: Dependence between variables, function graphs, interpreting graphs of functions, relationship between graphs and equations, recall and use of formulas, mean rate of change, and increments.
- *Geometry*: Longitude, area and volume measurements, graphic estimation, and recall of names of basic figures (triangle, square, rectangle, hexagon, cube).

For the advanced mathematics items, we identified the content knowledge in each one of the TIMSS categories:

- *Numbers, equations and functions*: Real and complex numbers, basic operations with these numbers, conjugation of complex numbers, operational properties, exponents and radicals (square root), prime factors, permutations and combinations, direct and inverse relationships between variables, variation range, approximation, estimation, rounding, precision, inequalities, posing and solving equations with real and complex numbers, linear systems of equations, rearranging formulas, equations of conic curves and equations of planes, logarithmic exponential and trigonometric functions, use of specialized notation and vocabulary.
- *Calculus*: Routine computation of limits with radicals (square root), the derivative and the integral (in speed and acceleration context, and their relationship), the derivative as a function (graphic representation and their relationship with the function), graphic representation of the integral as a primitive function, graphic representation of the integral as the area of a region, exponential and logarithmic functions and their relationship, trigonometric function and the use of some trigonometric formulas in problem solving situations, series (geometric and arithmetic), and the use of a more specialized vocabulary and notation. (We also distinguished some algebraic processes, like the conjugated product of binomials, and basic algebraic operations).
- *Geometry*: Representation of points on a plane, the distance between two points on the plane, longitude, area and volume measurement and units, properties of triangles (exterior and interior angles, the sum of interior angles, inscribed

right triangles in the circumference, congruency and similarity), parallelism and perpendicularity, regular polygons (their area), conic sections (identification of equations and graphics), Pythagorean theorem applications, lines and planes in space, graphic estimation of an area, spatial visualization, coordinate systems in two and tree dimensions, translation, reflection, rotation, invariance, vectors, vector representations on a plane, angle between vectors, sum, difference and product by scale operations with vectors, and correct use of a more specialized notation.

- *Probability and statistics*: Data representation and analysis, interpreting graphs, plots and charts, process of randomness, measures of central tendency, counting principles, scales, estimation, prediction and inference, classic probability, conditional probability, numerical probability; fitting lines, correlation, and specialized vocabulary.
- *Validation and structure*: There is only one item in this category related to formal logic.

The mathematics literacy items and the advanced mathematics items differ considerably in quantity, context, notation and vocabulary and content knowledge. For instance, the items related to geometry in the literacy mathematics test are fewer, and most of them are related to the names of geometric figures and measurement. In the advanced mathematics test, most of them are related to the properties of geometric figures, representations and spatial visualization.

Most of the items in the literacy test are in context; they incorporate situations and activities from daily life, while most of the items in the advanced mathematics test are in a mathematical context, or in a context that may not be very familiar to students; for example, item L3 related to the decomposition of a radioactive element.

The Mexican high school system is complex. There is a diversity of subsystems, both public and private. There are no standards or benchmarks to support the curriculum design and therefore, we found a diversity of curriculum guides. Although all of them include mathematics, some only in one part of the program and others during the whole program.

Within this diversity, however, there exist common mathematical contents, related to algebra, trigonometry and analytical geometry (Rivera et al., 2000). These contents relate well to the international curriculum that we described above (although TIMSS does not refer explicitly to algebra contents, some of them correspond to the algebra courses in Mexico); for example, solution of linear equations, trigonometric functions in problems that involve right triangles or the study of conic sections.

In general terms, our findings show that large differences do not exist between the curriculum contents in Mexican high schools and the international curriculum. There are, however, some differences. For example, numeric and graphic estimation is present in many of the TIMSS items, both in the advanced mathematics test and in the literacy test, but this notion was not identified in most of the Mexican high school curriculum contents.

Also, we found differences in the treatment of certain concepts. For example, item K5 in the advanced mathematics test refers to the identification of relationships between the function and their first and second derivative in a graphic representation (and some data in analytical representation). There are no instances of this type of treatment in the high school curriculum in Mexico. The derivative concept is present in the calculus courses (In Mexico, the students that take calculus courses, are those who expect to enter to college in areas related to mathematics and science studies or engineering). The treatment of this concept is in terms of the application of formulas to solve routine problems. The graphic representation of the derivative is only used as a way of introducing this concept in an intuitive form, such as for the slope of a tangent straight line to the graph of a function in a point. The second derivative is not present in the Mexican high school curriculum.

Although the content is only one part of the curriculum, it is an important one that allows us to appreciate the wealth of content knowledge and the expectations for students. The framework that TIMSS provides us is an excellent opportunity to reflect on these expectations.

References

Gonzales, P., Christopher C., Jocelyn, L. Mak, K., Kastberg, D., and Arafeh, S. *Pursuing excellence: Comparisons of international eighth-grade mathematics and science achievement from a U.S. perspective, 1995 and 1999*. Washington, DC: U.S. Department of Education, National Center for Education Statistics. (2000).

Haladyna, T. M. (1997). *Writing test items to evaluate higher order thinking*. AZ: Allyn and Bacon.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: The University of Chicago Press.

Martin, M. O., & Kelly, D. L. (1996). *Third international mathematics and science study technical report, Vol. I: Design and development*. Chestnut Hill, MA: Boston College, Center for the Study of Testing, Evaluation, and Educational Policy.

Mullis, I.V.S., & Martin, M.O. (1996). *Third international mathematics and science study: Quality assurance in data collection*. Chestnut Hill, MA: Boston College, Center for the Study of Testing, Evaluation, and Educational Policy.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Rivera, A., Santos, T. M., Alarcón, J., Bravo, A., Briseño, L., Díaz-Barriga, A., & Verdugo, J. (2000). *Propósitos y contenidos de la enseñanza de las matemáticas en el nivel medio superior en México*. Reporte de Investigación SEP-CONACYT, AMC, CCC. México D. F.

Romberg, T. A., & Collins, A. (2000). The impact of standards-based reform on methods of research in schools. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education,* (pp. 73-85). Mahwah, NJ: Lawrence Erlbaum.

Secretaria de Educación Pública (SEP) (1999). *Profile of education in Mexico.* México, D.F.: Secretaria de Educación Pública. (Available online at http://www.sep.gob.mx/Perfil_Ingles/index_english.htm).

Sternberg, R. J. Abilities are Forms of Developing Expertise. *Educational Researcher,* 27, (3), 11-20.

PROBLEM-SOLVING IN INTEGRAL CALCULUS: ONE ROLE OF METACOGNITIVE THINKING

Stephen J. Hegedus
University of Massachusetts – Dartmouth

Abstract: This paper reports one aspect of a larger study which analyzed the metacognitive thinking of undergraduate mathematics students solving problems from the Integral Calculus (Hegedus, 1998a). In that study the role of metacognitive thinking was highlighted as an important part of mathematical thinking with varying degrees of mathematical efficacy.

This paper illustrates one important aspect of metacognitive thinking, organizational thinking, which was evident in the study. The use of organizational skills in terms of mathematical problem-solving are highlighted by analyzing the verbalizations of undergraduate students. The result is a summary of one role of organizational thinking and how such metacognition is intimately bound up in the mathematics concerned.

Theoretical Framework

Metacognition has developed from a psychological construct (Flavell, 1976) to a pedagogical tool throughout the last 25 years. From classic Artificial Intelligent systems built around control-systems (Newell & Simon, 1972; Nilson, 1980) to metacognitive skills developed for students in problem-solving contexts (Mayer, 1998).

Metacognition, in an educational context, has mainly been concerned with the application of reflective strategies in classroom situations for the benefits of improving conceptual and procedural efficacy. Lester (1994) commented on how metacognition was losing its credibility at the turn of the 1990's because of the incoherence in its identity and its utility. Metacognition viewed as skills or strategies (Mayer, *ibid.*) lead to variations in definitions and thus problematic interdisciplinary research between fields such as (neuro-) psychology and mathematics education. In an attempt to grasp some overview from the fields of psychology and mathematics education we turn to two major sources.

Brown (1987) describes metacognition as being referred to as two cognitive aspects:

- Knowledge about cognition.
- Regulation of knowledge.

She attempts to describe these two concepts with very broad definitions:

> Knowledge about cognition refers to the stable, statable, often fallible, and often late developing information that human thinkers have about their own

cognitive processes; traditionally, this has been referred to as knowing that.

Regulation of Knowledge refers to cognitive processes. These include "planning activities (predicting outcomes, scheduling strategies, and various forms of vicarious trial and error, etc.)" (p. 67)

In mathematics education, Schoenfeld (1987a) summarizes the research on metacognition as focusing on three topics:

- Knowledge about your own thought processes. How aware are you in describing your own thinking?
- Self-regulation. How well do you keep track of what you're doing when (for example) you're solving problems, and how well (if at all) do you use the input from those observations to guide your problem-solving actions?
- Beliefs & Intuitions. What ideas do you bring with you? How does this shape the way you do Mathematics?

There is something quite clear in these various definitions. Firstly, metacognition is not just one thing; it incorporates a variety of thinking. Secondly, there is some agreement on two types of metacognition: knowledge of cognitive strategies, and self-regulation of cognitive strategies.

A pilot study (Hegedus, 1996) involving undergraduate mathematics students taking a final course in Calculus was conducted to observe if these two main types of metacognitive behavior was evident in the mathematical thinking of these students. It was found that self-regulatory thinking predominated much of the students' mathematical behavior to varying degrees of efficiency. This pilot study laid the ground work for the main follow-up study (Hegedus, 1998a). This large scale study developed a four-fold model for the description of self-regulatory problem-solving behavior of undergraduate mathematics students. This behavior was intimately bound up in the mathematical problems that the students were solving. The problems were from multivariable Calculus. Following a synthesis of the relevant literature ((see Brown, 1987; Schoenfeld, 1987a; 1987b for examples) a more detailed model of self-regulatory thinking was developed and analyzed in the pilot work. This was the first stage of a three-part empirical process. The model developed is called ROME and comprises of four distinct types of self-regulatory behavior:

- Reflecting (R)
- Organizing (O)
- Monitoring (M)
- Extracting (E)

This is the ROME model of self-regulatory thinking in the domain of undergraduate mathematics students solving problems in the integral calculus. There follows a brief description of the four types of self-regulatory behaviors that were evident in the study with examples of each type of mathematical thinking.

Reflection refers to the activity of referring to textbooks, notes, experiences and work in progress and the consequences of these activities. Once the solver has extracted a technique or algorithm, the solver might engage in some reflective activity to assess the local and global effects of its implementation.

Organization refers to the planning behaviors that a problem-solver engages in, in both the exploratory phases and the execution phases of a problem. For example, the order of integration and which co-ordinate system (e.g., spherical, cylindrical) to change to.

Monitoring refers to the business of 'keeping track of what you're doing' (Schoenfeld 1987a); monitoring the activities of choosing approaches to the problems and monitoring the implementation of new pieces of information into the problem. For example, why and when to choose a change of integrand or how might one manipulate the regions (or surfaces) which are being integrated over to improve problem-solving efficiency.

Finally, *extraction* refers to the activity of retrieving and allocating mental resources (lecturer's notes, experiences, and problem conditions) be they concepts, visual images or algorithmic techniques.

This paper now reports on one aspect of self-regulatory thinking described above as a metacognitive activity in Calculus problem-solving – *Organizational Thinking*.

Main Study

The main study comprised of two phases of work. In each phase, the work of a small group of undergraduates was investigated. These students were effective in thinking-aloud their thoughts and were chosen through problem-solving sessions outside class time and observation of their work. This was done over a period of a year and three groups of varying number were obtained. Specific problems from the multivariable Calculus were then chosen, and an exploratory study was conducted with each group to acquire some significant areas of metacognitive behavior with regards to the mathematical problem. The students were instructed to solve a problem previously unseen by them and to think-aloud their thoughts. These were recorded and transcripts produced for analysis. The researcher was present in the problem-solving session encouraging the student to talk and intervening to investigate the students' reflective thinking. Such areas of investigation were constructed as a topic agenda before the session, by observing "experts" solving the same problems. Such interventions were analyzed at every stage of the study. The students' written work was analyzed along with transcripts of their verbalization. Methods of protocol analysis developed by Schoenfeld (1985) were used in conjunction with an interventionist analysis to acquire some significant areas of metacognitive behavior with regards to *Reflection, Organization, Monitoring and Extraction*, which was intimately bound up, with the mathematics of the problems.

There were to main phases to the main work. Phase I comprised of two exploratory studies. These allowed the students to develop their ability to attend to and to verbalize their thinking. In addition, they illuminated certain characteristics of metacognitive thinking and areas, which affected their problem-solving behaviors. Phase II was a follow-up study aimed at further developing, but more so verifying, significant areas of the students' self-regulatory mathematical behavior, evident in phase I. In order to do this, a strict interventionist strategy was developed from the data in phase I that set up an agenda to sensitively direct the researcher's line of inquiry in phase II. Interventions were also analyzed to observe any bias in contriving metacognitive behavior. A detailed analysis of all of the exploratory and verification studies can be found in Hegedus (1998a) and summarized in Hegedus (1998b).

Data and Analysis

We now report on one aspect of the ROME model of self-regulatory thinking. It was evident that *organizational* thinking largely guided the problem-solving efficacy of the students. Thought patterns which directed what particular method to employ, which co-ordinate system to use and how a student might organize and attend to the various semantic parts of the Calculus problem gave rise to varying degrees of success in the students' work.

Each of the three groups of students studied exhibited various kinds of organizational problem-solving behavior. We now summarize the findings of the final main study, which sought to verify the metacognitive behavior, which was evident in the exploratory studies. Data from the main study is used to support the summaries of the analysis.

It was evident that organizational thinking comprised of conceptual and procedural thinking. Whilst conceptualizing the integral was largely to do with an initial reflection on the integrand and the region, procedural thinking was largely to do with the organization of resources for the solution of integral problems. Organizational behavior was a pre-occupation with the problem conditions and the order of the variables of integration.

For example, two students are working on solving the following integral:

$$\int_0^1 dx \int_x^{x^{\frac{1}{3}}} \sqrt{1-y^4\,dy}$$

Their initial thoughts in this problem-solving attempt attend to the problem conditions and the order of integration. The organization of the order of integration was effectively exhibited although not efficiently implemented.

G: Well, first of all I drew out the limits for the er ... the dy section .. of the integral, which is like ... y=x and y=$x^{1/3}$.

D: You drew the graphs of both of the ...

G: So you know what area you're trying to evaluate.

D: Yeh.

G: And you can sort of find the limits from that. But you've got to get the limits from x .. you see what I mean, they've changed around.

D: Yeh, yeh.

G: That's changed there instead of having $x^{1/3}$ you have y^3. By changing them around, so the limits have changed there.

D: So then you diff .. er .. integrate that in terms of x because you can't do it ....

G: And that's as far as I've got, just, you know, change the limits there ... and then I got stuck

Organization of the problem conditions into a suitable integrand was done via formulas in single variable integration and by the manipulation of variables in multi-variable integration. Whilst this seems relevant, problem-solving efficiency was flawed through insufficient reflection on the problem conditions and ineffective extraction of suitable resources. So whilst organizational behavior tended to occur early on in the solution attempts it was often inefficient because of a lack of looking forward on the student's part on how their solution might develop.

With regards to single-variable integration (e.g., spinning a region around an axis in the xy plane) the order was organized by which formula was more suitable to use, either in terms of x or y, rather than which direction of variability was more suitable to integrate in. Formulas were re-organized to suit the students' understanding of the problem conditions rather than through a process of re-organization and re-construction. The example of the abstract manipulation of formulas in order to obtain volumes of solids of regions revolved around a secondary axis rather than the primary one is discussed in this interchange between three students J, T and V. The researcher is S. They are attempting to solve the following problem:

Find the volume of the solid obtained by rotating the region $0 \leq y \leq 1-x^2$ about the line $y=1$.

J: [*To S*] Could you see where it crossed the x-axis and then take off the whole solid, the whole big rectangular cuboid ... revolution and then take off the smaller area or is that beating around the bush?

T: Don't know because it's going to be like rotating around the x-axis, isn't ...

J: ... up one.

T: But up one. Really it's going to be like rotating it about the x-axis. This area. If you do half of it then you've only got to double it afterwards haven't you? Cheating?

J: Are we going to rotate it around the x and then assume it's the same? Can you do that? [*To S*]

T: As long as you've got your x co-ordinate then it should be OK. Presumably, it's whatever x_1 minus .. x_2 is, maybe?

[*Pause*]

J: Could you just assume that you could rotate it around the x-axis and that it would be the same volume or ...? It is isn't it's got to be.

T: I reckon, as long as you let your x co-ordinate vary, it's just going to be like rotating the x but at this co-ordinate isn't it, here? [*Points vaguely to x axis*]

J: I'm not convinced.

T: You're not convinced. I reckon it would be the same, because if you have the same cut off point on your curve and then you shoved it down one and just imagine it going around the x-axis wouldn't you. You just know the height was 1. I have no formula with me ...

In multi-variable integration, organizational behavior was more a function of algebraic triviality and algebraic manipulation. With regards to the former, any variables that were pseudo-variables, i.e., constant in one direction in Euclidean 3-space, were simply evaluated and extracted outside the integral (e.g., $K\iint \ldots$). The long-term consequence of this was a condensation of algebra that led to problem-solving efficiency. Thus, algebraic manipulation was an organizational procedure. The later is exemplified best in the study of the double integral above, where the order of integration is reversed. Organization of the integral variables thus led to sufficient algebraic simplification to complete the problem, whereas it could not have been done otherwise. However, having noted this, the students still did not complete the problem. A theoretical understanding was not evident. Re-organization of the variables of integration had a direct effect on the region of integration and the manipulation of the limits.

Overall, there was little evidence of the organization of procedural skills when the students *finally* evaluated their integrals. This was mainly due to the misconceptions and inefficient problem solving. Evidence was often in the form of retrospection, since the students implemented procedures quite quickly with little verbalization. These problems were further enhanced when the students encountered algebraic difficulties. There was little evidence of an organized homogenous algebraic notation in effect that often led to confusion and a lack of simplification and manipulation (e.g., using signs and indices for square root).

Overall organizational procedures were evident and again were utilized with varying degrees of efficacy. Evidence of organization of a procedure of integration in single and multi-variable integration quite closely followed a set plan; that being:

- 'Parts', i.e., using the formula for integration of a product, then
- Make a substitution, then
- Implementation of trigonometric identities

Whilst a general procedural plan followed this cause of action:

- Visualize problem
- Obtain limits from region/sketch
- Investigate integrand, and
- Procedure (algebraic and algorithmic tools implemented)

Reflection on this generic procedural plan was also evident, since they got stuck at particular points (especially at point 2).

At a meta-level, not concerning the evaluation of the integral but the general method, the procedures implemented by the students were, in the main, inefficient, due to theoretical and conceptual difficulties.

It is thus questionable whether such an hierarchical approach is appropriate because of its lack of effectiveness. But it should be noted that such self-regulatory behavior is intimately bound up in the mathematics being solved. In fact the behavior is mathematical.

Procedural organization often determined whether the problem was effectively solved or not. The implementation of such techniques was a function of:

- The region of integration,
- The shape/form of the integrand
- Theoretical reflection on integration
- Reconstruction skills (abstracted generic formulas into problem-specific formulas)
- Algebraic difficulty.

In one of the main studies, a student exhibited effective problem-solving behavior through an efficient use of the metacognitive technique of procedural organization as outlined above, whilst solving a triple integral:

> A sphere whose equation is $x^2 + y^2 + z^2 = a^2$ has density which is proportional to the square of the distance from the plane $z + a = 0$, i .e. the plane which is tangent at the "South Pole". Find the total mass of the sphere.

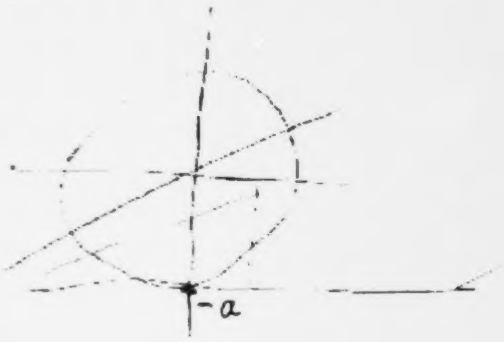
Through visualizing the region he deconstructed it into a 2D sketch which he used to organize and extract suitable resources to answer the problem. The sphericity of the region and the form of the integrand caused him to manipulate the problem conditions and organize a suitable procedure.

Here is evidence from his written work:

The student (J) and the researcher has the following discussion:

$x^2+y^2+z^2=a^2$

$z+a=0$



S: So what sort of question is it?

J: It's a triple integral .. uhm ...

S: Why's that?

J: It's density over a 3D region [*Pause*] Probably be easier in spherical …

S: Why do you need the conversion?

J: I'm just going to see if it's easier in the spherical.[*Pause*]

Such organizational thinking led him to set up a series of equations and 2D abstracted diagrams to assist him in his work

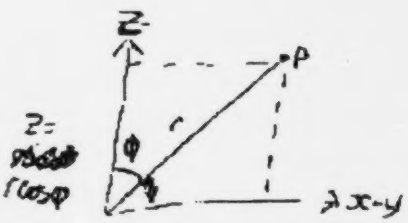
In the other main studies such metacognitive techniques were not as effective since the students were not so concerned with the efficiency of implementing certain

$x = r\cos\theta\sin\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\phi$

$r^2 = a^2$



BEST COPY AVAILABLE

procedures combined with a lack of forward reflection for suitability of method (i.e. in reducing algebraic difficulties). This was most evident when students were referring to generic formulas from experience, lectures or their textbooks to answer problems, which required suitable manipulation specific to the problem conditions.

Conclusion

The brief report highlights part of a larger study, which elaborated a model of self-regulatory thinking. This model attends to three other aspects of self-regulatory thinking including reflection, monitoring and extraction of mental resources (Hegedus, 1998b). Such studies help the growth of mathematical thinking by describing cognitive aspects of problem solving which are intimately bound up with the mathematics concerned. Through such studies one can look towards constructing informed pedagogic practices to assist the teaching of multivariable Calculus. This was proposed to the NSF recently as part of the *Assessment in Undergraduate Education Program.* The work aims to develop an assessment and teaching program that appreciates the types of problem-solving behaviors evident in the study discussed here.

References

Brown, A. L. (1987). Metacognition, executive control, self-regulation, and other more mysterious mechanisms. In F. E. Weinert & R. H. Kluwer (Eds.), *Metacognition, motivation and understanding.* (pp. 65-116). Hillsdale, NJ: Lawrence Erlbaum.

Flavell, J. H. (1976). Metacognitive aspects of problem solving. In L. Resnick (Ed.), *The nature of intelligence.* NJ: Lawrence Erlbaum.

Hegedus, S. J. (1996). Analysing the metacognitive behaviour of undergraduates: A pilot study. In *Proceedings of the 8th International Conference of Mathematics Education (ICME).* Seville, Spain.

Hegedus, S. J. (1998a). *Analysing metacognitive techniques in mathematics undergraduates solving problems in the integral calculus.* Doctoral thesis presented at the University of Southampton, UK. (Available online at http://www.soton.ac.uk/~heg/thesis).

Hegedus, S. J. (1998b). The ROME model of analysis: A methodological tool for use in investigating advanced mathematical thinking. *Proceedings of ICTM 98.* Samos, Greece: John Wiley.

Lester, F. K. (1994). Musings about problem-solving research (pp. 70-94). In *Journal for Research in Mathematics Education, 25,* (6). NCTM.

Mayer, R. (1998). Cognitive, metacognitive, and motivational aspects of problem solving. *Instructional Science, 26* (1/2), 49 – 63.

Newell, A., & Simon, H. A. (1972) *Human problem-solving.* Englewood Cliffs, NJ.

Nilsson, N. J. (1980). *Principles of artificial intelligence.* Palo Alto, CA: Tioga.

Schoenfeld, A. H. (1985). *Mathematical problem-solving.* Hillsdale, NJ: Lawrence Erlbaum.

Schoenfeld, A. H. (1987a). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematical education* Philadelphia: Lawrence Erlbaum.

Schoenfeld, A. H. (1987b). Cognitive science and mathematics education overview. In A. H. Schoenfeld (Ed.), *Cognitive science and mathematical education.* Hillsdale, NJ: Lawrence Erlbaum.

DOES THE ACQUISITION OF MATHEMATICAL KNOWLEDGE MAKE STUDENTS BETTER PROBLEM SOLVERS? AN EXAMINATION OF THIRD GRADERS' SOLUTIONS OF DIVISION-WITH-REMAINDER (DWR) PROBLEMS

Yeping Li
University of New Hampshire
yeping@math.unh.edu

Abstract: Fourteen third graders were given numerical computation and division-with-remainder (DWR) problems both before and after they were taught the division algorithm in classrooms. Their solutions were examined. The results show that students' initial acquisition of the division algorithm did improve their performance in numerical division computations with small whole numbers but not in solving DWR problems. Students' acquisition of division algorithm led some of them to perceive a DWR problem as one that can be solved using division procedure. However, all of them retreated from their initial perception of division procedure to the execution of alternative procedures for solutions. The use of alternative procedures led these students to achieve similar success rate and treat a "remainder" in a similar way when solving DWR problems before and after learning division algorithm in classrooms.

The development of students' proficiency in solving mathematics problems has been viewed as an important indicator of empowering students mathematically in school mathematics curriculum (e.g., National Council of Teachers of Mathematics [NCTM], 1989, 2000). Although it is generally believed that students' acquisition of formal mathematical knowledge can improve their competence in problem solving (e.g., Geary, 1995; Heffernan & Koedinger, 1997), previous studies have also shown that children and adults often use informal knowledge and strategies effectively in their problem-solving activities (e.g., Carraher, Carraher, & Schliemann, 1985; Scribner, 1984). Further explorations have indicated that the relationship between informal and formal knowledge can be either incongruous (e.g., Carraher, Carraher, & Schliemann ,1987) or complementary (Tabachneck, Koedinger, & Nathan, 1994; Vera & Simon, 1993). Although adequate use of formal knowledge can be much more mathematically powerful than the use of informal knowledge, informal knowledge can often help problem solvers in making sense of a mathematical problem and obtaining a solution. Different effects of using informal and formal knowledge suggest the need of further explorations on the role shift of informal and formal knowledge in students' problem-solving activities as they acquire more and more formal knowledge. Understanding such role changes can then provide a basis for informing classroom instruction. Along this direction, this study was designed to explore how students' acquisition of formal knowledge may change their uses of informal and formal knowledge in problem solving activities. In particular, this study explored how third graders' acquisition of division algorithm would affect their successes and sense-making behavior in solving Division-With-Remainder (DWR) problems.

Research Background

Middle school students' performance in solving DWR word problems in National Assessment of Educational Progress (1983) and several other studies (e.g., Silver, Shapiro, & Deutsch, 1993) exhibited that many middle school students solved a DWR problem by directly applying division procedure without interpretation of their calculation results based on the question being asked. It becomes clear in previous studies that many middle school students have no difficulty in understanding presented DWR problems and finding appropriate computation procedures for solution. Most errors happened in their final failure of interpreting computational results for answers (Silver et al., 1993). Missing link in connecting computational results and question asked at the final step indicated that many middle school students' solid acquisition of formal division algorithm led them to find an efficient solution strategy quickly but execute division procedure out of a problem context.

When a formal solution procedure was not ready accessible to students, a recent study on third graders' performance in solving a DWR word problem (Li & Silver, 2000) has illuminated students' super sense-making behavior with non-division solution strategies (e.g., counting up or down, multiplication). Although a DWR word problem was not the typical type of problems solved by third graders before their acquisitions of the division algorithm, third graders in this previous study showed their great success in solving a DWR problem. The contrast results between third graders and middle school students suggest the importance of understanding how the acquisition of formal division algorithm would likely affect students' sense-making behavior. Specifically, a study on the changes of third graders' success and sense-making behavior before and after their learning of division algorithm became feasible for pursuing this research inquiry.

Method

Fourteen third graders from a private laboratory school attached to a university participated in this study. This school is in an Eastern city. These children came from two classrooms taught by different teachers.

This study included two sections. The first section was carried out before these students' learning of division algorithm. The second section was given to the same group of third graders after their learning of division algorithm. These two sections were five month apart. In both sections, two types of problems were used. One type was word problems with whole numbers. The others were numerical calculation problems. In this report, only the DWR word problems and numerical division computation problems were included for analyses (see them below).

Relevant problems used in section one (before students' learning of division algorithm):

DWR 1: Mary has 22 tapes. She wants to buy some boxes to store all her tapes. Each box can store 5 tapes. How many boxes does Mary need to buy?

Numerical division computations:

$20 \div 4 = \quad 13\overline{)52}$

Relevant problems used in section two (after students' learning of division algorithm):

DWR 2.1: Tom has 28 tapes. He wants to buy some boxes to store all his tapes. Each box can store 6 tapes. How many boxes does Tom need to buy?

DWR 2.2: The Clearview Little League is going to a Pirates game. There are 540 people, including players, coaches, and parents. They will travel by bus, and each bus holds 40 people. How many buses will they need to get to the game?

Numerical division computations:

$30 \div 6 = \qquad 86 \div 12 = \qquad 252 \div 18 = \qquad 518 \div 30 =$

All tasks were administered to each student individually. Students were asked to think aloud when they solved the DWR problem(s), and their verbal explanations were recorded simultaneously for transcriptions. For the numerical division problems, students were asked to do computation on a piece of paper.

Students' solutions to numerical division items were coded as "correct", "incorrect" or "skipped". The transcriptions of students' verbal explanations of their solutions to the DWR word problems were analyzed both quantitatively and qualitatively. In particular, students' final answers were coded as correct or incorrect. Their solution process was subject to a fine-grained cognitive analysis. The cognitive analysis examined students' choice and execution of specific solution strategies in their solution processes. Four categories of solution strategies were developed and used in this analysis (taking students' solutions to DWR 1 as examples here):

1. Division (D). The student performed long division computation, 22+5, to get the problem's solution.
2. Multiplication (M). The student used multiplication such as 4×5=20 and then figured out the number of boxes needed for 22 tapes to solve the problem.
3. Additive Approach (AA). The student used this approach (including subtraction) as counting up by 1's or 5's to 20 (or 25) and then counting the number of boxes needed for 22 tapes; or counting out 22 first, then counting out by 5's to figure out the number of boxes needed for 22 tapes.
4. Unidentifiable (U). The student either used some unknown strategies to solve the DWR problem or was unable to tell how he/she got the answer.

Results and Discussion

Quantitative Results

Table 1 shows the percentages of students obtaining correct numerical answers in solving the two types of problems. In section one, 43% students obtained a correct answer for "20 ÷ 4" and 29% for $13\overline{)52}$, whereas 93% of these solved the DWR 1 problem correctly. The results indicate that the lack of division procedure affected students' successes in numerical division computations but not in solving the DWR problem. In section two, 71% students did correct in calculating "30 ÷ 6", 21% for "86 ÷ 12", 7% for "252 ÷ 18", and none for "518 ÷ 30". 86% students obtained a correct answer in solving DWR 2.1, a problem containing small whole numbers, but only 29% in solving DWR 2.2, a problem with large numbers. The results suggest that students' initial acquisition of division algorithm helped students to perform well in solving both types of problems with small size of whole numbers but not in solving problems with large numbers. Across the two sections, the results show that students' acquisition of formal division algorithm did improve their performance in numerical division computations (e.g., 43% correct in calculating "20 ÷ 4" in section one versus 71% correct in calculating "30 ÷ 6" in section two) but not in solving the DWR problems (e.g., 93% correct in solving DWR1 in section one versus 86% correct in solving DWR2.1 in section two).

Table 1. Percentages of Students Obtained Correct Solutions in Two Sections.

Section One	20÷4	$13\overline{)52}$	DWR1			
	43%	29%	93%			
Section Two	30÷6	86÷12	252÷18	518÷30	DWR2.1	DWR2.2
	71%	21%	7%	0%	86%	29%

Qualitative Results

Table 2 shows the percentages of students who tended to choose a specific strategy after reading a DWR problem in the two sections. The results indicate that the Additive Approach was the most commonly chosen strategy by these students after reading these DWR problems in both sections (71% for DWR 1, 43% for DWR 2.1, and 71% for DWR 2.2). No student was able to perceive the DWR1 problem in section one as a problem that requires division procedure, whereas 36% of them chose division strategy at their first look at DWR2.1 in section two. Such difference indicates that initial acquisition of division algorithm led some students to perceive a DWR problem as one that requires a division procedure.

Table 3 shows the percentages of students who actually used a specific strategy to obtain a solution for these DWR problems in both sections. The results show that the Additive Approach also was the most commonly used strategy for solving these DWR problems (71% for DWR 1, 71% for DWR 2.1, and 79% for DWR 2.2). Although

Table 2. Percentages of Students Who Considered a Specific Solution Strategy After Reading the Problem During Solving DWR Problems in Two Sections.

		Division (D)	Multiplication (M)	Additive Approach (AA)	Unidentifiable (U)
Section One	DWR1	0%	29%	71%	0%
Section Two	DWR2.1	36%	21%	43%	0%
	DWR2.2	0%	21%	71%	7%

Table 3. Percentages of Students Who Actually Used a Specific Solution Strategy in Solving DWR Problems in Two Sections.

		Division (D)	Multiplication (M)	Additive Approach (AA)	Unidentifiable (U)
Section One	DWR1	0%	21%	71%	7%
Section Two	DWR2.1	0%	29%	71%	0%
	DWR2.2	0%	14%	79%	7%

some students (36%) perceived DWR problem 2.1 in section two as a problem that can be solved using division algorithm, none of them actually executed division procedure to obtain an answer. All of these students retreated from their initial choice of division procedure to the execution of other alternative procedures for solutions. Their uses of alternative procedures in solving DWR2.1 in section two and DWR1 in section one led them to achieve similar success rates across the two sections. These students' behavior suggests that their initial acquisition of division algorithm was not solid enough to help them to get the problem solution. The possible impact of division algorithm acquisition on students' problem understanding was evident but also limited as none of them perceived and solved the DWR2.2 as associated with the division algorithm. As a result, students' initial acquisition of division algorithm showed some impact on their understanding of a DWR problem but no apparent impact on their executions of solution procedures.

It is important to understand how students handled the remainder when studying possible changes in their sense-making behavior (Li & Silver, 2000; Silver et al., 1993). With their exclusive uses of non-division strategies for finding answers for all DWR problems, however, these students treated the remainder effectively in a similar way in both sections. Specifically, based on their situated reasoning and use of non-division strategies, these third graders treated a remainder concretely and successfully. For example, in solving DWR 1 in section one, these students took the remainder as "left over" or "extra" tapes or some other equivalent expressions in their solutions.

S: (After reading the problem) Five.

I: How do you know it is five?

S: Because if you bought four and each box can carry five tapes, that will be twenty tapes. And you need another one because there are *two more left*.

The results from this study indicate that students' initial acquisition of division procedure had limited impact on their performance in solving a DWR problem. Without a solid acquisition of the division algorithm, these third graders showed some 'division' sense when understanding a DWR problem but folded back to use non-division strategies in solving the DWR problem. Evidence exists that middle school students' sophisticated acquisition of division algorithm led them to perceive and solve a DWR problem using division procedure, but often disconnected from a problem context (Silver et al., 1993). Thus, students' acquisition of the division algorithm can have a great potential to enhance their computational efficiency, it can also change their sense-making behavior. The more sophisticated acquisition of division algorithm students have, the less use of alternative/informal strategies can be observed in their sense making and solving a mathematical problem. Therefore, in order to empower students with formal mathematical knowledge, it seems necessary for teachers to help students to make meaningful connections between formal mathematical knowledge and various problem contexts. As there are different ways in classroom instruction that can help students to make such connections, it remains to be explored what impact different classroom instruction may have in facilitating students' learning and developing their problem-solving proficiency as expected.

Connections to the Goals of PME-NA

This study focused on the changes in students' sense-making behavior in solving mathematical problems before and after their learning of formal mathematics knowledge. It aims to provide a basis for understanding the role of formal mathematics knowledge in the development of students' mathematics competence and for generating pedagogical suggestions. Thus, this report connects to the goals of PME-NA in general and the coming PME-NA meeting in specific.

References

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the street and in schools. *British Journal of Development Psychology, 3*, 21-29.

Carraher, T. N., Carraher, D.W., & Schliemann, A.D. (1987). Written and oral mathematics. *Journal for Research in Mathematics Education, 18*, 83-97.

Heffernan, N. T., & Koedinger, K. R. (1997). The composition effect in symbolizing: The role of symbol production vs. text comprehension. In *Proceedings of the Nineteenth Annual Conference of the Cognitive Science Society* (pp. 307-312). Hillsdale, NJ: Lawrence Erlbaum.

Geary, D. C. (1995). Reflections of evolution and culture in children' cognition: Implications for mathematical development and instruction. *American Psychology, 50* (1), 24-37.

Li, Y., & Silver, E. A. (2000). Can younger students succeed where older students fail? An examination of Third Graders' solutions of a Division-With-Remainder (DWR) problem. *Journal of Mathematical Behavior, 19*, 233-246.

National Assessment of Educational Progress. (1983). *The third national mathematics assessment: Results, trends and issues*. Denver, CO: Author.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Scribner, S. (1984). Studying working intelligence. In B. Rogoff & J. Lave (Eds.), *Everyday cognition: Its development in social context* (pp. 9-40). Cambridge, MA: Harvard University Press.

Silver, E. A., Shapiro, L. J., & Deutsch, A. (1993). Sense making and the solution of division problems involving remainders: An examination of middle school students' solution processes and their interpretations of solutions. *Journal for Research in Mathematics Education, 24* (2), 117-135.

Tabachneck, H. J. M., Koedinger, K. R., & Nathan, M. J. (1994). Toward a theoretical account of strategy use and sense making in mathematics problem solving. In *Proceedings of the Sixteenth Annual Conference of the Cognitive Science Society* (pp. 836-841). Hillsdale, NJ: Lawrence Erlbaum.

Vera, A. H., & Simon, H. A. (1993). Situated action: A symbolic interpretation. *Cognitive Science, 17*, 7-48.

A COMPARATIVE STUDY OF THE SELECTED TEXTBOOKS FROM CHINA, JAPAN, TAIWAN AND THE UNITED STATES ON THE TEACHING OF RATIO AND PROPORTION CONCEPTS

Jane-Jane Lo
Cornell University
jl45@cornell.edu

Jinfa Cai
University of Delaware
caiji@gse.harvard.edu

Tad Watanabe
Towson University
tad@towson.edu

Abstract: In this study, we examined the introduction of ratio and proportion concepts in six textbook series from four different regions, China, Japan, Taiwan and U.S. When analyzing the definition of ratio and equal ratio as well as the types of ratio and proportion application problems included in each textbook series, we found similarities and differences among these textbook series both within and across regions. Future studies will be needed to examine these differences further.

The results of cross-national comparisons of mathematical performance such as TIMSS and TIMSS-R have received considerable attention by the educational research community as well as the general public. This type of analyses provide unique opportunities to understand the current state of students' learning and suggest ways future learning can be supported and enhanced (Cai, 2001). One of the findings from previous cross-national studies was that, in general, the U.S. students did not perform as well as the Asian students in mathematics. Because of the complexity of interpreting cross-national differences, we are just beginning to understand the possible factors that may contribute to the differences in mathematics. In an earlier study, we examined curricular treatments of arithmetic average in U.S. and Asian school mathematics. (Cai, Lo, & Watanabe, 2001). In this study, we focused on how the textbooks and teacher manuals were designed to facilitate students' understanding of the ratio and proportion concepts.

Theoretical Framework

Cross-national comparison of textbooks

Cross national comparisons of mathematics textbook are important because U.S. textbooks constitute a de facto national curriculum (Mayers, Sims, & Tajika, 1995). Therefore, examining the content and teaching methods of the textbooks provides a partial account of how mathematics is taught in different regions. For example, Mayers, Sims & Tajika (1995) compared the lesson on addition and subtraction of signed whole numbers in three seventh-grade Japanese and four U.S. mathematics textbooks. They found that Japanese books contained many more worked-out examples and relevant illustrations than did the U.S. books. And the U.S. books contained roughly as many exercises and many more irrelevant illustrations than did the Japanese textbooks. When comparing selected first-grade Japanese and U.S. textbooks, Samimy and Liu (1997) found that while the Japanese textbook was mainly characterized by its short explanation and equal distribution of both computational and word problems, the U.S. textbook was characterized by its detailed explanations, repetitions and pictures. Li (2000) compared the problems presented in selected U.S. and China middle school mathematics textbooks on the topic of addition and subtraction integers through three dimensions: mathematical feature, contextual feature and performance requirements. He found some striking difference between the U.S. and the China textbooks with respective to problems' performance requirement. For example, more problems in the U.S. textbooks (19%) required mathematical explanations than the problems in China textbooks (0%). All these analyses have identified important differences between the selected U.S. textbooks and Asian textbooks that might provide some explanations of the performance differences in the international comparative studies such as TIMSS and TIMSS-R. However, none of the above studies have examined textbooks' attempts to support students' learning of a major mathematical concept. Our previous study showed that such investigation was informative for identifying the difference in curricular emphases that might explain the difference in student performance (Cai, Lo, & Watanabe, 2001). The present study intended to provide further information by examining the textbooks' attempts to support students' learning of the ratio and proportion concepts by several U.S. and Asian textbook series.

Ratio and proportion concepts

We selected the ratio and proportion concepts to be the focus of this study because of their importance in school mathematics. Typically, students received their initial instruction on ratio and proportion as early as the fourth or fifth grade. Then they continued to study these topics with increasing complexity and formality throughout the middle school years. When solving ratio and proportion problems, students applied their concepts on multiplication and division of whole number and rational numbers. This process helped students build a solid foundation for more advanced mathematics concepts such as algebraic reasoning and analytic geometry. There has been a consensus that the concepts of ratio and proportion developed through stages. It took several years to develop the formal reasoning and computation skills that could be flexibly applied to various number structures and across different problem contexts (Lo & Watanabe, 1997; Vergnaud, 1988).

One challenge of teaching and learning of ratio and proportion concepts was the wide variety of problem types and contexts as well as number structures that could influence the students' performance (Harel, Behr, Post, & Lesh, 1991; Kaput & West, 1994; Lamon, 1993). In order to compare different textbooks within mathematics education research literature on ratio and proportion, this study examined the amount of variety included in the examples and exercises in each textbook series.

Method

The selection of the typical Asian mathematics textbook series in China, Japan and Taiwan was not difficult because all textbook series in these regions must conform to the curriculum set by the Ministry of Education at each region. The Asian textbook series chosen for this study included China series (Division of Mathematics, 1996), Japan series (Tokyo Shoseki, 1998) and Taiwan series (National Printing Office, 1999).

Finding a representative U.S. textbook, on the other hand, was much more difficult due to the lack of national curricular guidelines. In order to reflect this diversity, we decided to include both National Science Foundation funded "reform" curricular and "commercial" curriculum in our study. The two reforms curricular were *Mathematics in Context* (National Center for Research in Mathematical Sciences Education at the University of Wisconsin/Madison and Freudenthal Institute a the University Utrecht, 1997-1998), and *Connected Mathematics* (Fay, et. al, 1998). Because there were a wide variety of "commercial" textbooks available in the market, it was quite impossible to find a "typical curriculum" in the same sense as the Asian textbooks. As a result, we used a series that was adopted by a major school district in a Mid-Atlantic state, *Math in My World* (Clements, et. al., 1999).

Because of the developmental nature of the ratio and proportion concepts, it was necessary to examine the teaching of these concepts across several grade levels. Three levels of analysis were planned. First, we examined how the ratio concept was introduced by each textbook series through the examinations of the goals, examples, exercises presented by both the student books and the teacher manuals. Second, we studied the depth of the textbook presentation on ratio and proportion concepts by the type of connections to the other units in the same textbook series as identified by the teachers' manual. However, we found that frequently, for the ease of use, the teacher manuals presented only the vertical connections of a particular unit within a few grade spans, but not the horizontal connections. Therefore, we decided to also identify "other" ratio-related units there were introduced in the curriculum series but not identified by the teacher's edition as connected units. The goal was to capture the richness of the textbook presentation on ratio and proportion concepts. After these analyses were conducted for each textbook series, comparisons were made to highlight both the similarities and differences among these six textbook series. In this paper, we will present some results about the first level of the analysis, focusing on the introduction of ratio and proportion concepts.

Analysis

First, we will summarize the lessons that have been included in this analysis from each textbook series. The ratio concepts were formally introduced in all three Asian textbook series at the sixth grade. In China series, the concept of ratio (3 lessons) was introduced at the first half of the sixth grade within the unit of "Division with fractions". The unit on proportion that include direct proportion, inverse proportion, map reading and other application of proportion was introduced at the end of sixth grade (10 lessons). Similar to China's series, the ratio was introduced at the first half of the sixth grade (8 lessons) in Taiwan series. Proportion-related concepts were introduced in the second half of the sixth grade in three different units: direct proportion (6 lessons), inverse proportions (7 lessons) and scale drawing/map reading (8 lessons). In Japan series, the concept of ratio was introduced in the sixth grade (8 lessons), followed by lessons on scale drawing and map readings (9 lessons). The concept of proportion including direct and inverse proportion was introduced about three months later at the same grade (12 lessons). There were some rate-related problems embedded in a unit called "per-unit-quantity" in the fifth grade, prior to the formal introduction of ratio and proportion concepts.

In the U.S. commercial *Math in My World*, the ratio concept was first mentioned briefly together with the introduction of rate in the fifth grade (4 lessons). This topic was revisited with more elaboration and additional topics such as rate, proportion and scale drawings in the sixth grade (12 lessons). Different from the three Asian textbooks and the U.S. commercial, the definition of ratio in the *Connected Mathematics* was given explicitly only after the students were given some chance to explore the idea of "ratio" informally. The term "ratio" was first mentioned in the unit of "Stretching and Shrinking" (seventh grade, 22-23 lessons). Students were asked to investigate the relationship between length segments and angles of different computer-generated figures in order to study the idea of similar figures. In a subsequent unit "Comparing and Scaling" (seventh grade, 2-24 lessons) students revisit the idea of ratio together with the concepts of rate and percents in various non-geometrical contexts. Similar to the *Connected Mathematics*, the definition of ratio was given explicitly only after the students were given some chance to explore the idea of "ratio" informally in the *Mathematics in Context*. The concept of ratio was first introduced informally through the unit "Some of the Parts" a unit on Fraction (Grade 5/6, 3 lessons). Ratio table was introduced as a model use to increase or decrease the serving sizes of recipes, and later became a tool for organizing fraction calculation. The idea of ratio was continuously explored informally in the context of percent, decimals, and fraction in the unit of "Per Sense" in grade 5/6 (14 lessons). And in the unit of "Ratios and Rates" in grade 6/7 (19 lessons), the ideas of ratio and rates were explored more formally and fully. Concepts and terminology like "part-part ratio" and "part-whole ratio," "constant and variable relationship" were introduced to help students further explore the relationship among ratios, fractions, percents and decimals, as well as among ratios, rate and average. In the following, we will present the initial analysis in four sections "definition of ratio," "equal ratio," "application of ratio" and "application of proportion."

Definition of ratio

In Chinese, the word used for ratio is "bi." It means "to compare or comparison." When used in mathematical context, the multiplicative comparison of two quantities was emphasized. For example, in China series, the ratio was introduced by comparing the length and width of a flag, "Figure out the length of the flag is how many times as the width of the flag?" In Taiwan series, the ratio was introducing by comparing the number of cookies each of the two brothers have, "Figure out the number of big brother's cookies is how many times as the little brother's cookies." In both series, the equality notation $a:b=a+b=a/b$ was introduced. The notation "a:b" is called "ratio of a and b" or "a to b" and the "a/b" is called the "value of ratio." The relationship among ratio, division and fraction were emphasized through this equality. In Taiwan series, the ratio was restricted to the relationship between two quantities with the like measures, but in China series, examples of ratio included "distance: hours" of different measures.

In the Japanese textbook, the ratio concepts was introduced by asking students to determine whether the taste of the two dressings, one made at home and the other made at the school, would taste the same. The dressing made at home consisted of 2 spoons of vinegar and 3 spoons of oil. The one made at school consisted of 8 spoons of vinegar and 12 spoons of oil. Compared to both China and Taiwan series, the Japan series made more clearly distinction between the "ratio" and the "value of ratio." For example, the following paragraph was included in the teacher manual: "A ratio is a representation of the proportional relationship between two quantities using two whole numbers. On the other hand, the value of a ratio, that is, A is 2/3 of B, expresses the amount of A using one number when we consider B as 1. Thus, when we express the proportional relationship of two quantities by an ordered pair of two whole numbers, we have a ratio, while we express that proportional relationship using one number, we have the value of a ratio."

All three U.S. textbook series defined ratio as a way to relate two numbers: for so many of the first quantity, there were so many of the second quantity. Students were taught three ways of writing ratios: 2 to 3, 2:3 or 2/3. The idea of "multiplicative relationship" was not emphasized explicitly. Different from all three Asian textbook, there was no clear distinction made between "ratio" and "the value of ratio." They simply made 2:3 equivalent to 2/3, without bridging through division operation.

Equal ratio

China textbook series introduced the idea of equal ratio directly through the previous work on fractions and equivalent fractions. The unit of Equal Ratio started out with the following question, "We have learned the invariant property of quotient in division and the basic property of fractions, if we combine these two properties, think about it, what properties about ratio do we have?" The following principle was listed after the question: When multiplying or dividing both the front term and the back term by the same number (other than zero), the value of the ratio does not change.

Both Taiwan and Japan series gave detailed illustrations to connect the idea of equal ratio with the principle of outlined above by China series. For example, in Taiwan series, a ratio of "20:30" was first identified as the relationship between the width (20 cm) and the length of a rectangle (30 cm). Then the students were asked to use 5 cm as a unit to measure the width and the length of the same rectangle. As a result, the width became 4 units (of 5 cm) and the length became 6 units (of 5 cm), thus a ratio "4:6" can be used to represent the same width vs. length relationship. Lastly, the students were asked to use 10 cm as a unit to measure the width and the length of the same rectangle, and another ratio "2:3" was obtained. From here the relationship 20:30 = 4:6 = 2:3 was established. Two more similar examples were shown before the algorithmic principle of equal ratio was introduced in Taiwan series. Similar discussion and illustration were used in Japanese series.

Even though China series did not include pictures to facilitate the explanation, it contained the most variety of problems in their exercise sections that challenged students' thinking. For example, one exercise asked "If Li Ming is 1 meter tall, and his dad is 173 cm tall. Li Ming said that the ratio between his height and his dad's height is 1: 173. Is this correct? What do you think the ratio should be? " This exercise pointed to the students the importance of converting both quantities to the same measuring unit when forming a ratio relationship

The *Math in My World* introduced the idea of equal ratio by asking students to use two red counters and three yellow counters to represent the ratio "2:3." Then repeatedly added groups of two red counters and three yellow counters to generate equal ratios such as "4:6" "6:9" "8:12" etc. The idea of ratio table was introduced as one way to find equal ratios. Another way of finding equal ratio was to multiply or divide the numerator and denominator by the same number. No connections were made between these two ways of finding equal ratios.

The *Connected Mathematics* introduced the idea of equal ratio through a series of investigation on similar figures. When comparing width and length of a rectangular shape, the book introduced the word "ratio" as a way to relate two quantities. For example, the width was one unit wide and 2 units long, so the ratio "1 to 2" could be used to describe the relationship. It also equivalent "1 to 2" to fraction "1/2." Through a series of computer supported explorations, students were expected to discover the following three principles about similar figures: 1) Their general shapes are the same, 2) Their corresponding angles are the same, and 3) Their corresponding side lengths are related by the same scale factor. The idea of "equal ratio" was explored informally through the last principle: scaling up and down the corresponding side lengths by the same scale factor. Later the term "ratio" was introduced more formally as a way to compare two quantities. The concept of "equal ratio" was mentioned as "scaling up and down a ratio." The following example was given: "Suppose a shade of purple paint is made using 2 parts red paint to 3 parts blue. You would get the same shade of purple whether you mixed 2 gallons of red paint to 3 gallons of blue paint, 4 gallons of red paint to 6 gallons of blue pain, or 6 gallons of red paint to 9 gallons of blue paint." No rule was given explicitly in the student textbook.

Similar to all three Asian textbook series, *the Mathematics in Contexts* introduced the ideal of "equal ratio" through the context of equivalent fractions. Specially, "ratio table" was introduced as a model use to increase or decrease the serving sizes of recipes, and later become a tool for organizing fraction calculation. Fore example, the idea that the relationship of cups of sugar to cups of flour remained constant could be shown in the ratio table below:

Cups of Flour	1	2	3	4
Cups of Sugar	1/2	1	1 1/2	2

This model was formally introduced to students in the student textbooks with more examples . Several valid operations could be performed on the table, such as halving, doubling, multiplying, adding and subtracting. Notice that, unlike the other five textbook series, the *Mathematics in Context* regarded multiplying both the first term and the second term by a non-zero number as just one of many ways to find equivalent ratios

Application of ratio

In all three Asian textbook series, problems that require direct application of ratio concepts were included right after the units of equivalent ratio. There were remarkable similarities among three Asian textbooks in terms of the type of exercises included in this unit. There were two basic types. The first type gave an explicitly stated ratio relationship between two quantities, and the actual amount of one of those two quantities, the question was to use the ratio relationship to find the actual amount of the second quantity. For example, " The ratio between the number of boys and the number of girls in a school choir is 3:5. There were 21 boys? How many girls?" (from Taiwan series , sixth grade). The second type was to give the ratio relationship between two quantities, and the actual amount of the sum of the two quantities, the question was to use the ratio relationship to find the actual amount of each of the two quantities. For example: " Two brothers shared 1800 Yen. The ratio between the older brother's money and the little brother's money is 3:2. 1) The big brother's money is what fraction of the whole amount of money? 2) How much money will the big brother get? 3) How much money will the little brother have?" (from Japan series, sixth grade). China series also included worked-out example like the above two types. Furthermore, some examples in China textbook series included ratio among three quantities:

such as 2:3:5. However, in U.S. commercial textbook, we only found example of the first type of questions included in the sixth grade textbook, but not the second type.

Different from the three Asian textbook series and the *Math in My World*, the *Connected Mathematics* and the *Mathematics in Context* did not include exercises with simple statement and unique answer that required students to directly apply their ratio concepts. However, the *Connected Mathematics* included much more elaborated exercise to help students link the concept between ratio, fraction and percentage. And great emphasis was placed on selecting the best way to make comparisons based on the purpose of the comparisons. The *Mathematics in Context* also included elaborated contextual problems that helped students to make connections between the expression among ratio, fraction, decimal and percentage. In one exercise that based on the context of speeders vs. total number of drivers, students were asked to find the equivalent fractions, decimals, and percents for 1:2, 1:4, 1:5 and 1:15. The idea that only part-whole ratio can be converted to percentage meaningfully was first introduced in an earlier unit Per Sense was again been emphasized here.

Application of proportion

All the textbook series we examined treated proportion as an equivalence between two ratios. The idea of co-variation: the change of one quantity influenced the change of another quantity was required in solving proportion problems. We have identified the following four major types of problems to organize this part of analysis: 1) stretching and shrinking: these includes scale drawing and map reading, 2) direct proportion including the problems involving familiar rate measurements, 3) inverse proportion, 4) comparative ratio and rate problems.

Stretching and shrinking problems. All textbook series we examined except China series[2] included problems with stretching and shrinking in the units we examined. Among those five textbook series, the *Math in My World* has the shortest time coverage of the topic (1 lesson in grade 5 and 2 lessons in grade 6), while the *Connected Mathematics* devoted a whole unit "Stretching and Shrinking" (22-23 lessons) on this topic. In terms of content coverage, both Taiwan series and the *Math in Context* introduced the idea of stretching and shrinking through changing the size of the unit squares used in drawings. They then guided students to the principle that the resulting figures would have similar shape but larger or smaller size and there is a constant ratio (scale factor) existing between the corresponding sides. Both the *Math in My World* and the *Connected Mathematics* introduced an additional principle about similar figures "the corresponding angles stayed the same" to their students. However, the introduction of this principle in the *Math in My World* was brief while the students were guided through a series of computer-based activities to reach the same conclusion in *Connected Mathematics*.

All textbook series we examined included discussion and exercises on understanding scale such as the one on the map, floor plan or microscope. Most of the textbook series included simple application such as "Assume the scale of the map is 1:1000000. What is the actual distance between points A and B if the distance between points A and B on the map is 2 cm?" with various degrees of the contextual information. The *Mathematics in Context* had the most elaborated exercises on this topic. For example, one exercise asked students to compared the actual sizes of two drawings obtained from two microscopes that had different magnifying power.

Direct proportion problems. All three Asian textbook series gave this topic significant attention at the sixth grade level. Various examples were given to help students understand the concept of direct proportion "When one quantity grows k times, the other quantity grows k times also," the notation "y=kx," and the graph. Exercises were included to help students differentiate between the situation involving direct proportion (e.g., length vs. perimeter) and the situation not involving direct proportion (e.g., length vs. area).

All six textbook series we examined included some discussion and exercises on rate-related problems, such as speed and density. However, the degree of the complexity of these types of problems varied greatly among these textbook series. The simple ones were missing value type of exercises, such as, "A car drove 140 km in 2 hours. If the speed stayed the same, and it took the car 5 hours to travel from point A to point B. What was the distance between point A and point B?" Several different strategies were shown for solving this type of problems. For example, one solution was to find the speed by dividing the distance, 140 km, by the amount of time, 2 hours, than multiply by 5 hours to get 350 km. The second method was to recognize the existing equal ratio, and find the scale factor between 5 hours and 2 hours, which was 2.5. Then multiply 2.5 by 140 km to get 350 km. The third method was the "cross product." An equation x/5= 140/2 was set up. When cross-multiplying, 2x=140x5=700. Therefore, x= 350. Only the China series and the *Math in my World* showed this cross-multiplying method. The *Mathematics in Context* with its unique "ratio table" introduced a unique way of solving this type of problem. A student could set up a ratio table for the ratio 2 to 140. Doubled it to get 4 to 280. Halved it to get 1 to 70. Then added the ratio of 4 to 280 to the ratio of 1 to 70 to get a new ratio of 5 to 350.

Inverse proportion problems. All three Asian textbook series gave this topic significant attention at the sixth grade level. Various examples were given to help students understand the concept of inverse proportion "When one quantity grows k times, the other quantity grows 1/k times also," the notation "y=k ÷x" and the graph. Exercise were included to help students differentiate between the situation involving inverse proportion (e.g., time vs. speed when the distance was the constant) and the situation not involving an inverse direct proportion (e.g., length vs. area). None of the three U. S. textbook series we examined included this type of problems.

Comparative ratio/rate problems. None of the Asian textbook series we examined included exercises on comparative ratio/rate at the sixth grade level where the main part of the instruction of ratio and proportion were introduced. Japan textbook had some discussion of this type of question in a fifth grade unit titled "per-unit-quantity," which was before the main introduction of ratio and proportion units in sixth grade. Taiwan textbook series discussed this type of questions at seventh grade level within the formal discussion of algebraic ratio properties. China textbook series did not contain any this type of problems in the elementary or middle school. However, this type of problems was included in all of three U.S. textbook series did. The amount of emphasis varied from one lesson (*Math in My World*), to 15-22 *lessons (Connected Mathematics* and *Mathematics in Context*).

Conclusion

This study attempted to compare the introduction of ratio and proportion concepts in the textbook series from three Asian regions and United States. Similar to the previous studies on international textbook comparison, we also found great differences between the Asian and U.S. textbook series. However, we also noted that the one U.S. commercial textbook series *Math in My World* included this study was more similar to the Asian textbook series in their form of presentations and the structures of the topics than the U.S. reform textbook series. For example, all three Asian textbook series and *Math in My World* used contextual problems to support and motivate the introduction of the concepts and procedures. However those contextual problems tended to be short and specific to the concepts and or techniques. Both the *Connected Mathematics* and the *Mathematics in Context* used elaborated contextual problems with rich contexts and many connected sub-problems to support students to develop their own ideas and strategies among many different concepts. Finally, reasoning and communication were emphasized throughout both textbook series explicitly by require students to write extensively. Certainly, there are differences among three Asian textbook series, between Asian and U.S. commercial textbook series, between U.S. commercial and reform textbook series, and between the two U. S. reform textbook series. Further studies will be needed to examine these differences in depth.

We also noted many differences among these six textbook series in terms of their definitions of ratio, the way equal ratios were introduced and the type of application problems included in the books. For example, the idea of multiplicative comparison was emphasized explicitly by the Asian textbook series but not by the others. Also, *Mathematics in Context* discussed many different ways equal ratios could be generated such as halving, doubling, adding, subtracting, and scale multiplying/dividing, while all the other textbook series discussed only scale multiplying/dividing to generate equal ratios.

There was also significant difference in terms of sequencing. For example, *Connected Mathematics* introduced the idea of similar figures at seventh grade and before the main discussion of ratio and proportion. Yet most of the other textbook series introduced the idea of similar figure after the introduction of ratio and proportion. A future study that focuses on the sequence of certain big mathematical ideas should have great instructional implication. Finally, with all the emphasis on reasoning and connection in both U.S. reform textbook series, will the students that have gone through either textbook series showed greater ability in these two areas? Future studies are needed to see if we can identify the influence of these textbook differences on student performance.

Endnotes

1. Preparation of this paper was supported, in part, by a grant from The National Academy of Education. Any opinions expressed herein are those of the authors and do not necessarily represent the views of National Academy of Education.

2. China textbook series included topics on similar figures at the second part of seventh grade. This topic was discussed at a much more formal level both geometrically and algebraically. Therefore, we did not include it in our discussion.

References

Cai, J. (2001). Improving mathematics learning: Lessons from cross-national studies of U.S. and Chinese students. *Phi Delta Kappan, 82*, 400-405.

Cai, J., Lo. J., & Watanabe, T. (2001). *Intended curricular treatments for students' conceptual understanding in U.S. and Asian school mathematics: A case of arithmetic average*. Paper presented at the Annual Meeting of American Educational Research Association, Seattle, WA; April 10-14, 2001.

Clements, D. H., Jones, K. W., Moseley, L.G., & Schulman, L. (1999). *Math in my world. Teacher's Edition (Grade 5 and Grade 6).* New York: McGraw-Hill.

Division of Mathematics. (1996). *National unified mathematics textbooks in elementary school.* Beijing: People's Education Press.

Fey, J. T., Fitzgerald, W. M., Friel, S. N., Lappan, G., & Phillips, E. D. (1998). *Connected mathematics*. Menlo Park, CA: Dale Seymour Publications.

Harel, G., Behr, M., Post, T., & Lesh, R. (1991). Variables affecting proportionality. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth Annual Meeting of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 125-132). Assisi, Italy: International Group for the Psychology of Mathematics Education.

Kaput, J., & West, M. (1994). Missing value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 61-85). Albany, NY: State University of New York Press.

Lamon, S. (1993). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education, 24*, 41-61.

Li, Y. P. (2000). A comparison of problems that follow selected content presentations in American and Chinese mathematics textbooks. *Journal for Research in Mathematics Education, 31*, 234-241

Lo, J. & Watanabe, T. (1997). Developing ratio and proportion schemes: A story of a fifth grader. *Journal for Research in Mathematics Education*, *28*, 216-236.

Mayers, S., & Tajika, H. (1995). A comparison of how textbooks teach mathematical problem solving in Japan and the United States. *American Educational Research Journal, 32*, 443-460.

National Center for Research in Mathematical Sciences Education at the University of Wisconsin/Madison and Freudenthal Institute at the University of Utrecht. (1997-1998). *Mathematics in context*. Britannica.

National Printing Office. (1999). *Elementary school mathematics* (6th ed.). Taipei, Taiwan.

Sammy K., & Liu, J. (1997). A comparative study of selected United States and Japanese first-grade mathematics textbooks. *Focus on Learning Problems in Mathematics*, *19*, 1-13.

Tokyo Shoseki. (1998). *Shinhen atarashii sansuu (New elementary school mathematics)*. Tokyo: Author.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141-161). Reston, VA: National Council of Teachers of Mathematics.

TRANSFORMING STUDENTS' FRAGMENTED KNOWLEDGE INTO MATHEMATICAL RESOURCES TO SOLVE PROBLEMS THROUGH THE USE OF DYNAMIC SOFTWARE

Manuel Santos-Trigo
Cinvestav-IPN, Mexico
msantos@mail.cinvestav.mx

Abstract: It is well documented that the students' use of different representation becomes an important ingredient in their learning of mathematics. In particular, representations achieved via the use of technology might enhance students' learning experiences. This study documents the work shown by high school students while working with tasks that involve the use of dynamic software. It is shown that the systematic use of the software can help students explore and connect fundamental ideas that may not appear in approaches based only on the use of paper and pencil.

During the process of learning mathematics, students need to examine various types of information through the use of different representations (NCTM, 2000). That is, they often require drawing a figure, visualizing relationships, paying attention to particular part of some representation, or studying features of variation of a phenomenon or situation. In particular, the use of technology (computers and calculators) can help them explore such information from different perspectives or angles. However, what do students need (in terms of mathematics resources) in order to use technology as a mathematical tool? When does the use of technology become a powerful tool for students? To what extent do students' approaches to mathematical tasks, via paper and pencil, differ from technological approaches? Are these two approaches compatible? Are previous students' experiences transferable to technology approach experiences? These are representative questions of the research agenda which involves the use of technology in students' learning of mathematics. In this context, the present study reports an experience with high school students who used dynamic software to work on tasks that involve the use of various representations.

Conceptual Framework

In the last 15 years several curriculum proposals agree that students' learning of mathematics goes further than learning a set of procedures or rules to solve series of routine tasks. It involves that they develop a way of thinking consistent with the practice of doing or developing mathematical ideas. In particular, it becomes necessary for students to develop mathematical experiences in which they value processes of inquiring that involve examining information, statements, or situations from diverse angles or perspectives (Carpenter & Lehrer, 1999). What does it mean that students get engaged in an inquiring process while interacting with a mathematical task? This question is used as reference to document students' problem solving approaches in which the use of dynamic software plays a fundamental role. Specifically, there is interest to document both strategies (representation, visualization, validation, communication, etc.) and basic mathematical resources (definitions, results, notations, etc.) displayed by students during their interaction with tasks. "Dynamic computerized environments constitute virtual labs in which students can play, investigate and learn mathematics (Arcavi & Hadas, 2000, p.25)".

Participants, Tasks, Methods of Inquiry, and Procedures

The study takes place in a problem-solving course with twenty-five high school students. The course included two sessions per week (two hours each session). Participants had studied algebra, Euclidean geometry, trigonometry, and analytic geometry in their previous courses. Each student had access to a computer during the development of the sessions. During the first three weeks of the course, the instructor provided the initial task to be explored during the session. Later, it was common to observe students proposing their own tasks or questions to the class. Three complementary instructional activities appeared during the development of the course: (a) individual students' participation in which each student spent time working on his/her own; (b) small group participation (five students) in which they discussed their approaches to the task and often proposed to explore related connections of initial tasks. Here, the idea was that they could reach a consensus among the group regarding methods of solutions or strengths/limitations of their approaches; (c) whole group discussions in which a particular students' solution, selected by the instructor, was examined by the entire class. At the end of each week, each small group presented a written report to the instructor in which they identified both individual and small contributions to the solution of the tasks. The instructor coordinated the discussion with the whole class and provided orientation or help for those students working individually or in small groups. Data for the analysis came from students' reports (small groups), individual files, and analysis of videotapes of students working with the whole class. Thus, the work shown by one student was important while analyzing both the small group contribution and whole group discussions.

Presentation of Results

To what extent did students' interaction with tasks show strategies and resources that are consistent with mathematical practice? To what extent did the use of dynamic software provide students tools to visualize, organize, and eventually propose and support conjectures? These questions were used as a guide to organize and analyze the work shown by students throughout the development of the course.

Representation of figures via the use of geometric properties

An important result that emerged from the first students' interaction with the tasks was that in order to draw a figure they needed to identify basic properties of that figure with the use of software commands. For example, to draw a rectangle it was necessary to think of properties attached to it (pairs of parallel sides, four right angles, etc.) that eventually led them to draw an accurate representation. They also realized that by changing the dimension of one side, the new figure maintained all the initial properties. That is, with the initial construction they could examine a family of these rectangles. This dynamic property became a fundamental tool for students to examine and explore the behavior of distinct geometric configurations.

An exploration of π

Students were asked to construct a square and measure its perimeter and the segment that joins its center with one vertex. They calculate the ratio of the perimeter to length of the segment. What do you observe when the length of the side of the square changes?

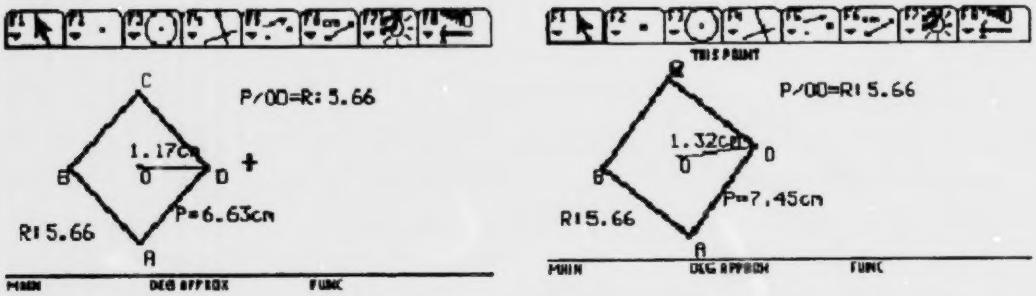


Figure 1.

Students observed that when they changed the length of the side of the square then the perimeter and segment OD changed but the ratio remains constant. Here, they were surprised that for any length of the side, the ratio between the perimeter and the segment that joins the center and one vertex was always the same number.

Students were asked to symbolize both the perimeter and the segment of a square with side s.

Some students explained that the value of $\frac{P}{OD}$ would not change because in the expression the symbol s was nullified.

$$p = 4s$$

$$OD = \frac{\sqrt{s^2 + s^2}}{2} = \frac{\sqrt{2}s}{2}$$

$$\frac{P}{OD} = \frac{4s}{\underline{\sqrt{2}s}} = \frac{8}{\sqrt{2}} = 5.6$$

BEST COPY AVAILABLE

What happens if we examine other regular polygons?

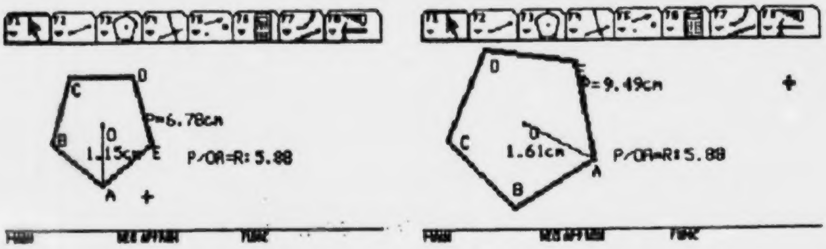


Figure 2.

They noticed that when the number of sides of the regular polygon increased the value of the ratio was closer to 2π. For example, they drew a regular polygon with 17 sides and observed that the ratio was 6.25 which gives an approximation of π = 3.12.

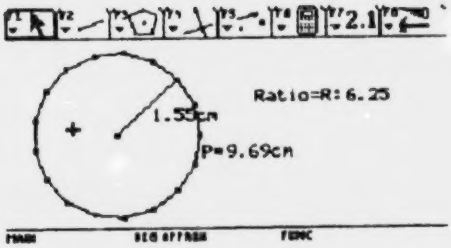


Figure 3.

Other students drew a regular polygon with thirty sides and reported that the ratio was close to what they were told the value of π was.

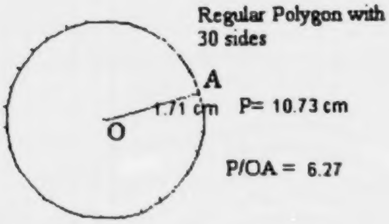


Figure 4.

In this case, students noticed that the value of π was 3.135 and explained that when the number of sides of the regular polygon increases then the ratio P/OA will be closer to 2π. Here, they also mentioned that the length of the circle could be thought as a polygon with many many sides and the segment OA as its radius, in this case the ratio would be 2π.

Formulation of questions during the tasks explorations. In general, students showed that a particular construction could be the source of other type of explorations. For example, while dealing with the representation of a rectangle one students asked the entire class the following question: Can we draw a rectangle if we know only its perimeter and its diagonal? Indeed, students' responses to this question generated

BEST COPY AVAILABLE

four distinct and complementary approaches. Two of these approaches were based on dynamic constructions and two involved the use of algebraic tools. Students here had opportunity to discuss mathematical qualities associated with each approach.

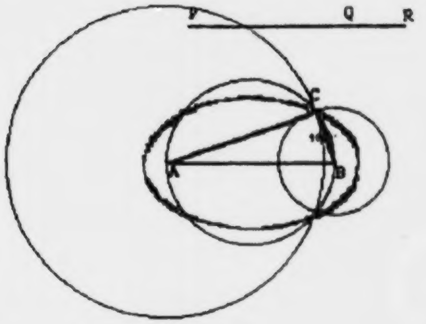


Figure 5.

This is a dynamic representation used by some students to draw a rectangle from the given data (perimeter and diagonal). They chose the diagonal AB as one side of a triangle and the other two sides, the length PR which represented half of the given perimeter. Then they noticed that when moving Q along PR they found a family of triangles with a fixed perimeter, here they focused on finding the right triangle to determine the desire rectangle. During the discussion, the idea of drawing a circle with center the midpoint of AB and radius half of AB helped them solve the problem. That is, the intersection point between the ellipse and drawn circle was the point in which triangle ABC was a right one. In this approach, students connected the construction of a rectangle with other themes (ellipse, circles and right triangles).

Typical approaches to the study of triangles involves the identification of basic properties that include conditions for their construction, their classification according their angles or sides and exploration of properties attached to particular types of triangles (Pythagorean Theorem). The use of the software helped students examine those properties in a dynamic sense and allowed them to make connections with other themes. For example while exploring conditions to construct a particular triangle, they could eventually be dealing with properties of ellipses as it is shown in the next task provided initially by the instructor:

> **Statement provided by the instructor:** Given two Segments. One representing the side of one triangle and the other the sum of the other two sides, how can you construct the triangle? Is this a unique triangle?

How can I draw a triangle? What data do I need? If DE represents the sum of two sides, then where should point C be located to construct the triangle? Can I move C along DE and draw other triangles? Is there any relationship between the sum of the two sides and the other side? How can I represent this problem with the software? These were some of the questions that students addressed and discussed during the initial phase of understanding the statement of the problem. Phases that describe Students' approaches to the task are described next:

- Segment AB represents one side of the triangle and segment DE = DC + CE represents the sum of the other two sides (Figure 6).

This representation was easy to achieve by all students and they also measured each segment.

- With the help of *compass*, students draw two circles: One with center on A and radius DC and other with center on B y radius CE. These circles get intersected at points P and Q (Figure 7). The idea here, they explained, was to construct a triangle with the required conditions:

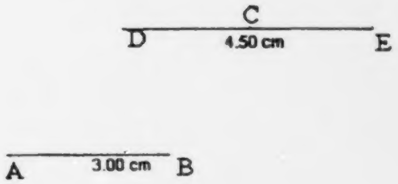


Figure 6.

First Observation

Students realized that triangles ABP and ABQ satisfy the required conditions. They also noticed that when C gets close to either D or E, eventually the circles would not intercept each other. Later, some of the students discussed the meaning of this in terms of the construction of the triangle. Here, the instructor asked: what would it be the locus of points of intersection P, Q if point C is moved along the segment DE? (Figure 8).

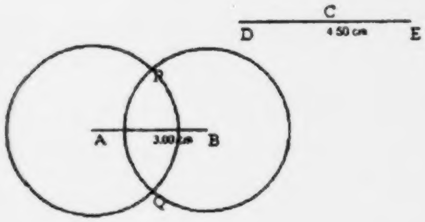


Figure 7.

Second Observation

With the help of the software, students reported that the locus is an ellipse where points A and B are the foci of the ellipse. That is, the locus is the set of points in a plane whose distances from two fixed points on the plane have a constant sum. Thus, they observed that it was possible to construct many triangles that satisfy the original conditions and there are some points on the ellipse where the construction of the triangle was not possible.

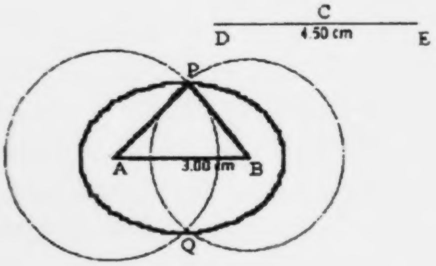


Figure 8.

Third observation

Students noticed that when point C (which is point P on the locus) is moved along segment DE, there is a part of the segment where the triangle will disappear. By measuring the sides of the triangle, students observed that in this case, the sum becomes less than the length of one side of the triangle. So, a necessary condition for con-

BEST COPY AVAILABLE

structing the triangle is that the sum of the lengths of two sides must be greater than the length of the third side (Figure 9).

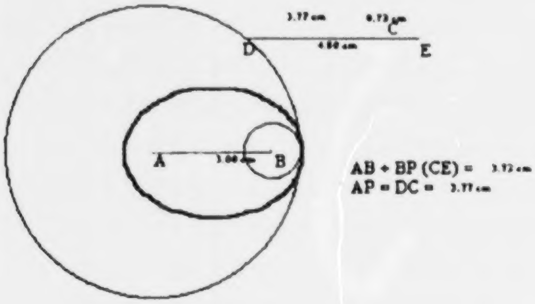


Figure 9.

Remarks

The use of the software offered clear advantages for identifying and exploring properties or relationships. What type of mathematical resources do students need in order to carry out explorations like those shown in the above examples? When students play with constructions, a lot of information might appear initially relevant to them. Students did not recognize initially that the construction of the triangle was related to the construction of an ellipse. Later, they mentioned that such construction provided certain meaning to those algebraic expressions they had studied previously. The teacher's role became crucial in directing students' attention to specific behavior (invariant, for example) that appeared during the interaction with the task. It was important that they discussed and expressed their observations orally and in writing. Here, the systematic use of the software provided students tools to identify and explore mathematical properties that might not easy to observe via paper and pencil approaches. It is also important to recognize that students need to develop diverse ways to communicate and support what they see through the use of these tools.

References

Arcavi, A., & Hadas, N. (2000). Computer mediated learning: An example of an approach. *International Journal of Computers for Mathematical Learning, 5*, 25-45.

Carpenter, P.T., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema & T. A. Romberg (Eds.*), Mathematics classroom that promote understanding* (pp.19-32). Mahwah, NJ: Lawrence Erlbaum.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: The Council.

BEST COPY AVAILABLE

ANALYSIS OF STUDENTS' STRATEGIES WHEN SOLVING SYSTEMS OF DIFFERENTIAL EQUATIONS IN A GRAPHICAL CONTEXT

María Trigueros G.
ITAM
trigue@gauss.rhon.itam.mx

Abstract: This study addresses the analysis of the strategies university that students use when working with systems of differential equations. The framework used for the analysis of the data coming from semi-structured interviews is based on the notions of triad and double triad from APOS theory. We present the detailed analysis of students' strategies while solving problems where they have to transfer information between different contexts. The analysis focuses on students' strategies to interpret each of the given tasks, to solve the problems, and to interpret their solutions. Results show that students use a wide variety of strategies and that their understanding of parametric functions, variation, and solution are central aspects in their success. We suggest that student strategies can be taken into account to help them move towards a deeper understanding of these concepts.

Introduction

The teaching of differential equations at the collegiate level has been the subject of study since the beginning of the Calculus' Reform Projects. Early research results have shown some of the difficulties and misconceptions of students when working with different topics in Differential Equations (Ramussen, 1998; Trigueros, 1993, 1994; Zandieh & McDonald, 1999). More research is needed, however, to inform the possible changes in curriculum and teaching strategies related to this important course, and to understand in depth the development of complex mathematical thinking.

One interesting problem, related to the integration of different concepts in Mathematics together with the use of many advanced mathematical representations and tools, is that of finding, analyzing, and using solution curves of systems of differential equations.

This study addresses the analysis of the strategies that students use when working with systems of differential equations in a graphical context. Previous research studies about student conceptions of the notion of the solution of differential equations, and of tangent fields (Ramussen, 1998; Trigueros, 2000; Zandieh & McDonald, 1999), together with research results on the integration of information when students deal with complex mathematical problems (Baker, Cooley & Trigueros, 1999; Baker, Cooley & Trigueros, 2000) were taken into account to formulate a theoretical model designed to help in the analysis of student strategies while solving systems of differential equations in a graphical context.

Theoretical Framework

This research study uses the notions of triad and double triad from APOS in order to analyze student solution strategies. Research results on student conception of the solution of Differential Equations and of Systems of Differential Equations suggest that the description of their strategies can be made in terms of two main schema: a schema for the representation of functions in parametric form, and a schema for systems of differential equations. A double-triad containing these two schema was thus developed and used in the design and analysis of student' interviews with the aim of understanding student strategies. It is important to remember that the nature of the stages involved in the development of a schema is functional and not structural.

The development of the parametric representation of functions can be described by means of the following stages: At the Intra-parametric level the student interprets the parametric representation of functions as two isolated functions, the possibility of elimination of the parameter and the representation of a function in a two dimensional plane, where the parameter is not explicitly shown causes confusion. At the Inter-parametric level, some relationships between the components of the function are found, but the difficulty of using and interpreting graphs for these functions persists. At the Trans-parametric level, students are able to describe the function and its different representations in terms of the parameter involved. Coherence of the schema is demonstrated by the student's ability to describe which parametric representations are possible for a given function and how they relate to different graphical representations.

The development of the schema for the solutions of systems of differential equations can be described by the following stages: At the Intra-solution level the student is able to solve a system but is unable to interpret it and to coordinate the system and its solution with its graphical representations. At the Inter-solution level, the student begins to relate some systems with their graphic representation, and to interpret the meaning of the solution to some systems, but it is not clear to him or her when a particular representation is convenient or even possible for many systems of equations. At the Trans-solution level, students are able to solve, interpret, and describe graphically different systems of differential equations. Coherence of the schema is demonstrated by the student's ability to discriminate between systems where the use of analytical methods is more appropriate from those for which graphical representations are better suited.

Methodology

Data were collected from two differential equations classes at a small private university. One of the courses was taught to 34 Applied Mathematics students, and the other to 37 Economics students. The two courses were taught by the author. During the semester three individual task-centered semi-structured interviews were conducted with nine students of each group. Interviews covered only topics taught in class. The interviewer was another researcher not involved in this particular project. The interviews were audio-taped and transcribed, and all the work done by the students was collected. The analysis of the data from the interviews was discussed with another mathematics educator who has also taught courses on Differential Equations and results were also compared with those obtained from the analysis of homeworks and tests for validation.

In this study we will present the detailed analysis of student strategies while solving problems on three specific tasks. 1) Given a mathematical model that describes the growth of two populations by means of a predator-prey model, and given a plot that shows a solution curve in phase plane for a specific set of initial conditions, draw the plots that show the growth of the populations and interpret the solution. 2) Given the solution curves for a different mathematical model, draw the phase plane representation and interpret the solutions found in terms of their representation in the phase plane. 3) Given a set of conditions for the solutions to a model, draw the solution curves in phase space and the x *vs.* t and y *vs.* t plots of the behavior of the solutions.

The analysis focused in student strategies. In particular strategies to make sense or interpret each of the given tasks, strategies to solve the problems, and strategies to interpret their solutions. The way students worked to transfer information from one representational context to a different one, and how they interpreted the meaning of solution curves and equilibrium solution in different representational contexts were also analyzed with care.

Results

In this section we will analyze some results obtained in the study. The classification of students was made on the basis of the analysis of their whole work. Here we can only show parts of the interviews that we think illustrate the use of the framework.

Students use different strategies to interpret and solve problems related with systems of differential equations. Some of the student strategies are different from the ones used either by the teacher or by the textbook.

Interpretation strategies

Interpretation of the tasks was not as direct as was expected. Students showed a tendency to focus mainly on the sign of the terms of the differential equations to make sense of the meaning of the system., Some students would take into account the value of the parameters, but they seldom took into account more general aspects of the system as its linearity or homogeneity. The conclusions students drew from their analysis of the system considered mainly the effect of the parameters on each of the equations of the system separately.

Fernanda, for example, analyzes the meaning of parameters in a predator-prey model:

F: …First in this equation, this term is positive, then this other term (showing one part of the equation) is positive, and since this is the equation for the prey, it means the prey is growing…and this one is negative so the predator decreases because I think…because of the predator, I mean, the prey, and then… in the other here, this is negative and this is positive so the predator is decreasing but also increasing…

I: What makes the predator population increase?

F: This, this is positive…it is because of the prey…

I: What happens with the prey? How do you know the predator's population increases because of the prey?

F: Well, here in this term, there is an x and a y meaning this is the prey and this is the predator, they interact…and…mmm…(pause)

I: Why does this term has the product of the two populations?

F: I only know this means interaction of the two, but I don't know why you have to multiply them.

I: OK, and what does these numbers tell to you, these constants.

F: These are parameters of the equation and mean… how each population grows or decreases because of the presence of the other species…

I: Can you tell something about the solution of this system just by analyzing its terms?

F: No…mmm…no I would have to see if it has equilibria and the signs for the regions and all that…

I: So you wold have to consider the solutions of the system in phase space to tell something about the behavior of the system? Which representation you would use?

F: Yes I suppose I need to plot that …although… I don't know really which is better. I think what I can do is the equilibria and regions in phase plane, and when I have that, I can try to go to the other plots, although I don't do it right every time, you know. I mean, in class, with others it is easier but…but here by myself I don't know if I can do it correctly…

Fernanda is an example of a student in the Intra-parametric- Inter-solution level. In this part of the interview she focuses on each of the equations separately, she does not see the system as such but only interprets each term according to its meaning for each equation, she does not see the structure of the system. However she is able to consider the solution of the system and knows how some systems can be represented graphically in different spaces. Her interpretation of the meaning of the system, however, is not clear.

Problem solving strategies

The strategies more frequently used by students in this study to solve the problems can be summarized as follows:

> Given a solution curve in phase space or solution curves in the *x vs. t* and *y vs. t* planes, and asked to plot the solution in a different space, students: a) focus on particular points on the curve and try to plot them on the other planes, b) use scales to consider intervals of points where the given graph has a specific behavior, follow "the movement" of the curve, that is, the changes of the curve in different regions of the plane, and to transfer this information to the other graphical representation, c) go back to the differential equations represented by the given curves to reconstruct the information needed to graph the solution curve; in this case, most of the students who use this strategy reconstruct the direction field in phase plane and forget about the particular solution they were supposed to work with, d) try to solve the equations and plot the solution curves once they know the solution e) consider intervals where the graph has a specific behavior and translate it to another representation.
>
> Given the properties of the solution of a system, the more frequently used strategies were: a) go back to the differential equations and solve them to be able to plot the solution curves, b) take into account some of the given conditions but not all, c) consider only the information about equilibrium points and nullclines, d) consider all the information given and plot condition by condition, e) analyze the given conditions and divide the phase plane in regions where they will plot the tangent field that is consistent with the information.

We illustrate the use of one of these strategies with the work of Lorena who was classified at the Inter-parametric, Trans-solution level.

L: …Given this information I have to draw the solution curve in phase space and in the x versus t and y versus t graphs…so here is the initial condition, but I will look at it later, what else?… Equilibria solutions are these four points and I draw them here…

I: Can you tell me before you go on what variables are on the axis?

L: Oh! Yes… x here and y here,(writes them on the phase plane)…afterwards I will draw the other graphs…so this are equilibrium points and these are nullclines in x, $x = 0$ and $y = x+2$, this line here, and the y axis, and y-nullclines are the x axis here and this is a circle, radius nine fourths, no, no, three halves. Then…this information means the regions to see directions, of tangent vectors…mmm (drawing directions)… and then it looks like this…

I: What about solution curves?

L: Right, I was forgetting, but here I only have to do one because I have this initial condition here.

I: All right. Is that all you have to draw on the phase plane?

L: I think so…Now the other graphs…I mean the ones with t, I follow this curve in phase plane, from here to here,…no…I don't know exactly because I don't have t here…but if this is the initial condition it should be from here to here…isn't it?…Here it starts at point five and goes up this way and then this other in y…starts at point two and also goes up.

I: The y component of the solution is going up all the time?

L: Yes, here more slowly it seems… but yes

I: What does this information mean in terms of the solution?

L: Both x and y are always growing.

I: Is this plot consistent with the information you have in plane phase? What about this equilibrium solution?

L: I drew this taking the information from the phase plane but…see, this is an equilibrium solution but the arrows point this way so it doesn't go to it, it must be a saddle or something like that…

I: What about here?

L: Here…in this part? This is small and here it might curve a little but then goes up.

Here the student focuses on the direction of the vector field she has drawn, but something is wrong and she does not pay attention to it. Her curve is not consistent with the direction of the nullcline and with the direction of one small region between the circle and the line nullcines. The behavior of the solution curve in the *x vs. t* and *y vs. t* planes is not consistent with her drawing in the phase plane. She shows difficulties in using and interpreting the parametric functions, she also showed difficulties in deciding about the direction of the solution curve in phase plane.

I: If you had the differential equation for this system, do you think you would be able to solve it analytically?

L: No the equation has to be linear, well… only if the two equations are not

> *related one with the other, but in general if that does not happen then you know this system is non linear because the nullcline is a circle.*

The student, however, shows a good understanding of the meaning of the solution curve, can interpret its meaning correctly in terms of her interpretation of directions in the phase plane, and is able to distinguish the structure of the system.

Analysis of solution strategies

When analyzing the solution of the systems, students showed a tendency to focus on specific regions of the planes, to consider only some pieces of information, or to focus on some details of the solution. Only a few students analyzed the results in terms of the long-term behavior of the solution curves, taking into account the direction of the solution curve in phase plane and its relationship with the equilibrium solutions. Equilibrium solutions were very frequently not taken into account.

While analyzing a solution curve in the *x vs. t* and *y vs. t* planes, Hector explained:

I: What can you tell me about the solution of this system?

H: y is growing here. x grows and decreases.

I: Is that all you can say? Can you be more specific and tell me about the meaning of that in terms of the solution to the system?

H: I know...well…t is the same for both, I mean they go like together, these are x and y the components of the solution, and one is increasing and the other has a maximum here at around 4…and the meaning…I don't know, it is difficult to see how these populations are together…I can see this population is growing and this decreases here but that's all I can say…

I: And you said x goes to zero and y is growing, will it always grow? You can use the information from the phase plane too…

H: Well, I don't know, here it is increasing more slowly and well…it seems it grows slowly but continues to grow.

Hector, who was classified at the Inter-parametric, Inter-solution level, recognizes the relationship between the components of the system but cannot interpret the meaning of the solution using that relationship. He also disregards the equilibrium solution and in his phase plane curves, he shows he has some confusion about the behavior of the curves near the nullclines and near the straight line solutions.

In general, student strategies show that, in many cases, they have not constructed strong links between calculus concepts such as variation, derivative and tangent line, and their representation in a graphical context, particularly when the geometric representation used is the phase space.

Concluding Remarks

Students show a tendency to describe behavior in phase space as static; they seem to loose track of the dynamics of the system and thus have difficulties to relate graphs of the same solution curve in different geometrical representations. Many students have a very weak understanding of parametric representation of curves and this fact can be on the basis of student' difficulties with the notion of variation.

A considerable number of students show a tendency to go back to the differential equations that describe the system in order to reconstruct the geometric representations. Then they fail, to make sense of the properties of the system when they can't work with the analytical form of the differntial equation.

The theoretical model used in this study has shown to be a helpful tool in the understanding of students' interpretations and difficulties when they work with systems of differential equations in a graphical context. This framework can also be used as a guide in the design of teaching strategies which can foster students' understanding of solution curves and equilibrium solution. Taking into account students' strategies and difficulties can guide the design of class and homework activities, which can help students reconsider some important concepts from calculus and algebra. This study shows that it is important to be sure that they have a stronger understanding of those concepts before one starts with the study of systems of differential equations. The treatment of the topic of systems of differential equations can be designed with the idea of helping students establish relationships among the different concepts used. Knowledge of the strategies they use can be used in class activities and discussion to help them move forward towards a better interpretation and understanding of situations where systems of differential equations are involved.

References

Baker, B., Cooley, L., & Trigueros, M. (2000). The schema triad: A calculus example. *Journal for Research in Mathematics Education, 31*, (5), 557-578.

Baker, B., Cooley, L., & Trigueros, M. (1999). On schema interaction: A calculus example. In F. Hitt & M. Santos (Eds.), *Proceedings of the Twenty-First Annual Meeting of the Psychology of Mathematics Education, North American Chapter*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Rasmussen, C. (1998). *New directions in differential equations: A case study of students' understandings and difficulties*. Paper presented at the annual meeting of the American Educational Research Association.

Trigueros, M. (2000). Students' conceptions of solution curve and equilibrium in systems of differential equations. *Proceedings of the Twenty-Second Annual Meeting of the Psychology of Mathematics Education, North American Chapter*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Trigueros, M. (1994). *Student's interpretation of variation problems in a graphical setting*. Sixteenth Psychology of Mathematics Education North American Chapter Annual Meeting. Baton Rouge, LA.

Trigueros, M. (1993). Representación gráfica y ecuaciones diferenciales. *Memorias del V Simposiun Internacional de Investigación Educativa* (pp. 239–344). Yucatán, México.

Zandieh, M. (1997). *The evolution of student understanding of the concept of derivative*. Unpublished doctoral thesis, Oregon State University, Corvallis, OR.

Zandieh, M., & Mc Donald, M. A. (1999). Student understanding of equilibrium solution in differential equations. In F. Hitt & M. Santos (Eds.), *Proceedings of the Twendy-First Annual Meeting of the Psychology of Mathematics Education, North American Chapter*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

BUILDING MATHEMATICAL MEANING WITHOUT TEACHER INTERVENTION: ANALYSIS OF THE DISCOURSE OF PRECALCULUS STUDENTS

Janet G. Walter
Rutgers University
jwalter@switchpoint.net

Abstract: This paper examines how precalculus high school students construct meaning and mathematical understanding in a student-centered culture when presented with a task about describing a spiral shell. A two-week summer mathematics institute, designed as part of a longitudinal study of the development of children's mathematical thinking and proof-making, made it possible to expand ongoing research to include students who had participated in the longitudinal study for more than a decade with students from other high schools. Analyses of data demonstrate that within relatively short periods of association and without teacher intervention, students construct meaning, sharpen understanding, and modify thinking as they mathematize a contextualized task.

Introduction

The National Council of Teachers of Mathematics suggests in *Principles and Standards for School Mathematics* (2000) that a major responsibility of mathematics education is to produce autonomous learners. Of course, learners are already autonomous agents. Thus, school mathematics programs can provide educational conditions that foster and support student-initiated inquiry and learning. Such conditions evoke the perception of teacher as facilitator, in which the supportive role for the teacher is to give serious attention to the growth of student ideas within the academic community of the classroom. In such a community, students are the featured participants in the learning process under conditions that provide durable opportunities for students to pursue student initiated questions, discuss ideas, develop and refine representations and draw conclusions.

The research reported here is a strand in a larger study of seventeen high-school students who attended an NSF supported two-week summer mathematics Institute and met in a simulated classroom setting for four hours each day to solve open-ended, contextually rich mathematics problems during July, 1999. This paper, based on student discourse, without teacher intervention, follows the development of student understanding by a focus group of four students with differing ethnic and educational backgrounds and who had no prior experience working together. The purpose of this paper is to describe how students build mathematical understanding without teacher intervention and how student understanding changes.

Theoretical Framework

A perspective on learning for understanding suggests that meaning is formed in and modified through the context of social- and self-interaction. Such a view implies that there is an interrelationship between the internalized social process, which focuses attention on which objects have meaning, and the requisite discourse in the formative construction and application of meaning. Meaning and understanding depend upon prior experiences and are attributed to objects through symbols. Symbol appropriation and use provide for participation in a discourse (Doerfler, 2000).

Speiser and Walter (2000) suggest that individually constructed understanding occurs within a social milieu of interactions which are aspects for consideration in attempting to support the growth of student understanding. Maher (1998) asserts that this supportive milieu includes the allocation of sufficient time for students to reflect on their own understanding, think deeply about mathematics and consider the reasonableness of their ideas. Meier (1995) also emphasized the importance of allocating sufficient time in the classroom to foster inquiry, community, respect, and responsibility among students.

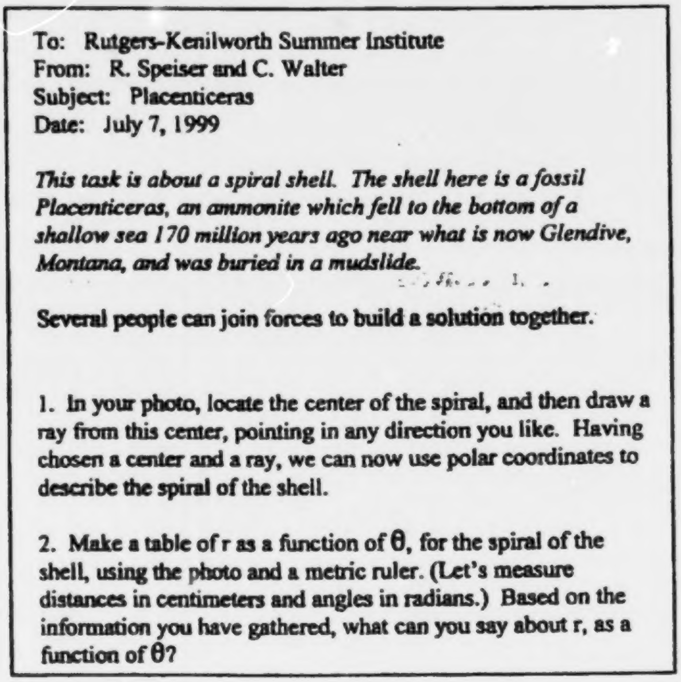
Participants

Students were assigned seating in groups of five or six with open invitation for members of different groups to interact. Four students compose the focus group for this paper. Victor, from the Dominican Republic, was the only student in the study who had completed a calculus course prior to the Institute. Curiously, his high school had placed him in a calculus class through scheduling error the previous year. He had been a participant in a related cross-sectional research study in grades four through seven. His seating assignment at the summer institute was at Table 3.

The other students in the focus group were assigned to Table 2 and all had completed precalculus during the prior school year. Benny, an African American, attended the same high school as Victor. Milin, of East Indian ancestory, and Matt had both participated in the longitudinal study up to grade six but neither attended the focus high school for that study. The classroom setting of the institute was the first opportunity for these particular students to work together as a group.

Task

Introduction of the task took about ten minutes, but no mathematics instruction was given to the students. A fossilized shell of an ammonite, Placenticeras, was provided for all of the students to handle and inspect during the middle of the second day of the institute. The spiral fossil had been cleanly sliced in half and the internal structure of the shell was beautifully evident on the face of each half. Students were given written copies of the task (Figure 1) designed in 1991 by Robert Speiser, which asked students to gather information, use polar coordinates to describe the spiral of the shell, and say what they could about r as a function of θ.



To: Rutgers-Kenilworth Summer Institute
From: R. Speiser and C. Walter
Subject: Placenticeras
Date: July 7, 1999

This task is about a spiral shell. The shell here is a fossil Placenticeras, an ammonite which fell to the bottom of a shallow sea 170 million years ago near what is now Glendive, Montana, and was buried in a mudslide.

Several people can join forces to build a solution together.

1. In your photo, locate the center of the spiral, and then draw a ray from this center, pointing in any direction you like. Having chosen a center and a ray, we can now use polar coordinates to describe the spiral of the shell.

2. Make a table of r as a function of θ, for the spiral of the shell, using the photo and a metric ruler. (Let's measure distances in centimeters and angles in radians.) Based on the information you have gathered, what can you say about r, as a function of θ?

Figure 1

Students were each provided with a photocopy and overhead transparency of the shell (Figure 2) enlarged to twice its actual size, graphing calculator, pens, rulers, blank transparencies, grid paper and plain paper.

Data Collection and Analysis

Data were collected from videotapes of student interactions, individual task centered interviews and personal interviews, student journal entries, student written task-based work and observer/participant field notes. Videotapes were compressed and digitized using Dazzle hardware and software, then stored on CDs as MPEG1 files using Create CD software. Descriptive narrative of all video and verbatim transcripts of critical events were produced and time coded to the video using vPrism, a software designed for analysis of video in qualitative research.

Qualitative analysis of student discourse and interactions yielded four broad coding categories:

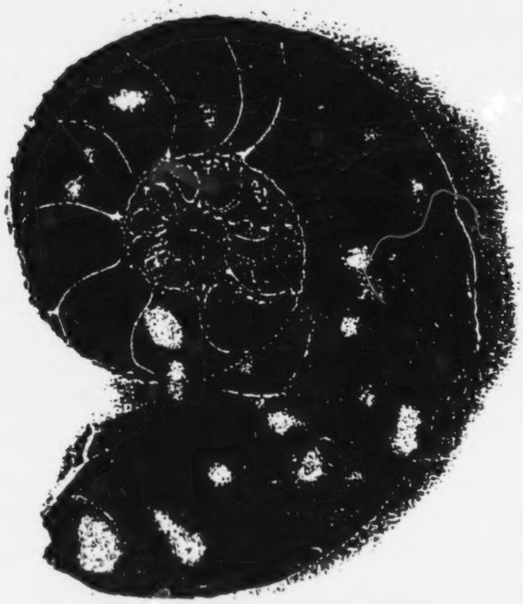


Figure 2.

Representation (R), Demonstration (D), Idea (I), and Question (Q). Emic perspective refinement of categories emerged during analysis. Representations include (1) inaccessible internal cognitive or mental representations; (2) numerical, algebraic, graphical, or textual inscriptions; (3) kinesthetic manipulation, gesture and body language; (4) verbal language; (5) enlarged photocopies of the shell; and (6) graphic calculator displays. Demonstrations consist of student explanations, justifications or proofs. Ideas include mathematical concepts that are stated or written such as notions of linear or angular measurements and unit conversion, calculator modes, patterns, inequality relationships, and ordered pairs. Questions may be (1) language patterns that check for attunement of one's own understanding or for attunement between participants (QA); (2) interrogatives for information (QI); (3) procedural (QP); or (4) confirmation requests regarding conceptual understanding (QC).

The focus episode occurs approximately thirty minutes after students began working on the task. The critical event analyses provided here are based on approximately six minutes of discursive interactions between Victor, Benny, Milin, and Matt and include time codes, coding categories, narrative descriptions and transcription.

Findings

After the task was introduced, Victor was quiet while other members of his group were discussing how to approach the task. He walked over to the table where Benny, Milin and Matt were seated and addressed his question to Benny.

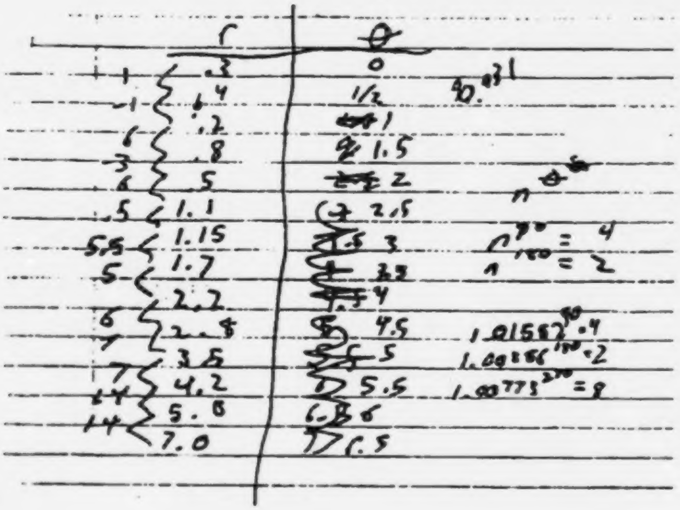
Time	*Code*	*Student*	*Transcript*
42:13	QI	Victor	What's goin' on?
42:18	D, R, I	Benny	I don't know. First we started off, this right here, we measured millimeters away from the center to each point. That's what this is (points to the photocopy of the shelL).
42:28	R	Victor	I see.
42:29	D, R, I	Benny	But, see, the reason why we got it like that (points to the table of measurements his group collected, Figure 3) 'cause we want it in centimeters.
42:34	QC, I	Victor	Okay. You know how one hundred eighty over pi is equal to one radian, right?
42:39	R	Benny	Yeah.
42:40	I, QP	Victor	Okay, then. My thing is, right, how do you go from radians to centimeters?
42:49	QC, D	Benny	That's why we didn't start out like that. We used-degrees, right? We're not even using radians, right?
42:53	I, D	Milin	No. We are [using radians]. Yeah, we are. Yeah, we decided that the calculator, right, doesn't work with-

			degrees that well.
43:02	I	Victor	Um, actually, if you change the mode you can use degrees. (Victor's facial expression changes as he listens to Milin's response.)
43:05	D, R	Milin	Yeah, I know. But it doesn't work that well in graph-theory 'cause you have to put out things. So, you have to erase it.
43:15	QA	Benny	You want me to just show you basically what we did, right?
43:19	R	Victor	Yeah, go ahead.

Victor and Benny both smile as they make eye contact. Victor is kneeling on the floor to Benny's right. Victor folds his arms and nods as Benny talks. Neither Milin, Matt nor others at Table 2 interrupt while Benny is explaining. Victor displays his agreement and understanding and gives encouragement to Benny with verbal "uh huh"s between every few words that Benny utters.

43:20	D, I, R, QI	Benny	We measured from the center to each point all the way around. Now this would be like, say from here, his closest point, right there would be, what's that? Three millimeters, thirty millimeters? (Benny points to a point on a graph and then to the table of numbers. He looks at Milin.)
43:40	QA	Milin	Um, alright, what, what? (Milin is busy working with the calculator.)
43:44	R	Benny	Right here (points to the table of data).
43:45	R	Milin	(Milin looks at the table of data prior to answering.) Yeah, thirty, yeah.
43:46	QC	Benny	Thirty millimeters, right?
43:47	D, I, R, QA	Benny	And that would be at zero. So then at pi over two, one-half pi, or ninety degrees, whatever you want to call it. You know what I'm saying? So it'd be forty.
43:57	R	Victor	Forty, right.
43:57	R, QA	Benny	Forty millimeters. So, ninety, right? Are you listening to me?

Benny points to student-created inscriptions of plotted points, axes, quandrantal angles, and radii (Figure 4) as he explains that the radius of forty millimeters corresponds to an angle of ninety degrees. Victor is not looking at this inscription and



r	θ
.3	0
.4	1/2
.2	1
.8	1.5
.5	2
1.1	2.5
1.15	3
1.7	3.5
2.2	4
2.8	4.5
3.5	5
4.2	5.5
5.8	6
7.0	6.5

1
-1
6
3
6
.5
5.5
5
6
7
7
14
14
$r^{90} = 4$
$r^{180} = 2$
$1.01552^{90} = 4$
$1.00386^{180} = 2$
$1.00773^{270} = 8$

Figure 3.

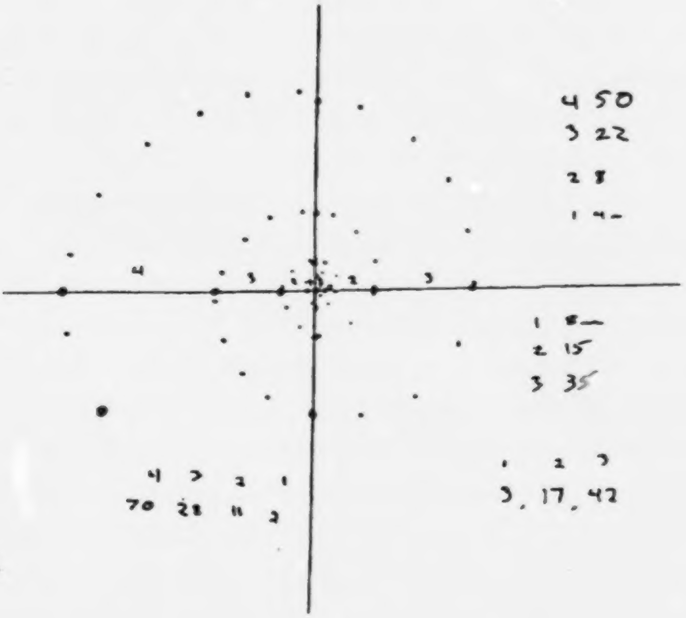


Figure 4.

Benny questions whether Victor is listening.

44:03	R, D, I	Victor	I'm listening. I'm also looking at the data.
44:05	R, D	Benny	So from here to here would be ninety, I mean it would be forty millimeters from here. (Benny pauses in his explanation and picks up the paper with rays and axes drawn on it.)
44:14	R	Victor	I'm paying attention, Benny.

BEST COPY AVAILABLE

44:16	R, D, Q	Benny	Nah, I'm trying to point it out. So, from here to here would be ninety millimeters. Then from, what's this? Pi, pi equals one, you know, one whole.

Benny again points to the paper with the axes. He then points to the table of data and his explanation trails off as he begins to repeat the values in the table. He is quiet for about twenty-eight seconds.

44:44	R, QI, I	Victor	I'm just looking at the data. Why you go from point three and then right here its point two?
44:50	D	Benny	Because it get, you know, it gets smaller as you go around.
44:54	QI	Victor	Your radius gets smaller?
44:55	R, D	Benny	Point two is one. (Benny refers to a radius of length point two corresponds to an angle of one radian.)
44:58	QI	Victor	But then why, okay, but why does it go bigger again, get larger?
45:01	R, D	Benny	'Cause right here, right here might be bigger in this area.

Benny points to the drawing of the axes and moves his pen to indicate the length of one of the radii. Benny is silent as he reaches out and picks up the transparency of the shell and points to the approximate location on the shell that corresponded to the radius he had just indicated. Benny then points to the table of data that contains the measurements of the radii and quadrantal angles. Victor stands up. Benny says that the transparency is not the right one and after laying it aside, finds a different paper of student-drawn axes and plotted points from data collected from the spiral.

45:30	R, QA D	Benny	This is not the right one. That's the right one. See here, look, here we go. You see that it gets smaller, right, its smaller right here. You see? Look, through here, how it gets bigger around the circle.

Benny points with the tip of his pen to the plotted points on the paper and gestures with a counter-clockwise motion of the pen. He then points to the table of data. Victor moves closer to the table, leans over and points at the same table of data.

46:00	R, QI, I	Victor	But my question is this. They move from point twoto an angle of one. Then you go from point four, that's less, that's one half. But then you have a point two and the r is one. What're you? (Victor raises his pen and rotates it clockwise as he questions.) And then

			right here, you've got point eight is one and a half. How does point two, the last (inaudible) point four, and you get.
46:23	QI	Benny	You think, you're saying they should be the same?
46:26	R, I	Victor	No, I think, I think that basically, this should be, the way they're doing looks right. (Victor points to the table of data.)
46:31	R	Benny	Um hum.
46:32	QA	Victor	I'm thinkin' that, you know what I'm saying?
46:33	QI	Benny	As you go down, it increases?
46:35	R, QC, I	Victor	As you go up on your radian (points to the radius length column of data), this would increase (points to the angle data column), right? That's what it seems like, right? But why would this be point two here, and then, assume, I would think that the points would be equal to zero point three to one half and point four to one. (Victor gestures with a sliding motion of his hand from the top of the chart toward the bottom.)
46:51	R, QI, D	Benny	I see what you're saying. (Benny explains to Matt.) Matt, this is what, this is what he's saying, he's saying, he said that point two and point four, right, is at, is at one-half, right here this point four, how is it at one point two?
47:06	QC, R	Victor	Oh, no. But wait. That seven, was at that seven, right?
47:08	R	Benny	Yeah.
47:10	D, R	Victor	Okay, you see how everything else here, right, as it increases the theta increases except right here? At point two, you've got that equal to one and point four equal to one-half and point three is equal to zero.
47:32	R, D, QA	Matt	I think its because, like this where, from where we started is why. You see out, see out here? (Matt points to the transparency of the shell.) We went out here and here.
47:31	R	Victor	Um hum.
47:31	R, D	Matt	So its wider. It starts here, gets wider here than it is here. So as you go around it gets smaller there, but from there on it increases. 'Cause that's the very side of the shell.
47:41	I, R,	Victor	Oh! So then, you're basically telling me is that its not,

	QC		its not in your measurements, it's exactly what you're measuring is distorted?
47:51	R	Matt/Benny	Yeah, yeah. (All three students are smiling and nodding at one another.)
47:52	R, I	Victor	Oh! I see that. See? That's what I wasn't seeing until you pointed that out.
48:00	R	Benny	Well, okay then.
48:01	R	Victor	Yu-eah! (Victor slaps Benny's shoulder as he leaves to return to Table 3.)
48:10	I, R	Benny	(Benny is smiling and laughing quietly as Victor departs.) I didn't see how that was why, but now that I look at it. But, like if you draw it though, it seem it'd be a circle, a regular circle right there.
48:24	R, D	Matt	It actually gets, it's actually thinner right there than it is here.

Conclusions

In this episode, when Victor addressed his first question to Benny, students at Table 2 had already drawn axes, plotted points, taken radii measurements for corresponding quadrantal angles, and organized a table of ordered pairs. Victor's second question, the sole procedural question --how to convert from radians to centimeters--had not been addressed in the work so far accomplished by the students at Table 2. Victor asked how to convert an angle measure to a linear measure because, as he demonstrated during the next few days, his solution to the task involved trying to find the length of the spiral. Benny's response indicated that he may have focused his attention on a different meaning for Victor's question—how to convert from radians to degrees. Victor's actual question went unanswered, but he was willing to listen to Benny explain what the group had done so far.

The task specifically instructed students to measure angles in radians. The decision to use radians instead of degrees was supported by Milin's assertion that the calculator works better in radians than it does in degrees. Neither Benny nor Victor used the calculator during this episode. It is interesting to note, however, that Benny's explanation included seamless conversion between radians and degrees, possibly as an interpersonal attempt to build community and to support Victor's suggestion that degrees could have been used just as easily as radians. Students excluded pi from the written numerical radian measurements for theta and simply listed 1/2, 1, 1.5, etc.

Victor's attention was caught by what seemed to be an incorrect measurement when Benny showed him the table of ordered pairs. Victor expected the pattern to be monotonic increasing; but the radii increased, then decreased and increased again for continuously increasing theta. It was not until the organically grounded representation of the photocopy of the shell was kinesthetically, verbally and cognitively linked to the student measurement inscriptions that Victor understood the measurements were correct. Additionally, Benny's understanding changed as a result of his involvement in the fairly lengthy discourse and led to the consideration of a circle as a possible model.

Students constantly compared multiple representations as they considered the reasonableness of their inscriptions and conclusions. Certain objects had meaning for the students, and the inscriptions they generated were symbols of their understanding. Student-created inscriptions and photocopy representations enhanced certain meaningful features of the shell and lessened the impact that other features might have on problem solving strategies. For example, quandrantal angles were apparently chosen for convenience and familiarity; but the terminal rays did not coincide with the growth chambers of the shell which were readily apparent in the enlarged photocopy. The actual size of the shell seemed no longer to be of consideration once the enlarged photocopy was used as the representation for the shell; measurements were not eventually scaled down to the actual size of the shell.

Individually constructed understanding evolved through interactive discourse in a social milieu unhindered by artificial time constraints. Within six minutes, twenty four questions were posed by students to one another: one (4%) procedural , six (25%) confirmation, seven (29%) attunement, and ten (42%) interrogatory. Data collection and representation work was created and critiqued by students to sharpen their understanding of a particular mathematical model they were constructing for the spiral. Students chose to listen to one another, gestured to clarify communication, and assertively changed body posture to underscore and pursue questions that occurred to them. All four students participated respectfully and responsibly as they persevered together to construct meaning, modify thinking and sharpen understanding They concluded this episode of learning in good spirits with displays of triumph.

References

Doerfler, W. (2000). Means for meaning. In P. Cobb., E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms* (pp. 99-131).

Maher, C. (1998). *Can teachers help children make convincing arguments? A glimpse into the process.* Rio de Janeiro: MEM; USU: NJ: Rutgers University.

Meier, D. (1995). *The power of their ideas: Lessons for America from a small school in Harlem.* MA: Beacon Press.

NCTM (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Speiser, R., & Walter, C. (2000). *Five women build a number system.* Stamford, CT: Ablex.

A DEVELOPMENTAL AND SOCIAL PERSPECTIVE ON PROBLEM SOLVING STRATEGIES

Judith S. Zawojewski & Guadalupe Carmona
Purdue University

Abstract: Analysis of small group problem solving with "model-eliciting activities" (described by Lesh, Hoover, Hole, Kelly and Post, 2000) suggests rethinking the conventional role of problem solving strategies (e.g., *draw a picture, identify the givens and goals)* as heuristics suggestions that teacher make for students when they are "stuck." In this paper we take a developmental and social perspective of these strategies, using examples from a transcript of students' working on such an activity to illustrate three points. The first point is that conventional problem solving strategies can be used descriptively to trace the development of students' mathematical models. The second point is that a social perspective on problem-solving strategies can be used for instructional purposes. The third point is that based on Vygotsky's work (1978), we can view students' natural use of problem solving strategies, as well as their monitoring and assessment processes, as developing from external, socially-based mechanisms, to processes that are internalized over multiple experiences in group problem solving with complex problems.

Background

The theoretical perspective described in this paper is based on small groups engaged in model-eliciting activities. Students' problem solving strategies are traced through transcripts of selected modeling sessions. This background section of the paper provides information about model-eliciting activities, our perspective on problem solving strategies, and the selection of transcripts that are analyzed for this purpose.

Model-Eliciting Activities

The type of group problem solving activities that are referred to in this paper are called model-eliciting activities (Lesh et. al., 2000). These activities present a problem, based on a real-life situation, to be solved by students in small groups. The solution calls for a mathematical model to be developed for an identified client who has a specified need. The problem is complex, calling for multiple cycles of interpretation and articulation of the solution in order for the model to be developed to the point that a client would find it understandable and useful. As a result, group members working on the activity need to engage in multiple cycles of interpretation and reinterpretation in order to describe, explain, manipulate, or predict the behavior of their model to each other. As in most real-life problems, there are multiple solutions, ranging from ones that are inadequate to those that optimally meet the needs of a client.

The model-eliciting activity used to illustrate the main points of this paper is The Paper Airplane Activity. This activity was adapted from a "case study" that was originally used in Purdue University's graduate program for aeronautical engineering. The engineering problem used a "wind tunnel" and was designed to encourage graduate students to develop more powerful ways of thinking (and quantifying) the concept of "drag" in various shapes of planes and wings. The problem was adapted for middle school students by creating a newspaper article and an activity based on a paper airplane contest. The newspaper article was about how to make a variety of different types of paper airplanes. Then, the activity provided data from three trial paper airplane flights for each of six teams. The data presented were: amount of time in air (seconds), length of throw (meters), and distance from target (meters) (Figure 1). Students were also given information about various paths that airplanes might take (Figure 2). This diagram was useful for differentiating between "length of throw" (which was the straight-line distance between the start point and where the plane landed) and "total path" (as illustrated in Figure 2). Winners of the contest were to be declared in four different categories: (1) best floater (i.e., going slowly for a long time), (2) most accurate, (3) best boomerang, and (4) best overall. Students were required to write a letter to students in another class describing how such data could be used to assess paper airplanes' performance in each of the four categories. The main challenge in the activity is to develop a mathematical construct for each category.

In the class from which our illustration transcript was taken, the middle school students worked in teams of three. Calculators, tape measures, and other potentially relevant tools were available. After reading the newspaper article, the students made their own versions of the featured paper airplanes, and tested the flight characteristics of their planes. For each paper airplane created by the students, three measurements – identical to those in the activity - were recorded, giving students an opportunity to thoroughly understand the meaning of the data table that would be used in the activity on the following day.

Problem Solving Strategies

The explicit teaching of problem-solving strategies as a goal for instruction was called into question in Schoenfeld's (1992) review of research on problem solving. Schoenfeld indicated that early studies in mathematical problem solving literature sought to show the usefulness of teaching general problem solving strategies, but he concluded that prescriptive use of such heuristics was not particularly helpful for improving problem solving performance or transfer. Yet, conventional school mathematics programs continue to include explicit instruction on heuristics such as "draw a picture", "make a table", "try a simpler version of the problem", "use a similar problem", "act it out", and "identify the givens and goals". These problem-solving strategies, originally based on Polya's (1945) work, have typically been used to help students connect the situation at hand to previously-learned mathematical procedures,

	Path 1			Path 2			Path 3		
TEAM	Amount of time in air (seconds)	Length of throw (meters)	Distance from target (meters)	Amount of time in air (seconds)	Length of throw (meters)	Distance from target (meters)	Amount of time in air (seconds)	Length of throw (meters)	Distance from target (meters)
Team 1	3.1	11	1.3	2.3	7.7	3.2	0.7	1.8	6.8
	0.1	1.5	8.7	0.9	2.9	8.6	1.2	3.7	6.7
	2.7	7.6	4.5	0.1	1.1	2.4	2.7	8.4	4.4
Team 2	3.8	10.9	1.7	3.2	9.2	4.6	2.3	8.1	6.1
	5.2	13.1	5.4	2.3	9.4	2.9	0.2	1.6	6.9
	1.7	3.4	8.1	1.1	2.7	8.8	2.1	6.9	5.2
Team 3	4.2	12.6	4.5	1.7	4	5.9	2.9	8.5	5.5
	5.1	14.9	6.7	3	10.8	3.1	2.4	7.7	8.7
	3.7	11.3	3.9	2	6	3.2	0.2	1.9	6.7
Team 4	2.3	7.3	3.25	1.3	4.9	4.4	1.4	4.9	4.9
	2.7	9.1	4.9	2.3	7.3	2.4	2.7	7.2	8.1
	0.2	1.6	9.1	1.4	3.7	4.1	0.2	1.1	7.3
Team 5	4.9	7.9	2.8	2.7	10.7	1.1	2.5	7.7	5.7
	2.5	10.8	1.7	3.3	6.3	2.5	2.1	9.8	9.8
	5.1	12.8	5.7	1.3	2.6	7.2	3.2	10.4	5.8
Team 6	0.2	1.8	8.8	1.8	3.9	4.2	0.1	1.2	8.2
	2.4	10.1	4.6	0.2	5.7	4.2	1.3	4.9	4.9
	4.7	10.3	5.4	1.6	8.5	3.4	1.8	3.5	2.7

Figure 1. Data Table
Note. "Length of throw" indicates the straight-line distance between the start point and where the plane landed. The "target" is the finish point where the plane should land.

BEST COPY AVAILABLE

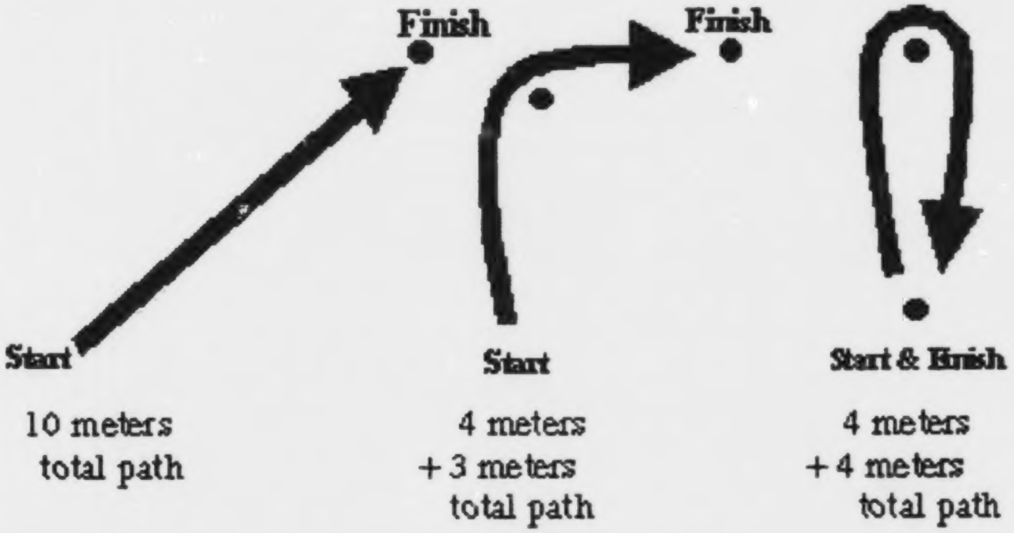


Figure 2. Measuring "Distance Covered"

which in turn presumably lead to a solution. Once students have learned these strategies, teachers make these heuristics suggestions when it appears that students are "stuck" during the problem solving process. The eventual goal is for students to internalize these suggestions in order to respond to their own question: "What can I do when I'm stuck?" Our work with model-eliciting activities concurs with Schoenfeld's finding that these heuristics are not particularly useful. When examining students' performance on complex, realistic problems that require mathematical modeling, very few instances occur when the problem solvers are actually "stuck" – i.e., have no relevant ideas to bring to bear. Most of the time students have a variety of ideas and procedures that they identify as relevant, but the ideas and procedures they identify as relevant are often in need of modification or adaptation to be useful. Thus, students have not temporarily lost a "tool" or a relevant idea. Instead the ideas and procedures they have identified as relevant, as well as their current interpretation of the situation, need to be extended, refined, or modified in order to meet some specified goal. In other words, students need strategies that help them revise and adapt the ideas and procedures they bring to bear on the problem.

On the other hand, problem-solving strategies have the potential to serve a useful function. Schoenfeld (1992) indicated that problem solving strategies serve as powerful descriptors of problem solving behavior. Students engaged in modeling-eliciting activities do indeed "draw pictures," "identify the givens and goals," and so forth. We find that observation of students' use of these strategies provides an opportunity to interpret students' thinking, and also an opportunity for us to rethink how problem-

BEST COPY AVAILABLE

solving strategies can be used for instructional purposes. These observations have led us to think about students' use of problem solving strategies (and assessment/monitoring processes) as developmental. For the purpose of this paper, we have traced three problem-solving strategies throughout the transcript of a group engaged in The Airplane Problem. The definitions we have used are:

> **Use a representation.** We have clustered a class of problem solving strategies into this category. When problem solvers represent a situation, they do so in a variety of ways. They may draw a picture, make a table, use a metaphor, or act the situation out.
>
> **Use a similar problem.** This strategy includes situations in which students identify and use a similar problem, or a simpler problem, to help them deal with the problem at hand. It also includes situations where problem solvers simplify the given problem.
>
> **Identify the givens and goals.** This strategy includes students' explicit attention to "What is given" and "What is needed." It also includes when students are actively engaged in clarifying the given information and the goals of the task, as students engage in multiple cycles of interpretation and model development.

Selection of Problem Solving Sessions

Through purposeful sampling, we have transcribed the conversation of small groups solving model-eliciting activities. We select problem-solving sessions, which are characterized by group interaction (in contrast to a collection of individuals working in parallel), and where students have exhibited sustained interest and work on the task. The selected sessions include examples in which students produce a variety of responses ranging from inadequate to optimal. Since the activities have been designed to require collaborative effort, the resulting transcripts provide opportunities to consider social aspects of problem-solving strategies used. Also, since the activities have been designed to require multiple cycles of interpretation and revisions, the development of students' mathematical models is usually evident.

The theoretical perspective described in this paper is illustrated by using transcript excerpts from one session of the Paper Airplane Problem. The transcript excerpts are from a group of three seventh grade African American boys in a remedial math class in an east coast inner city school. These students had already done five or six other model-eliciting activities in their class, although they hadn't necessarily worked together as a group of three. The group was videotaped, and then the conversation transcribed, and some observations incorporated into the transcript. Throughout this solving session, students went through cycles of construct development as they discussed, posed and refined their own definitions of floater, boomerang, and accuracy, and as they decided how to determine how to determine the "best" in each category. Specifically, students revised and refined mathematical ideas like units (time, length), magnitudes, average, maximum, minimum, rounding numbers and approximation, speed, etc. At different times, these students also used strategies to sort and organize, integrate, operate, disaggregate, compare, and simplify information. Each of the team members played different roles at different times within the session. At times, individuals were monitoring and assessing themselves and others, and at other times they were using problem–solving strategies that included: identifying the givens and goals, using a similar/simpler problem, and using a representation. The final solution that the students presented included complex (and clear) mathematical ways of defining best floater, best boomerang, most accurate, and best overall.

Social and Developmental Perspectives

We take a social and developmental perspective on problem solving strategies to establish three important points about students engaged in solving complex mathematical problems. First, problem-solving strategies are useful as descriptors of students' behaviors in tracing the development of groups' mathematical models. Second, a social perspective on problem-solving strategies can be used to guide instructional purposes. The third point, based on Vygotsky's work (1978), is that we can envision students' natural use of problem-solving strategies, as well as their monitoring and assessment behaviors, as developing from external, socially-based mechanisms, to internal mechanisms as students experience multiple opportunities for group problem-solving using complex problems.

Development of Mathematical Models

Tracing students' use of problem solving strategies provides an opportunity to outline the development of the team's mathematical model. Working in a group, students often create initial representations to describe or justify their solution to each other. As interaction among the team members takes place over the course of the session, the representations used are often modified, reflecting the revisions in the students' solution (i.e., mathematical model). For example, at first, the team members in the Paper Airplane Problem interpreted time as distance in the table of information given in the problem. Upon recognizing their confusion, they used color highlighting to prevent further difficulties – reflecting an increased understanding of the task at hand, and simultaneously allowing their model to develop from treating all the data columns as similar information to acknowledging qualitative differences between them. Later on, students circled subsets of data to group data from the same team, when they encountered the need to integrate this information. Thus, observations of students as they *used representations* revealed that their model was developing to a point where they could systematically differentiate among different types of data during their solution process.

Tracing the problem solving strategy, *use a similar problem*, also has the potential to reveal how a group's mathematical model evolves. As individuals in a group probe, question and challenge each other's points of view, their team interpretation and proposed solution evolves to one that increasingly takes into account more details and relationships. Students in this session used familiar (i.e., similar) problems to deal with the definition of floater, which as defined in the activity statement as "going slowly for a long time." The group's stated interpretation for this construct was "floaters go slow . . . for a long time," and they treated the speed and time aspects separately. Powerful mathematical constructs from baseball were brought to bear, as students talked about how the speed of a baseball related to the problem. They interpreted speed in this session as "time divided by distance," which is the inverse of speed conventionally defined, but consistent with what is commonly done in describing the speed of baseballs. The *use of a similar problem* also surfaced as students searched for ways to deal with the second part of their definition of "floater" - ". . . for a long time." The students referred to a modeling-eliciting activity they had previously encountered which called for "scaling up." They said, "If you stay up five times longer… then you oughta get five times more credit. … So, you oughta scale up, like in that "big foot" problem we did." Thus, students dealt with the complexity of the construct of "floater" by developing a procedure where they divided time by distance, and then multiplied by time (which means: time^2/distance). This is a very complex definition for floater, especially for seventh grade students. In this case, students refined their initial definition of "floater" as "floaters go slow… for a long time," by tapping and integrating their experiences with baseball and a previous model-eliciting activity, and out of this developed a mathematical formula that would take into account their verbal definition, and integrate the information they had available (amount of time in air and length of throw).

The problem solving strategy, *identify the givens and goals*, also revealed this group's developing mathematical model. For each of the categories (floater, boomerang, and accuracy), these students revised and refined their definitions (givens), and once they agreed upon the definition, they proceeded by using this definition in finding the 'best' for each one (goals). In addition, throughout the session, these students frequently questioned themselves if what they were doing (goals) was helping them solve the problem, thus, they constantly went back to read the original problem statement (givens).

A Social Perspective

Taking a social perspective on the use of problem solving strategies in classroom instruction suggests that we need to rethink the teacher's conventional role in which problem-solving strategy suggestions are made to students during the solution session. We suggest that the teacher's role is to create a natural need to use problem-solving strategies in small group, complex, problem-solving situations. Students naturally use problem-solving strategies when they need to communicate with each other, offering different perspectives, posing test solutions, and articulating their ideas aloud to see if others understand their point of view as one sees his or her own point of view.

Using a social perspective, the strategy, *use representations*, may be recast as: *use representations for the purpose of communication*. The Paper Airplane problem-solving session had numerous examples where it was clear that students were representing their ideas for the purpose of communicating. For example, when students decided to highlight different units with a text marking pen, it allowed them to communicate the distinction to the others in the team.

> Al: Yeah, you're right. Which ones are times? Mark 'em. .. {pause} Here, we can use this, this, … Where is it? Here. {referring to a yellow text marking pen}. … {long pause while he marks the time columns in the table} ... Look. These here are times. {referring to the first column of times in the table}. These aren't .. {garbled} …
>
> Bob: How about these? {pointing to the column of time measurements for Path #2). Aren't they times too? … Yep. They're times. So, mark 'em. … So are these. {pointing to the column of time measurements for Path #2).

Individuals in a small group problem solving initially hold "similar, yet different" interpretations of a complex problem, even though each typically assumes that peers "see the same thing." It is only through comparing and contrasting these "similar yet different" perspectives of the same problem that students come to recognize that each sees the problem in a different way. Further, it is through considering these "similar problems" that students are able to refine and modify their initial interpretations. Thus, using a social perspective on problem solving strategies, *use a similar problem*, might be recast as: *consider the given problem from different points of view.* Students considered "similar problems" (i.e., different perspectives of the given problem) in the illustrative transcript when they were developing a definition of "boomerang," and they commented on the different experiences that each of them had with one.

> Al: What's a boomerang?
>
> Bob: You know, it's what those guys in Australia do … for hitting kangaroos and stuff. I saw 'em on TV last week. You know.
>
> Carl: Yeah, I've see one. They're cool. You throw 'em and they come back. … I saw a real one once. My uncle Tony had one. It's hard to throw. … I never did get it to come back. … He could do it a little.

Similarly, *identify the givens and goals,* takes on a social flavor when considering small group model-eliciting activity. As students explain their points of view to each other, they naturally clarify the givens and goals in the problem – while simultaneously revising their proposed solution. From a social perspective, this problem solving strategy might be recast as *explain your problem interpretation to someone else*, since this naturally leads students to articulate their interpretation of the givens and goals and also facilitates the development of their mathematical model. This was evident in the transcript when they were developing a definition of a "floater". In this excerpt, Al gives a definition of floater by moving his hand; Carl relates his own experience with paper airplanes and adds something to Al's definition; finally, Bob adds his own experience, which modifies Al's and Carl's definition.

Bob: So, what's a floater?

Al: I think it's. It's like this. {Al is using his hand to suggest a plane that is moving very slowly through the air}. … Goin' real slow.

Carl: Mine was a pretty good floater. … But it didn't go very far. It crashed pretty fast.

Bob: Yeah, mine didn't go slow. It went fast. But, it crash right away too.

The teacher's most important role, thus, becomes one in which the teacher selects the tasks and the combination of students to put into a group. While students are working, interventions are best kept to a minimum, since the teacher-student communication pattern is so easily changed to "trying to figure out what the teacher wants us to do." When interventions appear necessary, they should be those that help focus students' attention to understanding the problem at hand by communicating among the peers and reflecting back on previously considered problem interpretations. The instructional perspective taken is that the teacher's main responsibility is to set up the situation in which students will use representations for the purpose of communication, will need to consider each other's perspective of the problem, and will naturally engage in explanation of the problem interpretation to peers (Zawojewski, Lesh & English, in press).

A Social and Developmental Perspective

The social dimension is described by Vygotsky (1978) as critical to the development of individual learning. We believe his view of learning, as developing along external-to-internal lines, can be extended to students' development in problem solving strategy use and monitoring/assessment processes. When students are initially introduced to small group model-eliciting activities, their use of problem solving strategies and monitoring/assessment processes serve a social purpose – to communicate with group members. Over multiple experiences in small-group complex problem-solving situations, the external social functions have the potential to become internalized, in which students learn to communicate with oneself as if one were a group of students solving a problem, or to communicate with oneself at a later time by referring back to previously recorded information. For example, students initially *use representations* to assist their explanation about their perspective to peers. With experience, the external social function may become internalized, such as when individuals experienced with complex problems can be seen to use representations to "communicate" about different perspectives to oneself, or to look back at their previous points of view by revisiting their prior records. In the real world, individuals experienced in working with complexity often communicate with themselves using representations, while also seeking to communicate with others in their community.

Using the social perspective of problem solving strategies described above, when small groups *use a similar problem*, they are often considering the given problem from their peer's different points of view. These "similar, yet different" problems emerge because each group member inevitably holds a different perspective. Again, over multiple experiences, individuals have the potential to internalize active "perspective-taking" that was originally external in the social setting, developing from an external to an internal process. Similarly, as students explain how they each *identify* (i.e., interpret) *the givens and goals*, they learn the social process of explaining their perspective to someone else. Eventually, students may begin to value the process of articulating, revising and rearticulating their own interpretation, and in so doing, frequently revisit the given information and goals statement in the problem.

The social context of small group model-eliciting activities not only facilitates the use of problem solving strategies, but also provides natural opportunities for students to monitor and assess each other's points of view, processes, procedures, and outcomes. In the illustrative session of the Paper Airplane Activity, this was quite evident. For example, in the transcript excerpt below, Al monitors Bob's way of thinking when explaining how to calculate an average.

Bob: The average is, like, in baseball. ... It's the one in the middle. ... It's like you got three stacks and you gotta make 'em all the same. You move 'em up or down a little – if they're too high or low. ... Here, I'll show you. You gotta even 'em up. ... Take these. {1.8, 8.7, and 4.5}. ... Now. Move ... uh ... move 7 from the big one {i.e., 8.7} to the little one {i.e., 1.8}. ... No, just move two.

Al: What are you doing? What are you doing? I don't get it.

Bob: OK, watch. ... I'm trying to make these {indicating the three numbers 2.0, 8.5, 4.7} the same size.

Al: Why?

Bob: Cause that's the middle. ... That's the average. ... Look. Take 4 from these 8 {referring to the 8.5}. So, that makes ... let's see ... four-point-five here ... and six here. ... No, that was too much. ... Just take three from here {8.5} and put 'em here {2.0). ... That makes ... uh .. six-point-five here ... and four here.

Then, Carl gets a calculator, and confirms Bob's result.

Carl: OK. Got it. It's. It's. It's. It's five-point-zero, zero, zero.

Bob: That's what I got. Five. That's what I got.

This session was characterized by three boys who engaged in substantive group interaction and also maintained sustained interest and work on the given activity. It was natural for them to monitor each other, and they did so throughout the activity session. Considering that they have had experiences with previous small group model-eliciting activities, perhaps they have developed these exemplary monitoring and assessment processes over time. Further, if we extend Vygotsky's social perspective of learning to monitoring/assessing processes, it is reasonable to think that with multiple experiences, these students are internalizing these important metacognitive processes.

Summary and Implications

Small group problem solving with model-eliciting activities requires students to work collaboratively, and to go through multiple cycles of interpretation and reinterpretation of their proposed solution. The development of their mathematical model that results of these cycles can be revealed as students externalize their thinking in order to communicate with each other as they work collaboratively. The use of problem solving strategies is natural in these contexts, when communication is needed to work on a task that requires group effort to deal with the complexity. Thus, one of the most important roles for teachers is to provide the environment where small groups of students have opportunities to interact together in a sustained manner as they work through a complex situation. Giving students multiple opportunities to engage in model-eliciting activities may facilitate their development in the use of problem solving strategies and monitoring/assessment processes along an external-to-internal dimension.

References

Lesh, R., Hoover, M., Hole. B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education.* Mahwah, NJ: Lawrence Erlbaum.

Polya, G. (1945, 2nd ed., 1957). *How to solve it.* Princeton, NJ: Princeton University Press.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grows (Ed.), *Handbook of research on teaching and learning mathematics.* New York: Macmillan.

Vygotsky, L. (1987). *Mind in society: The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

Zawojewski, J., & Lesh, R. (in press). A models and modeling perspective on problem solving strategies. In H. Doerr & R. Lesh (Eds.), *Beyond constructivism: A models & modeling perspective on mathematics problem solving, learning & teaching.* Mahwah, NJ: Lawrence Erlbaum.

Zawojewski, J., Lesh, R., & English, L. (in press). A models and modeling perspective on the role of small group learning activities. In H. Doerr & R. Lesh (Eds.), *Beyond constructivism: A models & modeling perspective on mathematics problem solving, learning & teaching.* Mahwah, NJ: Lawrence Erlbaum.

COLLEGE STUDENTS' PREFERENCES FOR PROSE AND TABULAR REPRESENTATIONS WHEN CONSTRUCTING FUNCTIONAL EQUATIONS

Susan S. Gray
University of New England
sgray@.une.edu

In describing mathematical representations, Bruner (1966) distinguished among three types. Enactive representations are models, objects, or written forms upon which actions can be performed. Iconic representations are internal mental structures that result from experiences with enactive representations. Symbolic representations are the language or symbols used to describe mathematical relationships. Bruner believed that experiences with enactive representations are required to form the iconic representations that enable an individual to express mathematical relationships symbolically.

Mathematics educators have stressed that translations among prose, tables, graphs, and equations should be a focus of algebra instruction (Kaput, 1989). Translating functional relationships from prose to equations occurs within Bruner's (1966) symbolic representational realm. Students may not be able to make these translations successfully if they have not formed adequate mental representations. Tables of values, with pairs of corresponding values serving as examples of the functional relationship, can potentially serve as one of Bruner's enactive representations. As such, tables of values may facilitate the construction of equations to represent functional relationships described in prose.

Past studies reported low success rates when high school and college students constructed equations for functional relationships represented in either in prose or with tables of values. Common errors were identified as equations in which the variables were reversed (Clement, 1982; Clement, Lochhead, & Monk, 1982; MacGregor, 1991; MacGregor & Stacey, 1993). However, there is little research examining college students' equation constructions when prose and tabular representations of functional relationships are presented simultaneously.

In the current study, thirteen college students enrolled in a developmental algebra course were interviewed as they constructed equations for additive, unit-rate, and non-unit-rate functional relationships represented with both prose descriptions and tables of values. The goals of the study were to document the number of times subjects referred to prose, tabular, or both representations, the frequencies of correct responses associated with these referrals, the types of errors subjects made in their constructions, and the ways in which subjects used the prose and tabular representations.

Most subjects in this study used only tables of values or both prose and tables in their attempts to construct functional equations. Subjects who used both representations were more successful in constructing equations than were those who used only tables. The two subjects who used only prose were the least successful. Nonunit-rate functional relationships presented the most difficulty for all subjects. In addition to reversed equations, a variety of other errors were observed. Most subjects used pairs of values from the tables to check their equations, a process that allowed some to identify and correct their errors.

The results of this study suggest that tables of values may assist students in constructing equations for simple functional relationships described in prose and may prompt students to check their equations with pairs of values from the tables. The combination of prose descriptions and tables of values appears to facilitate the construction of additive and unit-rate functional equations for this small sample of college students who have experienced difficulty with algebra. Additional research with greater numbers of students of varying mathematical backgrounds is needed to test and extend these conclusions. The difficulties with nonunit-rate translations observed in this study may indicate that these relationships present a different challenge than do additive and unit-rate relationships for this population of students.

References

Bruner, J. S. (1966). *Toward a theory of instruction*. Cambridge, MA: Harvard University Press.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 13* (1), 16 -30.

Clement, J., Lochhead, J., & Monk, G. G. (1981). Translation difficulties in learning mathematics. *American Mathematical Monthly, 88,* 286-290.

Kaput, J. J. (1989). Linking representations in the symbol system of algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the teaching and learning of algebra* (pp. 167-194). Hillsdale, NJ: Lawrence Erlbaum.

MacGregor, M. E. (1991). *Making sense of algebra: Cognitive processes influencing comprehension*. Geelong, Victoria, Australia: Deakin University Press.

MacGregor, M. E., & Stacey, K. (1993). Cognitive models underlying students' formulation of simple linear equations. *Journal for Research in Mathematics Education, 14* (3), 169-181.

BUILDING MEANING FOR MATHEMATICAL IDEAS

Tania A. Myrtil
Rutgers University
queenzu @yahoo.com

Alice Alston
Rutgers University
alston@rci.rutgers.edu

This report focuses on mathematical thinking of teachers and their students. Among the questions that were of special interest to the researchers were: (1) How would the teachers approach a generally stated problem, related to a "real-life" situation, in order to make sense of the situation? (2) What difficulties would the teachers encounter and what strategies and solutions would they develop to deal with the problem? (3) What forms of the problem would the teachers present to their students? (4) What strategies and solutions would students use and how would these compare to those of the teachers?

This study is a part of ongoing research informed by the theoretical perspective that learners select from alternative mental representations in building solutions to problems (Davis, 1984; Davis & Maher, 1997). Dõrfler, (2000), contends that meaning, for the learner, develops as he or she constructs representations that become prototypes with the capacity to guide future mathematical activity.

Some 30 elementary and middle school teachers were presented the following task: "You have a certain number of animals. Some of them are rabbits and some are chickens. The total number of animals is A and the total number of feet is F. How many rabbits and how many chickens do you have? How would you solve this problem? Include particular strategies that you used to develop a convincing solution. After you have completed the task yourself, use any variation of it that is appropriate with your students."

Analysis of the written work and reflections of the teachers and their students documents a variety of approaches used by both teachers and students. A significant number of the teachers, especially those teaching middle and secondary grades, first approached the problem symbollically with no reference to its context. These teachers were unable to develop complete logical solutions that were convincing to themselves and others. A number of teachers set up systems of linear equations. However, they either became lost in meaningless symbolic calculations or concluded, as one teacher that: "X+Y=A (number of animals) and 4X+2Y=F (number of feet). You can go on to solve for X=A-Y and Y=A-X and substitute that .....then F=6A- 4Y-2X. But I don't think that will give us a solution because there are too many variables. So, I'm stuck." Other teachers successfully approached the problem by creating tables, holding one or more of the variables constant and systematically examining possibilities for numbers of rabbits, chickens, feet and animals. In their reflections about the problem, both for themselves and for their students, teachers consistently said that specific numbers for recognizable entities were necessary in order to understand the situation. A number of the teachers modified the problem, selecting a particular pair of numbers, one for the number of animals and the second for the number of feet. After developing their own solution, they presented the modified problem to their students. Strategies used by the teachers and the students were similar; frequently including tables or charts where various combinations of numbers were recorded until a correct solution was found. A second frequently used approach involved pictures.

This study evidences the importance of building representations of problem situations that make sense. The results document instances among adults and children where this building occurs. Additionally, the actions of this typical group of teachers indicate the importance of such experiences in teacher preparation. Encouraging learners to build general mathematical ideas for themselves on the basis of real experience is essential if we are to make progress toward the goal of meaningful mathematics instruction.

References

Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education.* Norwood, NJ: Ablex.

Davis, R. B., & Maher, C. A. (1997). How students think: the role of representations, in L. English (Ed.), Mathematical *reasoning: Analogies, metaphors, and images.* Mahwah, NJ: Lawrence Erlbaum.

Dõrfler, W. (2000). Means for meaning. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design.* Mahwah, NJ: Lawrence Erlbaum.

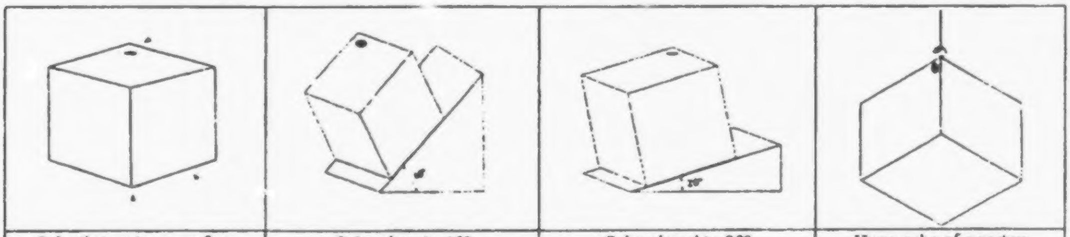
FOSTERING STUDENTS' MATHEMATICAL THINKING THROUGH THE USE OF QUALITATIVE TASKS

Armando Sepúlveda
Universidad Michoacana de San Nicolás de Hidalgo
asepulve@mail.cinvestav.mx

What types of tasks do students need to approach in order to develop strategies and mathematical processes consistent with the practice of doing mathematics? This is a crucial question that needs to be analyzed in terms of providing examples that encourage students to engage in those mathematical practices. NCTM (2000) suggested that students should work in tasks carefully selected by teachers that allow them to acquire knowledge in a variety of contexts. It is aimed at fostering students' mathematical thinking through tasks that allow the acquisition of abilities, so the students can particularize, generalize, and discover patterns and relationships to make conjectures and to justify results (Mason et al., 1987).

Polyhedrons of Chocolate

We take this task provided in the Balanced Assessment for the Mathematics Curriculum *High School Assessment Package 2* (1999). This is a learning situation of space imagination, in which the students are asked to draw three dimensional polyhedrons that result when varying the positions of a cube that are exactly half filled with chocolate, which, when cooled, take diverse forms that the student has to identify and to analyse. From the resulting polyhedrons of chocolate, students were also asked to explore the formula of Euler.


Cube that rest on one face | Cube place to 45° | Cube placed to 20° | Hung cube of a vertex

Learning situations were applied to five teams of students as part of the pilot study. Not all of them received previous instruction to evaluate the mathematical potential of the task. The high school students had serious difficulties in explaining what they had drawn in the third and fourth position of the cube; none was able to describe the polyhedrons formed in the last position. Those who answered coincided in drawing the polyhedron as if the cube was a hung octahedron of a vertex; none was able to obtain the formula of Euler. The students of Architecture still had difficulties. They had almost no trouble drawing the polyhedrons in the first three positions. Two of them described the polyhedron formed in the fourth position almost perfectly, one found the formula of Euler and another didn't find the formula.

References

Balanced Assessment for the Mathematics Curriculum. (1999). *High school assessment package 2*. White Plains, NY: Dale Seymour.

Mason J., Burton L., & Stacey K. (1987). *Thinking mathematically*. New York: Addison Wesley.

National Council of Teachers of Mathematics. (2000). *Principles and standards in school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

VERBAL PROBLEMS CLASSIFICATION OF SENIOR HIGH SCHOOL'S TEXTBOOKS WITH TECHNOLOGICAL INDUSTRIAL EDUCATION ORIENTATION IN MEXICO

Jorge Villa
DGETI, Mexico,
s36villa@prodigy.com

José Guzman
Cinvestav, IPN, Mexico,
jguzman@mail.cinvestav.mx

Bednarz and Janvier (1994, 1996) classify arithmetic and algebraic verbal problems according to relationships between two or more quantities: *comparison*, *transformations* and *rate*. Therefore, it is possible to identify schemata with similar features. They classify arithmetic problems as "connected"; i.e., problems where it is not necessary to state an equation to obtain direct results of unknown values. A contribution of this theory is the identification of different types of problems: *unequal sharing, transformation and rate.*

Methodology

We classify arithmetic and algebraic problems found in textbooks in accordance to the complexity of schemata rather than difficulty-level. The multiple combinations make hard to classify them. Analysing algebraic processes is more complex than analysing arithmetic ones. However, since the purpose of this study was on teaching, the classification was divided in two parts and seven "classes". The four belonging to the first part include problems without rate-type relations; the three remaining include problems with rate-type relationships.

Exploration Test

Five problems -one of each class-, considered as a high difficulty level were proposed: the first is an unequal sharing interconnected problem; the next four are "rate type". Example*: a car has a 15 km/hr speed more than a bus and covers 220 km in one an a half hour less than the necessary for the bus to cover the same distance. Find each transport speed.*

Test Implementation

The sample included 51 (16-17 years old) students. They spent 1:50 hours working on the test.

Analysis of Test Results

The students' success includes: 80, 96, 20, 8 and zero, respectively. This indicated us the great difference in its complexity level, mainly in the "rate-type". Concerning the students who had a good performance with the equations, the percentage of success for the five problems was 73. The great variety of possible combinations in the schemata would be hard to classify, that's why it was determined only under seven 'labels'.

Conclusions

Results seem to show that "rate-problems" difficulty does not depend on the scheme structure, but it is related to the concepts involved in them. The students showed a good performance in algorithmic procedures. We identified a "spiral-type" classification.

References

Bednarz, N., & Janvier, B. (1994). *The emergency and development of algebra in a problem solving context: A problem analysis. Proceedings of the International Conference for the Psychology of Mathematics Education*. Lisbon, Portugal.

Bednarz, N., & Janvier, B. (1996). *Emergence and development of algebra as a problem solving tool: Continuity with arithmetic.* Université du Québec à Montréal: CIRADE.

MULTIPLE EMBODIMENTS IN MODEL-ELICITING ACTIVITIES

Edith Gummer and Hollis Duncan
Oregon State University

In this poster session we present the results of the implementation of model-eliciting or problem solving activities with middle school students in grades 6 through 8. The model-eliciting activities require students to work in small groups on problems based in the real world (Lesh, Hole, Hoover, Kelly & Post, 2000). The activities require students to interpret the problem statement, devise a plan of action, create a model to analyze the data, and report findings. Problems stem from real world situations and require students to construct their own data interpretation plan. Students must justify their reasoning for inclusion of all data. In addition they will explain the solution(s) in detail. Multiple reasoning strategies have been documented and characterized. These multiple reasoning strategies are cognitive instantiations of Dienes's principles of multiple embodiments (Dienes, 1960).

By multiple embodiments, Dienes meant the concrete materials that a young student might manipulate to provide the structural scaffold that enables her to build understandings of more abstract ideas in mathematics. A cognitive equivalent of a concrete manipulable is the mathematical model that the student constructs as she goes about solving a particular problem that is communicated in writing or verbally to some audience. The communication or explanation of the model makes the student's understanding more concrete. The Model-eliciting activities provide the context within which the student can develop increasingly powerful explanations that provide a solution to the problem as it is interpreted. These explanations are increasingly powerful as they provide the student with more confidence about her solution. The student can use the track record of her models for problem solution to reflect about and justify her solution path.

The context for model-eliciting activities is students working together in groups of three. These results come from a constructivist/interpretivist study conducted with students who were engaged in a summer enrichment program, the Saturday Academy at Oregon State University. Students met with an instructor for 4 sessions that lasted for 3 hours per session. Students were audiotaped as they engaged in the solution of the model-eliciting activities and transcripts of the tapes were analyzed for cycles of student problem-solving models. A total of 4 problems were given to the students over the course of the study. During the first problem-solving session, students were introduced to the nature of the problems and were informed that the instructor would be periodically asking them to briefly summarize their thinking and to reflect on how their thinking had changed from one period of time to another. This study focuses on the responses to such questions for the final 2 problem solving sessions to minimize the novelty factor of students having to verbally explain their thinking.

Examples of models that students construct demonstrate the changing nature of the characteristics upon which the student focuses as the final model emerges. Students describe an initial interpretation of the problem statement and then search for aspects of the problem data that will allow them to begin a solution strategy. Students move from consideration of surface characteristics of the problem solution strategy to a re-interpretation and refinement of the problem statement. This is frequently precipitated by one element of the data that the student cannot fit into the model that is the solution strategy.

References

Dienes, Z. P. (1960). *Building up mathematics*. New York: Hutchinson.

Lesh, R., Hoover, M., Hole, Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education.* Mahwah, NJ: Erlbaum.

PROBLEM SOLVING IN MATHEMATICS: BARRIERS TO PROBLEM-CENTERED LEARNING IN AN EIGHTH GRADE CLASSROOM

Loan H. Nguyen & Amy Roth-McDuffie
Washington State University

Purpose

This research project addressed current concerns over students' lack of problem solving skills. We conducted an action research project (Hubbard & Power, 1993) to investigate the challenge of helping students develop problem-solving skills. I specifically planned to implement a conceptual approach to facilitate use of reasoning and problem solving behaviors. The project was motivated by the incongruities I observed between classroom practices and my reading of educational research on problem solving. My observations in middle school mathematics classes raised concerns because, though teachers aggressively integrated problem solving into the curriculum for several months, student work demonstrated little improvement. A majority of class time was devoted to teaching specific problem solving strategies, yet students demonstrated no ability to transfer the strategies meaningfully to problem solving applications.

Theoretical Perspective

As reported by the National Commission on Mathematics and Science Teaching for the 21st Century (2000), students' disturbing lack of mathematical ability can be attributed to the unchanging teaching method in American mathematics classrooms where teachers illustrate problems and algorithms, then assign drills of low-level procedures. When the curriculum centered on problem solving, however, students were better able to construct meaning and application for their procedural knowledge. Studies of problem solving in elementary and middle school classrooms found that problem-centered curricula, which revolves around challenging mathematical tasks to develop concepts or apply learned concepts to new situations, increased students' ability to construct meaning and solve practical problems (Goldberg, 1999; Ridlon, 2000).

Methods

During the first author's student teaching internship as part of a Master in Teaching degree program, we planned to implement a problem-based curriculum in a middle-grades mathematics classroom and hoped to integrate teaching metacognition and problem solving strategies through an immersion system of modeling, self-directed "thinking" speech, and collaborative problem solving to support teaching those processes explicitly. We also planned to opt for problems and explorations that centered on the required content and hoped to teach problem solving as part of developing the mathematics content. We evaluated students' use of modeled strategies and self-directed speech to assess the success of the interventions, evidenced by records kept in a researcher journal, student work, and videotaped lessons. We also videotaped pre-intervention interviews in order to track any progress in students' problem solving ability.

Results

Because of a rigid classroom structure established by the supervising teacher, unanticipated barriers to implementing problem solving instruction existed. The supervising teacher required that his traditional class format consisting of several math worksheets each day (a heavy emphasis on drilling skills) be adhered to during the intervention period. Both the highly structured daily schedule and the type of work assigned were integral parts of his classroom management strategy and could not be changed by the first author as an intern. We hoped to integrate different teaching strategies into that structure, but we found that these conditions set by the supervising teacher completely conflicted with a conceptual approach to learning and problem solving. The structure limited the amount of time necessary for students to reflect and reason through problem solving situations. The specific conditions of this traditional classroom not only were contrary to the development of problem solving skills, but the teaching of problem solving as an add-on to otherwise traditional methods hinders student learning of the mathematics content as well.

Relationship to Goals of PME-NA

These findings have implications both for placement of student teachers in internships and for teachers who attempt to retro-fit problem solving instruction without significantly changing their overall approaches to teaching and learning. Thereby, this study serves to further our understanding of efforts to teach and learning problem solving.

References

Goldberg, P. (1999). *Increasing problem solving through the metacognitive skills of planning, monitoring, and evaluating*. Chicago: Spencer Foundation.

Hubbard, R. S., & Power, B. M. (1993). *Handbook for teacher research.* Portsmouth, NH: Heinemann.

National Commission on Mathematics and Science Teaching for the 21st Century. (2000). *Before it's too late.* Washington, DC: U.S. Department of Education.

Ridlon, C. L. (2000). Christi makes sense of sixth-grade mathematics. *Mathematics Teaching in the Middle School, 5* (6), 367-373.

MIDDLE SCHOOL STUDENTS' SENSE MAKING OF SOLUTIONS TO MATHEMATICAL WORD PROBLEMS

Clara Nosegbe
Georgia State University
cnosegbe@gsu.edu

There is ample research evidence in the literature that students often provide numerical answers to word problems solved without considering the situations described in the problems. This study describes and examines how sixth-grade students make sense of numerical answers to mathematical word problems, and examines the impact of an instructional program designed to help them develop an inclination toward making sense of solutions to problem-solving tasks.

Seventy-six students (age 10.5 – 12) from three sixth-grade (n = 25, 25, 26) classes participated. Five problems were administered to the students during pretest, posttest, and retention test. Twenty-seven problems were used during the two-week instructional phase that followed the pretest. The study also involved pre- and post interview sessions to determine how selected students solved word problems on the pretest and posttest.

All word problems used in this study contained real-world situations. While some of the word problems can be solved without critical analysis of the story situations using straightforward and simple arithmetic operations with the given numbers, most of them required the students to make use of both their common-sense knowledge and conceptual knowledge learned in school.

An analysis of the pretest revealed that the students do not make use of their common-sense knowledge about everyday situations that is useful in solving mathematical word problems. Moreover, the study found that students' failure to make use of this common-sense knowledge could be attributed to classroom norms and teacher expectations. However, after two weeks of "actual" instruction, the students learned to use their common-sense knowledge to solve a variety of word problems. This finding supports the notion expressed by Verschaffel and De Corte (1997) that it is possible to develop in students a disposition toward sense making of mathematical word problems. Suggestions for future research are discussed.

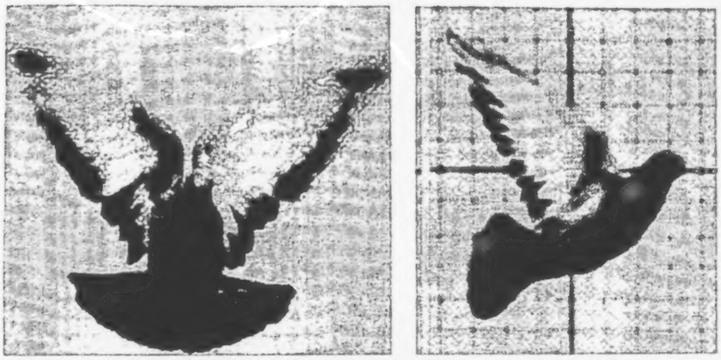
Reference

Verschaffel, L., & De Corte, E. (1997). Teaching realistic mathematical modeling in the elementary school: A teaching experiment with fifth graders. *Journal for Research on Mathematics Education, 28* (5), 577–601.

ISBN 0-88119-001-2

9 780881 190014

562



PME-NA XXIII

Snowbird, Utah
October, 2001

PERMISSION TO REPRODUCE AND
DISSEMINATE THIS MATERIAL HAS
BEEN GRANTED BY

D.Owens

TO THE EDUCATIONAL RESOURCES
INFORMATION CENTER (ERIC)

1

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION
CENTER (ERIC)

☑ This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality.

- Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

ERIC Educational Resources Information Center
Clearinghouse for Science, Mathematics,
and Environmental Education

Proceedings of the
Twenty-Third Annual Meeting

North American Chapter of the International Group for the

Psychology of Mathematics Education

Volume 2: Research Reports, Short Oral Reports, Poster Presentations, Author Index

PME-NA XXIII

October 18-21, 2001
Snowbird, Utah

Editors:
Robert Speiser
Carolyn A. Maher
Charles N. Walter

ERIC
Educational Resources Information Center

Clearinghouse for Science, Mathematics, and
Environmental Education, Columbus, OH

Cite as:

Speiser, R., Maher, C. A., & Walter, C. N. (Eds.). (2001). *Proceedings of the twenty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vols. 1-2). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

ERIC Clearinghouse accession number: SE 065 164

International Standard Book Number 0-88119-001-2

Clearinghouse Production Team:

Executive Editor: David L. Haury

Copyeditor: Linda A. Milbourne

Cover Design: Two Samurai Graphics

Printed in the United States of America

Order from: ERIC/CSMEE Publications, 1929 Kenny Road, Columbus, OH 43210-1080. Price per two-volume set is $45.00 plus shipping, handling, and relevant tax. All orders must be prepaid with delivery to a single address. Prices are subject to change without notice. To order by credit card, call 1-800-276-0462 or 614-292-5680.

Note. The *Proceedings* were produced with financial support provided by the U. S. Department of Education under contract no. ED-99-CO-0024. Opinions expressed in this document do not necessarily reflect the positions or policies of the U. S. Department of Education.

Contents

Reasoning and Proof 575

Research Reports

Growth in Student Mathematical Understanding Through Precalculus Student and Teacher Interactions 577
Daniel R. Ilaria & Carolyn A. Maher

Investigating the Teaching and Learning of Proof: First Year Results 585
Tami S. Martin & Sharon Soucy McCrone

Beliefs About Proof in Collegiate Calculus 595
Manya Raman

Posters

Metaphors of the Novice: Emergent Aspects of Calculus Students' Reasoning 605
Mike Oehrtman

Socio-cultural Issues 607

Research Reports

An Examination of the Interaction Patterns of a Single-Gender Mathematics Class 609
Amy Burns Ellis

The Role of Intellectual Culture of Mathematics in Doctoral Student Attrition 617
Abbe H. Herzig

Group Tests as a Context for Learning How to Better Facilitate Small Group Discussion 631
Judith Kysh

Learning Paths to 5- and 10-Structured Understanding of Quantity: Addition and Subtraction Solution Strategies of Japanese Children 639
Aki Murata & Karen Fuson

An Attempt to Clarify Definitions of the Basic Concepts: Belief, Conception, Knowledge 647
Erkki Pehkonen & Fulvia Furinghetti

Using Technology in the Mathematics Classroom and Its Impact on Girls and Boys: Teachers' View 657
Sonia Ursini, Gabriel Sanchez, Mónica Orendain & Cristianne Butto

Posters

Numeral Forms in Early Elementary Grade Levels 669
M. Faaiz Gierdien

Students' Beliefs About Mathematics after Moving Out of Reform Curricular Experiences Into More Traditional Curricular Experiences 671
Amanda Jansen Hoffman

On-line Math Teacher Conversations: Graphical, Statistical and Semantic Analysis 673
Eric Hsu

Cooperative Learning in a Fifth Grade English as a Second Language Mathematics Class 675
Theresa L. Jardine & Amy Roth McDuffie

Factors Contributing to Middle School Girls' Success in Mathematics: A Statistical Study 677
Michelle P. Longest & Sarah B. Berenson

Teacher Beliefs **679**

Research Reports

The Beliefs and Instructional Practices of Mathematics Teaching Assistants Participating in a Mathematics Pedagogy Course 681
Thomas C. DeFranco & Jean McGivney-Burelle

What Impacts Teachers as They Implement a Reform Curriculum? The Case of One Fifth Grade Teacher 691
Theresa J. Grant & Kate Kline

The Story of First-Year Teachers' Struggle to Maintain a Reform Perspective After Experiencing Integrated Mathematics Content and Methods in Their Teacher Preparation Program 699
Lynn C. Hart

Using Discourse Analysis to "See" Teacher-Positioning: A Tale of Two 8th Grade Mathematics Classrooms 709
Beth Herbel-Eisenmann

A Teacher in Conflict: Robin and Open-Ended Assessment 729
Wendy B. Sanchez

Short Oral Reports

Analyzing Professional Development by Thinking of Teachers as Learners 732
Jeffrey M. Choppin

Toward a Coherent Network of Learning Goals for Pre-K-Grade 5 734
Karen C. Fuson

Criterion Referenced Testing: An Agent of Change in Mathematics Education 736
Jean J. McGehee & Linda K. Griffith

Teachers' Conceptions of "Doing Mathematics" 738
Neil Portnoy & Karen Graham

What to Teach About Volume? Some Mexican Primary Teachers' Beliefs 740
Mariana Saiz & Olimpia Figueras

Posters

The Professional Road Less Traveled 742
Bridget Arvold

Believe It or Not 743
Jason K. Belnap

Affective Changes Tied to a Secondary Teacher's Reflections on Discourse 745
M. Lynn Breyfogle

Teachers' Beliefs and Expectations: Influences on Mathematical Learning 747
Lecretia A. Buckley

Teacher Education **747**

Research Reports

Classroom Coaching: An Emerging Method of Professional Development 751
Joanne Rossi Becker

Curriculum Reforms That Increase the Mathematical Understanding of Prospective Elementary Teachers 761
Irene Bloom

The Relationship Between Preservice Elementary Teachers' Conceptions of Mathematics and Mathematical Reasoning 771
Linda A. Bolte

The Effects of Staff Development Using a Study Group Model 779
Daniel J. Brahier

Enhancing Prospective Teachers' Understanding of What and How to Teach Through Small-Group Collaboration 787
Laurie O. Cavey, Sarah B. Berenson, Matthew R. Clark & Katrina Staley

Learning Mathematics Teaching 795
Brent Davis & Darren Stanley

Conceptions of Ratio in a Class of Preservice and Practicing Teachers 807
Karen Heinz

Using Methods Questions to Promote Mathematical Understanding in a Probability Course for Teachers 815
Todd M. Johnson

Building on Each Other's Ideas: K-3 Teachers' Mathematics Learning 823
Ok-Kyeong Kim & Joy Whitenack

Questioning Strategies in Two Middle School Mathematics Teachers 829
Karen Koellner Clark & Roberta Y. Schorr

What Do You Expect? How Learning Expectations Influence Preservice Teachers in a Field Experience 839
LouAnn H. Lovin

A Case Study of Change in Teaching High School Algebra 849
P. Michael Lutz

An Exploratory Study of Problem Posing From Numerical Expressions by Preservice Elementary Teachers 859
Armando M. Martinez-Cruz & José N. Contreras

Development of a Rubric to Measure Mathematics Teacher Reform 867
Doug McDougall, John A. Ross & Ann Le Sage

What Mathematical Knowledge Do Pre-service Elementary Teachers Value and Remember? 875
Mercedes A. McGowen & Gary E. Davis

Mentoring Styles in Mathematics: Two Contrasting Cases 885
Blake E. Peterson & Steven R. Williams

Pre-Service Elementary Teachers and Place Value: Written Assessment Using a Digit-Correspondence Task 897
Sharon H. Ross

An Analysis of the Teaching Practices of a Group of Fourth Grade Teachers 907
Roberta Y. Schorr, William Firestone & Lora Monfils

Perceived Experiences and Attitudes of Preservice Mathematics Teachers 917
Julie Sliva & Cheryl Roddick

Elementary Teachers' Ability to Analyze Student Errors Made When Solving Part-Whole Division Problems 929
Walter E. Stone, Jr.

Classroom Reform: Theory and Practice in Reflective Action 937
Sharon B. Walen & Kathy M. Ivey

Short Oral Reports

Dr. Robert B. Davis, the Teacher 945
Carol W. Bellisio

Preservice Elementary Teachers' Performance on Nonroutine Word Problems Involving Addition and Subtraction of Whole Numbers 947
José N. Contreras

Spatial Sense vs. Counting on Frank: Preservice Teachers' Comparative Analysis of Mathematical Software 949
Sandra Crespo

Mathematical Discussion During Student Teaching: Who Helps, Who Hinders, and Why 951
Vari Nelson Durrant

Math Journals: Tools for the Growth of Mathematical Understanding 953
Hari P. Koirala

Developmental Process in the Transitional Period: Preservice Mathematics Teachers Becoming Classroom Mathematics Teachers 955
Hea-Jin Lee

Insights From a Mathematics-Specific Field Experience for Elementary Education Majors 957
Denise S. Mewborn

Confronting Constructivist Pedagogy: Interpreting Two First-Grade Teachers' Conceptions of Innovative Pedagogy for Arithmetic 959
Jo C. Olson & Jeffrey Barrett

Representations as a Resource for Teacher Learning 961
Miriam Sherin & Andrew Izsák

Lesson Study and Collaboration in College Algebra 963
Emily Dixon Simpson

Preservice Teachers' Attitudes, Beliefs and Concerns About the Teaching & Learning of Mathematics 965
Cynthia Smith & Lisa Hitchcock

Posters

Fostering Teacher Education Through Research on Children's Problem Solving 967
Emily Dann

Pre-Service Elementary Teachers' Understanding of Place Value 969
Bradley E. Garner, Margaret Kinzel, & Sharon B. Walen

The Changes Teachers Encounter in Applying State Standards to Lesson Planning 971
Catherine Pomar Herzog & Kenneth L. Shaw

The Use of Video to Facilitate Teacher Reflection 973
Debbie Junk, Erin Turner & Susan Empson

A Common Agenda for Content and Methods Instruction 975
Gavrielle Levine & Elliott Bird

Implementing Standards-based Instruction in a Mathematics Sequence for Preservice Elementary Teachers 977
Cheryl Roddick & Julie Sliva

Tracing a Preservice Teacher's Growth of Understanding of Ratio, Proportion and Rate of Change 979
Katrina N. Staley & Clarence E. Davis

Efficacy of Preservice Elementary Mathematics Teachers 981
Robert G. Wenta

Meeting the Challenge of New Mathematics Standards: Exploring Teachers' Understanding of Geometry, Statistics and Probability 983
Joseph T. Zilliox & Neil A. Pateman

Technology **985**

Research Reports

Students Mathematical Explorations Via the Use of Technology 987
Luis Moreno-Armella & Manuel Santos-Trigo

The Effects of Multiple Linked Representations on Student Learning in Mathematics 997
S. Asli Ozgun-Koca

Visualizing Geometry With a Calculator 1005
Marco Antonio Santillán Vázquez &Luis Moreno-Armella

Short Oral Reports

Examination of a Mathematics Teacher's Exploration in a Technology-Supported Classroom 1013
Doug McDougall

Posters

Developing Pedagogy for Wireless Calculator Networks 1015
A. Louis Abrahamson, Douglas T. Owens, Michael Meagher & Marlena Herman

New Activity Structures Exploiting Wirelessly Connected Graphing Calculators 1017
James J. Kaput & Stephen Hegedus

Calculus and the Wired Classroom: Exploring the Interstices of Social Technologies 1019
Robert M. Klein

Learning to Facilitate and Analyze Students' Problem Solving With Technology 1021
Hollylynne Drier Stohl

Use of Technology in Data Analysis of a Long-Term Longitudinal Study 1023
Elizabeth Uptegrove

Author Index **1025**

Reasoning and Proof

GROWTH IN STUDENT MATHEMATICAL UNDERSTANDING THROUGH PRECALCULUS STUDENT AND TEACHER INTERACTIONS

Daniel R. Illaria
Carolyn A. Maher

Abstract: This paper investigates the role of teacher interaction in the development of mathematical understanding of five students who worked together on a math-modeling task. The dialogue between the teacher/researcher and students is analyzed. Preliminary findings suggest that where the mathematical thinking of the students was understood, interventions helped develop students' thinking.

Introduction

The students, engaged in conversation with the teacher, often give explanations for their ideas. A question arises as to what influence, if any, the teacher's response to those explanations have on student progress. This report examines dialogue between teacher and students and seeks to investigate the effect on students' growth in mathematical understanding.

The data come from a two-week summer institute that was a component of a longitudinal study on the development of proof making in students[1]. The students worked in groups on precalculus level mathematics problems. This paper focuses on one group of students and one of the problems they examined.

Theoretical Framework

Communication is an essential part of the mathematics classroom. Communication provides a means for students to express their ideas and explain their thinking (NCTM, 2000). Through communication students can share ideas and discoveries about the mathematics on which they are working. The communication process helps students create meaning for their ideas. NCTM (2000) includes communication as one of the standards in *Principles and Standards for School Mathematics*. Because of the need for communication in the classroom, an environment can be created where teachers and students engage in very important dialogue. Dialogue is important because it helps teachers assess the student's mathematical understanding and allows the students to clarify and express their ideas. Towers (1998) developed several themes to describe teacher interaction with students and illustrated how these interactions occasion the growth of students' understanding. A teacher should be skilled in interacting with students in order to gain access to students' mathematical understanding. Teacher questioning can help students justify and extend ideas, make connections, and generalize their conjectures (Dann, Pantozzi, & Steencken, 1995). The development of these skills is not immediate for the teacher, but once gained the teacher has an effective way to facilitate the growth of a student's understanding (Martino & Maher, 1999).

Being a participant in the classroom discourse, the teacher has an important function. In describing a classroom where students are working in small groups on a task, Maher, Davis, and Alston (1991) indicate that the teacher plays many roles: listening to children, offering suggestions, asking questions, facilitating discussions, drawing out justifications. When students discuss with their teachers the meaning of mathematical notions, students are expected to think about concepts, their meanings and their interrelations (Vinner, 1997). If students do think about concepts, they are in a conceptual mode of thinking (Vinner, 1997). If students do not think conceptually, but still produce answers which seem to be conceptual, then Vinner (1997) states the students are in a pseudo-conceptual mode of thinking. The teacher must continuously assess whether or not the students have learned the mathematical concept, truly understands the reasoning behind their problem solving approach, and can adequately support and defend their conclusions using their previously learned mathematical knowledge. In a regular classroom, it is not always possible to observe what a student does after an interaction with the teacher. Because this observation is not always possible, it is difficult for a teacher to determine if the interaction was beneficial to the student. Videotape data that follows the student when the teacher leaves make possible gaining a better understanding of a student's actions. Interacting with students is a challenging task for a teacher, who has to make instantaneous decisions. The researcher, who has the benefit of studying and referring to videotape data, however, can learn from the interaction after the fact. What the researcher learns from the interaction can be shared with the teachers, who can reflect on their actions, and facilitate a growth in students understanding.

Methodology

Participants

Five students seated at the same table (four males and one female) and one teacher/researcher were subjects in this study. All of the students were entering their fourth year of high school. The teacher/researcher involved in the interaction is an experienced professor of mathematics and mathematics education at the university level.

Task

The students were given a picture of a fossilized shell called Placenticeras. The first part of the task, designed in 1991 by Robert Speiser, was to draw a ray from the center of the shell in any direction. Then with polar coordinates as a way to describe the spiral of the shell, the students were to make a table of r as a function of theta. After creating the table, the students were asked what they could say about r as a function of theta. The students had graphing calculators, transparencies, rulers, and markers at their disposal for completing this task.

Data Collection and Analysis

The data come from a two-hour videotape session during the third day of a two-week Institute. The interactions were coded to consider perspectives of the teacher and the students. For the students, the following codes were developed and used. S(i): Student ignores the suggestion made by the teacher; S(c): Student asks the teacher for clarification of a statement; S(a): Student attempts the teacher's idea or suggestion; S(e): Student engages in conversation for the purpose of explaining their own views. For the teacher: T(r): Teacher restates the problem or returns to an old idea; T(f): Teacher follows the student's idea or suggestion; T(n): Teacher introduces a new idea; T(c): Teacher asks the student to clarify their statements or idea. The codes were used to follow the choices of the teacher and the resulting action by the student. When students became engaged in a conversation, their words were examined for evidence of their understanding.

Findings

The students' own words demonstrate where mathematical understanding occurs, and where their growth in a solution to this problem appears. The teachers' insistence on reiterating previous ideas, as well as, moving on to new ideas and following the student's suggestions, allows for the opportunity for the students to advance their understanding. For example, a lack of understanding about their fourth power regression solution modeled using the TI-89 was observed when both students agreed that their model is a parabola.

00:52:35:17	E S(e)	I think it does. I mean if you look, if you look at the regression. It's just like a parabola. And uh your data.
00:52:42:19	Mi	It is a parabola
00:52:43:12	Ma S(e)	It is a parabola. A very nice parabola And like you know. I mean you can't use anything behind past zero on the x obviously because it can't have negative growth. That doesn't make sense. So you can't do that. But I mean the way, the way it goes up and the reason why it goes sharply up is just the fact that. I mean even from here to here like say the distance is 6 then all of a sudden it is 40. It's not going to keep on going little by little. Eventually it's get ting wider like this. And that's why it's jumping so high up. It's not the fact that it's off or it's not predicting anything. It's just the numbers are getting larger and larger. It has to go higher and higher. So that's why it goes that steep angle like that.

00:53:32:16	B T(f)	Ok well. I am still interested in that earlier part because. Are you saying that this animal really started growing where we're saying theta equals zero is.
00:53:46:10	Ma	Um hmm
00:53:48:21	B	How do you know that
00:53:51:02	Ma S(e)	It's gotta start somewhere. And it doesn't start. You can't start with. You can't start.
00:53:55:24	Mi	You can't start with anything negative
00:53:57:24	Ma	anything past nothing.
00:53:57:24	Mi	Yeah
00:53:59:09	Ma	You know
00:54:01:05	Mi	Cause then it doesn't exist. In which case it's not there.

The students do not look further into the data beyond a visual fit of a scatter plot and their curve. The teacher/researcher listens the discussion about the model by focusing on where the shell started growing. The teacher/researcher returns to the idea about how the model describes the start of the growth of the shell.

00:57:20:24	B T(r)	See then I am wondering about that fourth power model cause if you go to the left on it. You are sort of going inward on the shell right. You are going backward in time.
00:57:32:04	Ma	Yeah
00:57:33:05	B	But then suddenly as you keep going left it goes up.
00:57:37:10	Ma S(e)	Oh but there is nothing there though. That is the thing. Like you have to set limitations somewhere because some things are just physically impossible you know.
00:57:45:24	B	I think we're beginning to understand each other.
00:57:48:02	Ma	Yeah
00:57:48:19	B	Okay, Umm
00:57:51:09	Ma S(e	I mean its like. I guess its like certain things like if you figure out like differences with like electricity or something or like in physics. Like you can't have things that are. Sometimes you can't have things that are negative. There are things that are just physically impossible to have. And that to have something, to have an animal or a living thing that is a negative distance would mean that it isn't there. So it's not physically possible to have that anything past that zero. You know. It just wouldn't be there. This animal would not be there if there was a negative number. Basically.

The teacher/researcher continues to question the students. He asks the students to clarify their ideas in order to allow them to provide evidence for building their understanding of why their model does not work for certain values.

01:01:47:12	B T(c)	Oh, so there's a place. Okay then you are agreeing that there's a place where the regression doesn't model the animal
01:01:48:01	Mi S(e)	You can put it so that the restriction has to be greater than zero.
01:01:52:25	Ma S(e)	Yes, but that's necessary for other things too. There's limitations.
01:01:56:10	B	Okay, Okay.
01:01:57:13	Ma S(e)	like like the first graph we did with uh with the running thing, with the uh, with the thing you had to put limitations on it cause there were certain things that went past a certain time.

The teacher/researcher and the students continue the discussion by focusing on the accuracy of the model outside the range of their collected data. The question of what a model would look like if the data were collected again moves the conversation topic to the model's general shape. After this discussion, the teacher/researcher returned to the left side of the student's model. By the left and right side, the teacher/researcher and students are using the origin of the coordinate plane as their reference point. Therefore the left side would refer to negative values of time, and the right side would refer to positive values of time. When the teacher/researcher returned to the left side of the student model, the teacher/researcher and students revisited discussing the model's accuracy during negative values of time.

01:06:37:25	B T(r)	So it looks like we are making sense on the right and then we got questions on the left. Is that fair.
01:06:44:25	Ma	Sure, why not.
01:06:46:10	B	Okay
01:06:46:18	Mi S(c)	What possible questions could have on the left. It's dead. It doesn't exist.
01:06:51:02	B	Well I just don't.
01:06:51:18	Ma S(e)	Not even that. It's not even born yet
01:06:51:19	B T(r)	It's very hard for me, yeah. It's very hard for me to believe that this at some point in the distance past
01:06:57:05	Mi	It doesn't exist
01:06:57:29	B T(r)	That that it was very large as the fourth power, as that fourth power curve suggests.

01:07:04:08	Mi S(e)	Alright fine, we'll do a third power curve, it'll be very small. It'll be gone.
01:07:07:14	Ma S(e)	No, you know what. You know what you
	S(a)	got to do. You set a we do that?

The students have provided a way to adjust the model so that it does not show a large shell when time is negative. Though the two students believe the left side of this model does not accurately portray the growth of the shell, their methods of correcting the inaccuracy are different. Mi wants to change the regression curve to the third power model, which would result in a new equation that models a different rate of change, and continues the inaccuracy of the model before the shell began to grow. Ma's explanation shows that he wants to remove the left side, but believes that the right side correctly models the growth of the shell.

Conclusions

The students' understanding of their model grew because they have provided justification for why the model should not represent the shell before it started to grow. However the students' understanding of the rate at which the shell is growing did not grow during this interaction. The dialogue showed the students used a fourth power regression to create a solution to the task. In the discussion, three students used the word parabola to describe the curve. Their early classification of the graph as parabolic demonstrated a limited understanding of the rate at which the shell grew. This is because parabolic and quartic curves represent different rates of growth. Later when Mi and Ma recommended changes for their model, they provided different methods for a correction. Mi suggested changing their regression to a third power, and Ma suggested restricting the left side of the model. Since Mi's correction used a different regression model, Mi did not make a connection between the rate of growth and the type of curve needed to model that growth. Neither of the students provided evidence as to why the model is quartic. E stated, "I mean if you look, if you look at the regression. It's just like a parabola." Mi and Ma both followed with "It is a parabola." Mi and Ma accepted the visual interpretation of the model by E. The students' earlier understanding about rate of growth did not grow during this interaction because they have not provided justification for the model's shape beyond the visual inspection.

Despite this misunderstanding, the teacher/researcher did not correct or criticize their comments. Rather the focus of the teacher/researcher was to discuss the start of the growth of the shell using the students' model. The students' explanation showed an understanding about their model around zero. When the teacher/researcher returned to the growth of the shell around zero, the students' level of understanding increased through engagement in the conversation. The student connected the limitations on the model to other physical situations. This showed a growth in understanding by providing a justification for why the limitation exists. However, the student still did not give an explanation for how to adjust the model, which provided room for further growth. Despite a growth in understanding about how the shell is growing, the teacher/researcher asked the students to clarify their ideas. The two students expressed awareness that their model has limitations. When Ma referred to “the running thing” he drew on prior experience of why limitations are needed and therefore provides a mathematical grounding for his reasoning. The student referred to an earlier problem from this workshop, which has more meaning because it directly involved the student. This connection to another physical representation is more powerful than the representations mentioned earlier. The teacher/researcher returned to the idea of the left side of the model after a discussion of the general shape of the model. During this engagement, the students explained how to change the model so the left side did not exist. Mi stated, “It doesn’t exist”, but suggested the group use a third power regression to fix their model. Ma suggested, “You set a limitation on the graph so there is no left side and then we won’t have this problem”. The students demonstrated another growth in understanding by providing a method to correct the model. Previously the students explained why the shell could not exist for negative values, but have now moved forward to provide possible methods for representing the limitation on the model. Both of the students agreed the model does not accurately portray the growth of the shell by suggesting methods to alter their model. However, their understanding of the reasoning behind the inaccuracy differed which resulted in multiple methods for correcting the model.

Through interaction between the teacher and students, the students made public their level of mathematical understanding. By examining the episodes presented, one can see that the teacher/researcher consistently returned to the idea of how the model demonstrated the growth of the shell over the entire domain of the students’ model. Additionally, the students’ ideas are followed or they are asked to clarify their statements. Using this method of questioning, the students were given the chance to make connections and reorganize their thoughts in order to provide justification for their conclusions. In providing this justification, their understanding grew because the students suggested bases for their conclusions. By using previous knowledge as a method of justification, the students connected the current problem with other situations and showed a growth in understanding. The opportunity for growth occurred because the teacher continually returned to old ideas. As a consequence, the students had multiple possibilities to become engaged in conversations and articulate their understanding of the mathematics.

This research provides a foundation for continuing a dialogue about the affects of teacher and student interactions in the classroom. These preliminary findings imply that teacher interaction helps the student to express their mathematical understanding. Further research can help to indicate whether the teachers/researchers can learn from their choices during interactions to see if they are constructively contributing to students’ progress. More research is needed to provide a better understanding of how teacher intervention, particularly questioning, can contribute to students' mathematical understanding.

References

Dann, E., Pantozzi, R., & Steencken E. (1995). Unconsciously learning something: A focus on teacher questioning. *Proceedings from the North American Chapter of the International Group for the Psychology of Mathematics Education.* Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Maher, C., Davis, R., & Alston A. (1991). Implementing a "thinking classroom" in mathematics. *Journal of Mathematical Behavior*, *10*, 219-224.

Martino, A., & Maher C. (1999). Teacher questioning to promote justification and generalization in mathematics: What research practice has taught us. *Journal of Mathematical Behavior*, *18*, 53-78.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Towers, J. M. (1998). *Teachers' interventions and the growth of students' mathematical understanding*. Unpublished doctoral dissertation, University of British Columbia, Vancouver.

Vinner, S. (1997). From intuition to inhibition – Mathematics, education, and other endangered species. *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education.* Lahiti, Finland.

Note

[1] This work was supported in part by National Science foundation grant #REC-9814846(directed by C. A. Maher) to Rutgers, The State University of New Jersey. Any opinion, findings, conclusions, or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation.

INVESTIGATING THE TEACHING AND LEARNING OF PROOF: FIRST YEAR RESULTS

Tami S. Martin
Illinois State University
tsmartin@ilstu.edu

Sharon Soucy McCrone
Illinois State University
smccrone@ilstu.edu

Abstract: Although proof and reasoning are seen as fundamental components of learning mathematics, research shows that many students continue to struggle with geometric proofs. In order to relate pedagogical methods to students' understanding of geometric proof, our three-year project focuses on two components of student understanding of proof, namely, students' beliefs about what constitutes a proof and students' proof-construction ability. The classroom environments in the first year of the study were generally teacher-centered learning environments in which proof was logical exercise rather than a tool for establishing a convincing argument. Students harbored several ill-founded beliefs including: general claims may be established on the basis of checking critical examples, the form of an argument is more important than its chain of logical reasoning, and proofs are only valid for their associated diagrams, even if specific features of the diagram are not incorporated into the proof. In addition, students had great difficulty constructing proofs unless the key relationships necessary to establish the proof were outlined for them.

Introduction

Proof is fundamental to the discipline of mathematics because it is the convention that mathematicians use to establish the validity of mathematical statements. In addition, the teaching of proof as a sense-making activity is fundamental to developing student understanding in geometry and other areas of mathematics. Despite the fact that student difficulty with proof has been well established in the literature, existing empirical research on pedagogical methods associated with the teaching and learning of geometric proof is insufficient (Chazan, 1993; Hart, 1994; Martin & Harel, 1989). Our work in this area has begun to address the need for research into the pedagogy of geometric proof instruction. We focus on geometric proof because geometry is traditionally the course in which students are first required to construct proofs. We have begun a three-year study to develop an empirically grounded theoretical model that relates pedagogy to student understanding of proof.

In order to assess the effectiveness of the pedagogical methods used by participating teachers, the project focuses on two components of student understanding of proof, namely, students' beliefs about what constitutes a proof and students' proof-construction ability. Specifically, the first year of the project has addressed three objectives:

1. To document student understanding of proof in order to update and expand existing research in this area;

2. To characterize evolving student beliefs about what constitutes a proof in proof-based geometry classes and to link these characterizations to aspects of the pedagogy including sociomathematical norms, nature of the activities, and teacher beliefs; and
3. To characterize students' evolving proof-construction ability in proof-based geometry classes and to link these characterizations to aspects of the pedagogy including sociomathematical norms, nature of the activities, and the teacher's instructional philosophy.

Perspectives

Existing research documents students' poor performance on proof items and identifies common, fundamental misunderstandings about the nature of proof and generalization in a number of mathematical content areas (Chazan, 1993; Harel & Sowder, 1998; Hart, 1994; Martin & Harel, 1989; Senk, 1985). In trying to make sense of students' difficulties with geometric proof, Dreyfus and Hadas (1987) articulate six principles which form a basis for understanding geometric proof. These principles address many of the student misunderstandings of proof cited in the literature.

The theories guiding our research come from proof-related research projects including the work of Harel and Sowder (1998), Hoyles (1997), and Simon and Blume (1996). Some researchers (Balacheff, 1991; Harel & Sowder, 1998; Knuth & Elliott, 1998) have proposed similar theories that describe increasingly sophisticated strategies used by students to construct proofs. At the least sophisticated level, students appeal to external forces for mathematical justification. At the next stage, students base their justifications on empirical evidence. Finally, students are able to use more abstract and mathematically appropriate techniques when proving statements. The findings from our three-year study will be used to make connections between pedagogy and various levels of student understanding.

Methods

During the first year of the study, we collected data in the classrooms of two teachers in a large high school in the mid-western United States, recording the beliefs and proof construction ability of the students as well as the beliefs held and the pedagogical methods used by their teachers. The teachers participated in a summer workshop, prior to the school year, during which they read and discussed existing research on geometric proof and experienced methods for investigating proof. Teachers also worked collaboratively with the researchers and graduate assistants to plan for and reflect on classroom events. In order to capture classroom events, we conducted daily observations and videotaping of classroom activities during the four months in which proof was a major focus of the curriculum.

In order to document beliefs, we revised and extended Dreyfus and Hadas' (1987) six principles, then constructed the Proof Beliefs Questionnaire that assessed students' agreement with these revised principles. It was necessary to add more detail to Dreyfus and Hadas' (1987) principles, in order to address a broader set of beliefs and to reliably map items to particular principles. Questionnaire items consisted of items modified from instruments used by Chazan (1993), Healy and Hoyles (1998), and Williams (1980), as well as some original items.

To assess students' ability to construct proofs, we developed a performance assessment instrument, the Proof Construction Assessment. The Proof Construction Assessment included items in which students must construct partial or entire proofs, as well as generate conditional statements and local deductions. In addition to some original items, the instrument includes items modified from Healy and Hoyles (1998), Senk (1985) and from the Third International Mathematics and Science Study (TIMSS) (1995).

The Proof Beliefs Questionnaire was given to all students during the first semester, about three weeks after proof had been introduced. We conducted follow up interviews with six focus students in each class to clarify their beliefs. These students were selected on the basis of their performance on the questionnaire and teacher recommendations. During the second semester, the Proof Construction Assessment was administered to all students in the two classes, and another set of interviews was conducted with the 10 of the 12 focus students.

The multiple sources of data helped us to learn about the context for the development of students' beliefs about what constitutes a proof and their ability to construct proofs in order to interpret this information and connect it to pedagogy. Other data sources included audiotaped planning meetings with researchers and teachers as well as interviews with the classroom teachers.

Results

In order to get a sense of the classroom environments in which the research took place, we first describe general features of the two classrooms as well as some of the typical classroom practices. One of the two participating teachers had been teaching for 5 years and the other for more than 20 years. Despite the difference in years of experience, there were several commonalities in the teachers' classroom practices. Both teachers followed the order and scope of the textbook quite closely. The typical daily routine involved discussing homework, introducing new material, and practicing new material. At the suggestion of researchers, student desks were arranged in pods of four to facilitate student dialogue.

In analyzing videotapes and field notes of classroom sessions, we have identified several features of the classroom environment, including social norms, sociomathematical norms, and other factors, that may have influenced students' learning.

The social norms, or standards of social behavior in the classroom, included:

- *The teacher was the mathematical authority in the classroom.* Teachers provided counterexamples to student conjectures, rather than remaining neutral or turning conjectures back to the class. They also posed rhetorical questions so that students were essentially asked to agree with the correct answer.

- *There was limited time for thinking and answering questions.* Teachers often asked and answered their own questions. Wait time was very short. In-class work time for groups was very limited. Teachers often interrupted this time with hints, advice, and examples.

The sociomathematical norms, or standards for mathematical behavior in the classroom, included:

- *There were few opportunities for sense-making.* Students appealed to facts such as "you can't divide by zero," but were not generally asked to explain or make sense out of these facts. Students were able to make claims about geometric relationships without justifying their claims.
- *Problems and proofs always worked out nicely.* Problems had solutions and proofs contained all the necessary information to prove the desired result. In instances where this was not the case, it was due to a "typo" either in the text or on teacher-made worksheets. Students were directed to fix the mistake and solve the problem or complete the proof.
- *It was not clear that there was a need for proof.* There was very little opportunity to make conjectures or prove conjectures. Proofs that students had to construct were generally proofs of given "facts."

An additional factor that may have influenced the students learning environment was:

- *Teachers' pedagogical practices were limited by their content knowledge.* The less experienced teacher rarely strayed from teacher-directed activities. When she did, errors in reasoning and in logical structure were documented. The more experienced teacher made fewer errors and was more willing to follow up on students' mathematical suggestions.

These general features established an environment in which the teacher had most of the responsibility for constructing convincing arguments and the students were left to mimic the expert practices of the teachers.

Beliefs About Proof

We have used our revisions of Dreyfus and Hadas' six principles (1987) as a framework for our analysis of student beliefs. The Proof Beliefs Questionnaire, which is aligned with these six principles, formed the basis of students' self-reported beliefs about proofs. By synthesizing Proof Beliefs questionnaire data with interview data and classroom observations, we have developed some preliminary findings. These findings are organized by principle.

Principle 1: A theorem has no exceptions. Some students claim to believe that this is true. During clinical interviews with students regarding their beliefs, students stated that theorems don't always have to be true and there can be some exceptions to the theorems. However, they sometimes allow for exceptions to the rule when faced with counterexamples. In the classroom, the teachers used counterexamples to respond to students when they made a false claim. (For example, one student asked if AAA was a congruence theorem. The teacher sketched a pair of similar triangles to show that it was not a theorem.) However, the teachers did not take the opportunity to emphasize that here was a counterexample being used to refute a statement.

Principle 2: The dual role of proof is to convince and to explain. Despite students' claim that proofs are required to establish validity, they are often unconvinced by general proofs. In fact, they often claim that examples are more convincing than proofs. It is not clear that the explanatory role of proof has hit home with these students. The statements that they were asked to prove in the classroom were generally statements they already believed to be true. In other words, the explanatory role of proof was not a critical role for students, because, in their own minds, they had already ascertained the validity of the statement. Opportunities for students to make conjectures then prove these conjectures were rare, and generally out of the comfort zone for the participating teachers.

Principle 3: A proof must be general. Students believe that empirical evidence constitutes a proof. They also believe that checking critical cases (e.g., an isosceles triangle, a right triangle, an obtuse triangle, etc.) satisfies the requirements for generality in an argument. They view a specific triangle as a reasonable representative for all triangles in the classification. In the classroom, teachers appealed to specific examples to help demonstrate the validity of a statement and the application of a statement (not necessarily clearly distinguished), possibly contributing students' belief that examples constitute a convincing argument.

Principle 4: The validity of a proof depends on its internal logic. Students claimed to prefer two-column proofs to any other style of formal proof (e.g., paragraph or flow chart). They believed that two-column proofs were more organized and easier to understand. In assessing the relative value of multiple proofs, they appealed to form over internal logic. In addition, when checking proofs, some students were not particularly attentive to issues of logical order, with the exception of the location of given statements in a proof. In the classroom, the teachers almost always used two-column proofs for direct reasoning and reserved the paragraph format for indirect proofs. Although the teachers experimented with flow chart proofs, they often aligned them as if they were two column proofs and sometimes misrepresented the logical connections.

Principle 5: Statements are logically equivalent to their contrapositives, but not necessarily to their converses or inverses. Students appealed to context to determine the validity of various forms of a statement and not to the logical equivalence of the form of the statement. If the context is a nonsense context, students will translate the context to a "real life" context in order to reason in context. In the classroom, converses, inverses, and contrapositives were treated as an independent section at the beginning of the school year. They were not connected to later treatments of proof, which were generally focused on proving positive statements. There was no link to these forms during the section on indirect proof either.

Principle 6: Diagrams that illustrate statements have benefits and limitations. Students believe that a diagram is valuable to forming a proof. However, many students are unclear about which aspects of a diagram are general (i.e., meant to represent a class of figures) and which are specific. Some students also believe that a proof is only valid for its accompanying figure, or at least its accompanying class of figure (e.g., obtuse triangles), even if the specific features of the figure (e.g., obtuse angle) are not incorporated into the proof. The role of diagrams as general representations was not explicitly discussed in class.

Proof Construction Ability

Student proof construction ability was determined using three types of data collected during the project year. First, the Proof Construction Assessment instrument was developed to measure students' varying levels of ability to engage in formal logical reasoning. Second, data was collected during classroom observations. Observers took field notes and video recorded classroom sessions of proof instruction as well as students working in groups or with technology to develop proofs. Third, a set of ten focus students participated in clinical interviews with researchers. The interviews focused on some aspects of the Proof Construction Assessment and required focus students to create at least one original proof during the session.

The Proof Construction Assessment included items with varying amounts of support in order to assess proof construction ability at four levels. Items at the first level, which offered students the greatest support, required students to fill in the blanks in a partially constructed two-column proof. Items at the second level of support addressed specific components of proof construction. The first type of item at this level addressed students' understanding of conditional statements. Students were asked to separate the *if* and the *then* components of a conditional statement in order to identify which component was associated with the relationships that were given and may be assumed to be true and which component required proof or justification. The local deductions items were also at this second level of support. These items assessed students' ability to draw one valid conclusion from a given statement and to justify the conclusion. This was less supported than the fill-in type item because the students were required to draw the conclusion themselves, without being told what they were to justify or what the justification was for a missing statement. This type of task is equivalent to producing and justifying only one step in a logical argument. Items at the third level of support required multi-step reasoning. These items required students to construct proofs for which hints were provided. These hints identified some of the key elements in the proof's logical chain of reasoning. At the fourth level of support, students were asked to generate a complete, multi-step formal proof, independently.

Results from the Proof Construction Assessment are found in Table 1. Student performance on the instrument suggests that the students in both classes seemed to have the greatest difficulty with items 2 and 4, which provided the least amount of scaffolding. These items required students to write original proofs of statements based on given conditions. Even though students also needed to write a proof for item 5, they were provided with ideas for outlining the proof. Strong student scores on item 5 also might be due to the fact that students were most familiar with the content (similar triangles) since they had just completed a unit on similar triangles in class. Student performance was best on items 1 and 3, which provided the most scaffolding. For item 1, students were asked to fill in the missing statements or reasons for a proof that had been developed for them. For item 3, students were required to write a conditional statement and then use this statement to determine what information was given and what was necessary to prove if asked to justify the conditional statement. By synthesizing Proof Construction Assessment data with interview data and classroom observations, we have developed a few preliminary findings.

Content knowledge is a major factor in student proof construction ability. Student performance on the Proof Construction Assessment was discussed during clinical interviews with students. During these interviews, several students claimed to have difficulty with those items whose content was unfamiliar or whose content was from lessons earlier in the school year. When the geometric content was somewhat familiar to the students, they were able to talk through aspects of the given diagram (or provide their own diagram) that eventually led them to at least an elementary understanding of what was needed to write a proof. When the content was unfamiliar, at

Table 1. Student Performance on the Proof Construction Assessment.

Item Number	Average Score as a percent for Mrs. A's students	Average Score as a percent for Mrs. C's students
1	61.6	69.5
2	22.3	27.4
3	66.5	76.7
4	33.1	30.5
5	52.3	70.5
6	42.3	39.0
Total	46.7	51.1

least one student was unable to provide even one valid conclusion from a given statement. Field notes of classroom observations indicate that the classroom teachers often "brainstormed" with students about a given situation and wrote an outline for a proof prior to requesting that students write a formal proof on their own. These "brainstorming" sessions helped all students recall the content needed to complete the proofs.

Format for writing formal proofs was over-emphasized in class (understanding the need for proof was under-emphasized). At the start of instruction on proof writing, both classroom teachers modeled a variety of proof writing techniques and allowed students to write proofs in flow chart form, paragraph form, or two-column form. However, after about two weeks of proof instruction, the teachers only showed proofs in two-column form. Thus, this became the accepted method for writing a formal proof. By this point students had also begun to "validate" their proofs by checking that their statements and reasons matched those demonstrated by the teacher in number and content. Students were often convinced that their proof was valid if the number of steps in the proof matched the number of steps in the proofs constructed by other students in the class. These ideas relate to the belief that the form of the proof is more important than the substance of the proof.

Students were not given many opportunities to explore mathematical ideas and proof writing on their own. One goal of proof writing is that the writer will come to a deeper understanding of the mathematical concepts involved. For this to happen, the writer must see proof development as a logical process that begins with exploration of mathematical ideas. Often, students in the research classroom were merely given new geometric ideas such as the fact that parallel lines have the same slope, and expected to use these ideas to prove statements or situations that were provided for them. Moreover, the teachers frequently demonstrated how to complete a proof using the new ideas before allowing the students to explore the ideas or to write similar proofs on their own.

Conclusions

Some of the findings from the first year of the study echo the results of earlier studies such as the fact that many students believe that a set of examples constitutes a proof (Chazan, 1993; Harel & Sowder, 1998). In addition, students' poor performance on writing original proofs supports Senk's (1985) findings. It is our investigation of the classroom environment and its connection to students' understanding that sets our work apart from the existing literature.

The classrooms we studied were teacher-centered environments in which successful proof-writing consisted of using an acceptable format to link a collection of definitions, postulates, and theorems in a repeatable pattern of sequenced steps. In this environment, students developed some beliefs that were contrary to their teachers' expectations and to generally accepted principles of proof understanding. The classroom's social and sociomathematical norms gave rise to specific classroom practices that were generally detrimental to developing student understanding. For example, there were few opportunities for sense-making. Because students were not expected to reason about geometric relationships in an informal context in the classroom, they never practiced proof-writing as a form of making sense of geometric relationships. Although they claimed to believe that proofs were useful for explaining relationships, students judged the validity of arguments based only upon a proof's format or on whether they believed that the statement to be proved was true.

The teacher's role as the mathematical authority in the classroom also impacted students' beliefs. In particular, when the teacher led classroom discussions, students were easily distracted, because they had little responsibility for making mathematical decisions. They were also reluctant to investigate or make conjectures because there was usually not long to wait before the teacher would provide the correct answer or provide the next step in a proof.

An environment in which all problems can be solved and we only prove true facts also undermines the value of proof as a sense-making tool. It is not surprising that students do not see a need for proof in a situation in which everything we try to prove is true and if it cannot be proved, then we can safely assume that needed information was inadvertently omitted.

Our first-year results show that environmental aspects of the classrooms certainly have the potential to impact students. Social norms and sociomathematical norms can give rise to classroom practices in which students' main goal is to generate work that looks like the teacher's examples. The effect of this is that students may be less likely to do their own thinking about the given situation and more likely to simply follow the format provided, even if they experience little success in implementing the practice. If we take the position that students must construct their own knowledge by doing and experiencing, this model for teaching proof construction may have a detrimental effect on student learning.

References

Balacheff, N. (1991). Treatment of refutations: Aspects of the complexity of a constructivist approach to mathematics learning. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 89-110). Dordrecht, Netherlands: Kluwer.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, *24* (4), 359-387.

Dreyfus, T., & Hadas, N. (1987). Euclid may stay - and even be taught. In M. M. Lindquist & A. P. Shulte (Eds.), *Learning and teaching geometry, k-12* (pp. 47-58). Reston, VA: National Council of Teachers of Mathematics.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *CBMS issues in* *mathematics education*. Providence, RI: American Mathematical Society.

Hart, E. W. (1994). A conceptual analysis of the proof-writing performance of expert and novice students in elementary group theory. In J. J. Kaput & E. Dubinsky (Eds.), *Research issues in undergraduate mathematics learning: Preliminary analyses and results (MAA notes 33)*. Washington, DC: Mathematical Association of America.

Hoyles, C. (1997). The curricular shaping of students' approaches to proof. *For the Learning of Mathematics, 17* (1), 7-16.

Knuth, E., J., & Elliott, R. L. (1998). Characterizing students' understandings of mathematical proof. *Mathematics Teacher, 91* (8), 714-717.

Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education, 20* (1), 41-51.

Senk, S. L. (1985). How well do students write geometry proofs? *Mathematics Teacher, 78* (6), 448-456.

Simon, M. A., & Blume, G. W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *Journal of Mathematical Behavior, 15*, 3-31.

Third International Mathematics and Science Study. (1995). *Released advanced mathematics items: Population 3.* The Hague, Netherlands: International Association for the Evaluation of Educational Achievement.

Williams, E. R. (1979). *An investigation of senior high school students' understanding of the nature of mathematical proof.* Unpublished doctoral dissertation, University of Alberta, Edmonton, Alberta, Canada.

BELIEFS ABOUT PROOF IN COLLEGIATE CALCULUS

Manya Raman
University of California, Berkeley
manya@socrates.berkeley.edu

Abstract: The broad aim of this research is to characterize the views of proof held by college calculus students and their two types of teachers—mathematics graduate students and professors. The analysis is based on an examination of the ways in which people in all three groups produce and evaluate different types of solutions to a proof-based problem from a college calculus course. Initial results indicate a subtle but fundamental difference in the way university level teachers and students view proof. To the teachers a proof is the connection between an idea and the representation of that idea—the representation used being a function of the norms of a particular mathematical community. To the students, who have not had many mathematical experiences outside of school mathematics and who may not understand the underlying mathematics, the proof is simply the representation.

Introduction

The aim of this research is to characterize the views of proof held by college calculus students and their two types of teachers—mathematics graduate students and professors. I do so by examining the ways in which people in all three groups produce and evaluate different types of solutions to a proof-based problem from a college calculus course. Most work on proof at the university level has focused largely, if not entirely, on students. The results of those studies have yielded many interesting insights, one of which is to suggest that the views of proof are different from—and possibly even in conflict with—the views held by their teachers (e.g., Alibert & Thomas, 1991; Harel & Sowder, 1998). However, those studies by themselves fall short of determining if—and if so, *how*—those views conflict, because they only hypothesize about the views held by mathematicians. The study reported here is designed to address these questions, using a data collection method similar to one developed to compare student and teacher views of proof at the high school level (Hoyles & Healy, 1999), but on a smaller scale and with more in-depth interviews.

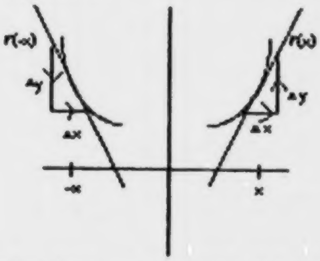
This study is situated within a broad literature indicating that students' mathematical difficulties are not only *cognitive*, e.g., they do not connect concept images with concept definitions (Vinner, 1991), but also *epistemological*, e.g., their view of what constitutes knowing may affect how they reason (Hofer, 1994), and *social,* e.g., the kind of mathematics arguments they generate are constrained by their expectations of school mathematics (Balacheff, 1991; Schoenfeld, 1992). Recent research has indicated that cognitive, epistemological, and social factors are related. For instance, certain perceptions about the nature of proof (e.g, what is the appropriate level of rigor) lead to difficulty in producing a proof (e.g. in an exam situation) (Moore, 1994). And some beliefs, such as what kind of answer is expected on an exam, may conflict with others, such as what kind of answer would demonstrate the best understanding (Elby, 1999). Continuing this line of research, an overarching goal of this research project to better understand the role of cognitive, epistemological, and social factors in shaping people's views of proof.

Methods

Using a task-based interview protocol, 11 students, 4 mathematics graduate students, and 5 mathematics professors were individually interviewed. Each interview lasted 1-1.5 hours and were both audio and videotaped. The students came from three different sections of a first semester calculus course at a top-ranked university. The calculus course was traditional, in the sense that there was a strong emphasis on rigor (the textbook was Stewart (1998)) and lectures were closely aligned with the textbook. The professors who gave the lectures and graduate students who led discussion sections were among the subjects.

The central question for the interview was to *prove that the derivative of an even function is odd*. Participants were first asked to answer this question on their own. They were then shown different responses to the question and asked to evaluate them. Before seeing the responses, participants were asked to discuss their work, focusing on why they chose a particular method and how convinced they were of their response. Next they were shown five responses to the question, not all of which were correct, which came from pilot studies and textbooks. Response #1 was empirical (checking $y=x^n$ for n from 1 to 6), #2 was graphical, #3 was a textbook-like proof using the definition of derivative, #4 was a short proof using the chain rule, and #5 was a false formal-looking proof. In much of rest of the paper I focus on people's views about #2 and #3 so they are reproduced in Figure 1.

Response 2:



If f(x) is an even function it is symmetric over the y axis. So the slope at any point x is the opposite of the slope at -x. In other words f '(-x) = - f '(x), which means the derivative of the function is odd.

Response 3:

Want to show if f(x) = f(-x) then - f '(x) = f '(-x).

f '(-x) = f '(x) by the definition of derivative

$= \lim_{h \to 0} \frac{f(x-h) - f(x)}{h}$ since f is even

Let t=-h

$= \lim_{t \to 0} \frac{f(x+t) - f(x)}{-t}$

$= -\lim_{t \to 0} \frac{f(x+t) - f(x)}{t}$

= -f '(x) as desired.

Figure 1.

BEST COPY AVAILABLE

Participants were asked to judge each response based on different criteria such as: (1) Is it convincing? Why or why not?, (2) How many points would this get on an exam? Why? (3) What response (or parts or combinations of responses) do you prefer? Why?, and (4) What response (or parts or combinations of responses) would demonstrate the best understanding? Why?

A second round of interviews was conducted to address issues of validity and scope that arose after a preliminary analysis of the first round data: how reliable are the professed beliefs, how typical are the beliefs/understandings of the individuals in the study, and how well do the five responses shown to people in the study provide access to people's beliefs and understandings about proof? A sample of participants from the first round (1 professor, 1 graduate student, and 2 undergraduates) was chosen from those who volunteered to be interviewed again. The participants were interviewed one semester after they were originally interviewed. They were shown some of the results of the study (which included comments from un-named participants in each group about each of the responses) and were asked whether the views expressed in those comments seemed typical of view of people in each of the three groups, and whether the views expressed in those comments reflected his or her own personal views.

Results

To get a rough sense of how student and teacher views compare, we can look at which responses they found deserved full credit (Figure 2), and which demonstrated the best understanding (Figure 3).

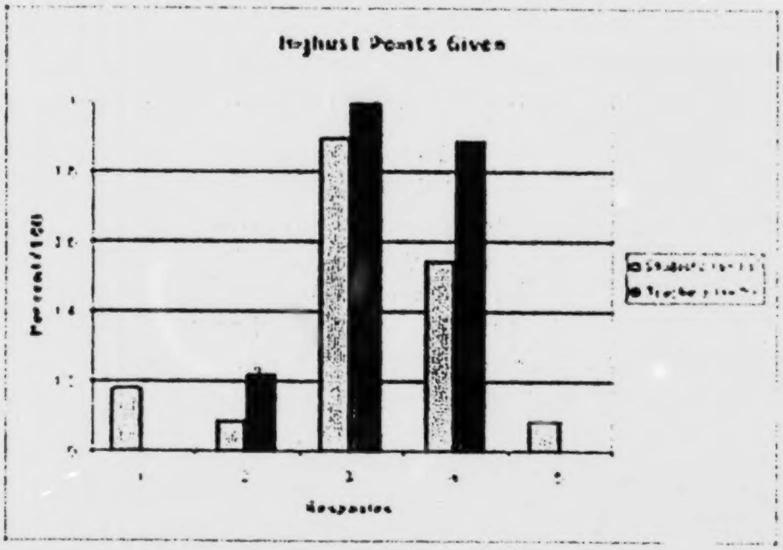


Figure 2.

ST COPY AVAILABLE

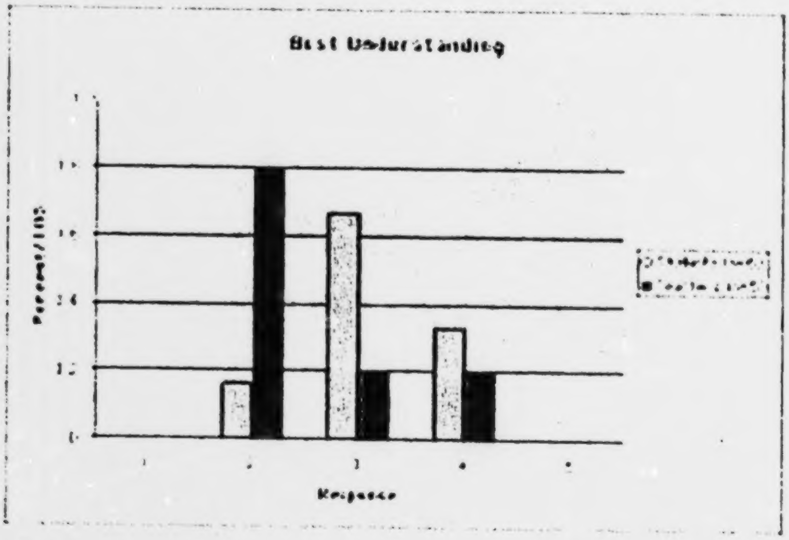


Figure 3.

Although both the students and faculty gave the textbook-proof (#3) full credit, they did so for different reasons. The students tended to like #3 because it stated the definition of derivative and used formal language.

Student CH[1]: I like that one (#3). Yeah, when I was doing the proof, I was trying to think how I could use limits. I couldn't really see how I would do the odd functions and the even functions, but yeah, this looks like something he would take as a proof.

Students thought #3 deserved full credit even if they admitted that they didn't completely understand it.

Student KY: So they proved it (#3) I think. I would give them full credit, 10 out of 10.

I: And what would your professor give?

Student KY: I also think the professor would give 10 out of 10. As long as he understands a little bit more, because I got lost a little bit.

Both students and teachers seemed to recognize #3 as the type of proof that could appear in a textbook. The students seemed to think a proof looking like a textbook-proof made it a better response than #4 or #2.

Student AN: The response 3, I gave it full credit because it makes sense. Actually I like a proof like that because it derives from something we know and gets to the point what we want. And its very useful for me to look at it. I get it right away. […] I'll remember it. And this is something that deserves credit,

BEST COPY AVAILABLE

deserves full credit because that is how the textbook does and it is really easy for a person to recognize it too. It is really useful.

Several students went so far as to say the proof was perfect. However, several teachers felt that #3 was not an ideal response.

I: And what did you think about response #3?

Prof A: (laugh) Wildly too…Way too complicated, but correct in every detail. You get points for correctness even if it is not the shortest proof. […] This is like Stewart. A long proof of something very simple.

Several people commented on some combination of #2 and #3 being preferable in some way, but again the reasons differed between students and teachers. For the teachers, #2 and #3 seemed to be saying roughly the same thing, differing only in the amount of rigor. The combination seemed to provide a combination of understanding and rigor, with the algebraic proof being more general or rigorous and the picture proof supplying a sense of understanding.

Prof B: Well, the problem from my standpoint is that it (#2) is not a proof. If I were going to use that picture, I would take it and turn it into a proof. Although if you do that, it comes down to pretty much this (#3). The two together might be the best proof from the standpoint of the students. The problem with this (#3) is that the manipulations are not transparent to them. This (#2) is a lot more transparent. The two together may make maybe a good proof for them.

Some students also saw similarities between #2 and #3. However, there was a general tendency among students to value the algebra, not just because it is more general or rigorous, but also because it, more than the picture, provided a sense of why the claim was true.

I: Ok, so now looking back over all of them, what kind of response, or combination of responses, or something that is not even here would be the best if you wanted to really show that you understood this problem?

Student CH: Probably a combination of these two (#3 and 2) Drawing the graph, you know, of this particular function and then proving it you know based on this (#3). So then you have a visual that people can see and comprehend (based on #2). And then you give the reason why (based on #3). […] I guess you could have added the two. Like, um, put the picture and then you know use the definition of the derivative and then stated that at the very end, combined them. But just the picture alone doesn't say much about the answer.

It is as if the picture is only of heuristic value; it says nothing about the truth of the claim, quite different from the teachers' point of view. Moreover, people's views about the algebraic proof appeared to be a function of understanding, as witnessed by this student who at first doesn't seem to understand #3 (and has a view similar to CH) and then seems to have an insight (and changes to a view more like that of Prof A):

Student KY: So in this case (#3), the understanding by a graph isn't necessary. I mean, not completely necessary.

I: Why do you say not completely necessary?

Student KY: Um, if you want to understand. If you, um...wow… yeah, man… I guess there are two different understandings. Like, what it does to the numbers, I guess and what it does to the picture.

[…]

I: Why do you say wow?

Student KY: I'm just seeing, I guess, the two different tracks you can take... the two different roads you can take when teaching this. I understand your studies I guess (laugh).

I: Can you say more about that?

Student KY: Well, I mean... its just that this (#3) seems more anal. And this (#2) seems a little bit more relaxed. But they both kind of show the same thing. More difficult (#3) . Less difficult (#2) . Um, um, hmm... I think if you want to go into math, I mean if you are a math major, both of them are necess… both of them.... I think you should know both of them. But this (#3) more so than this (#2).

Some teachers (especially graduate students) were reluctant to consider the textbook proof a complete proof, a picture being needed to make the proof complete.

Grad A: And if someone did this proof (#3) they would get full credit too. I might make a remark, like you could draw a picture too. So the true proof would be this (#3) and a picture. They would get extra brownie points or something.

At the same time, several teachers were reluctant to give a graphical proof full credit, because they did not think is was a real proof, or at least not a rigorous one. At least one graduate student expressed some angst over this reluctance.

Grad B: (About #2) I would like to give it full credit, but somehow I feel I'm just not allowed to. That a picture isn't good enough. It doesn't look like there is any math written down. I know, that is so stupid.

Several teachers commented on a difference between the type of proof expected in an informal situation such as office hours or a discussion section, and what they would expect on an exam.

Grad C: So, if it's the discussion section, I would definitely first try a couple of examples before actually launching into the main proof. Because sometimes proofs are proofs. They are fine, except they are not very illuminating. And a better way of actually convincing yourself that a result is actually true is to work hands on an example. Which might not give you a clue as to what the most general rigorous proof might be, but at least it convinces the student that ok, at least what we want to prove is correct as we can see by examples. So if it is the discussion section, I would first start maybe with a little bit of this and then actually ask how to prove it in the general situation. In an exam of course that is not going to be… that's the place where you just write down what is correct.

And some teachers (compared with no students) preferred the graphical proof over the textbook proof, not just for demonstrating understanding, but even as an answer on an exam.

Prof C: #2 I would simply accept, even if it is not rigorous. Somehow I think it is so useful that a student could think in these terms, I think, for his or her career, I think it is more important than their ability to write down these polynomials. So #2 I think I would simply accept without many questions.

Perhaps significantly, Prof R was the only professor in the study who had never taught freshman calculus (he almost exclusively teaches advanced graduate courses.)

The teachers distinguished between a formal proof which is not "very illuminating" (Grad C) and an intuitive proof which is more "transparent" (Prof B). This seems to indicate the existence of some idea, which in this case is more closely represented by a picture like in #2 than by #3. The intuitive proof is valued more for the purpose of convincing oneself of the a claim (ascertaining, in Harel and Sowder's language) and the formal proof is valued for its ability to establish the truth the language and level of rigor expected of a particular community (persuading, to Harel and Sowder). Consider the comments of this professor after he generated two different proofs for the odd/even question, one with a picture and one with the chain rule.

I: So now you have two different approaches. How do those compare in terms of how convinced you are?

Prof A: Oh, the first one (picture proof) convinces me completely that it is right, it is right. The second one (chain rule proof) is how you present it if you want to convince somebody else. It doesn't have… (sigh, look to side) your currency. My currency is kind of… my currency is like pictures. But the general currency that works for everybody is a formula.

This professor, by the way, preferred the chain rule proof on an exam to a picture because he thought it was easier to grade.

The teachers' comments seem to suggest that what they believe to be a true proof is the connection between the idea and the representation of that idea, but what counts as an appropriate representation is a function of their audience. Even though the teachers made distinctions between different types of proofs, they clearly see how they are related "If I were going to use that picture, I would take it and turn it into a proof. Although if you do that, it comes down to pretty much this (#3)" (Prof B). And some even went so far as to say the picture was necessary for the proof to be complete, "So the true proof would be this (#3) and a picture." (Stud KY) Many teachers favored an algebraic representation (Prof B, Grad B, Grad C) at least in the context of a course. However, I claim that what actually makes them call the algebraic proof (#3 or #4) a proof is that they can connect it to their idea of a proof.

In contrast, the students—many of which do not appear to really understand a proof like #3—are not able to connect the idea to the representation. Another way of saying they do not understand the proof is to say they do not have the idea, something that completely convinces them of the veracity of the claim. So they have only the formal algebraic proof to value as proof. " But just the picture alone doesn't say much about the answer." (Stud CH) So unlike the way in which teachers view the algebraic proof—that its authority derives from a connection to a main idea, to the students, the algebraic proof appears to stand alone, almost disconnected from the idea: "the understanding by a graph isn't necessary. I mean, not completely necessary." (Stud KY) This is a subtle, but I think fundamental, difference in perspective because while on the surface things may look the same (e.g. both value the algebraic proof on an exam) the surface behavior belies significant differences in not only how well people understand, but possibly also what they take understanding to be.

References

Alibert, D., & Thomas, M. (1991). Research on mathematical proof. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 215-230). Dordrecht: Kluwer.

Balacheff, N. (1991). Benefits and limits of social interaction: The case of teaching mathematical proof. In A. Bishop, S. Mellin-Olsen, & J. Van

Dormolen (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 175-192). Dordrecht: Kluwer.

Elby, A. (1999). Another reason that physics students learn by rote. *American Journal of Physics*, Physics Education Research Supplement, 67(7 Supplement 1), S52-S57.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinksy (Eds.), *Research in collegiate mathematics education* (Vol. 7, pp. 234-283). Providence, RI: American Mathematical Society.

Hofer, B. (1994, August). *Epistemological beliefs and first-year college students: motivation and cognition in different instructional contexts.* Paper presented at the American Psychological Association annual meeting, Los Angeles.

Hoyles, C., & Healy, L. (1999). *Justifying and proving in school mathematics* (End of Award Report to ESRC). London: Institute of Education, University of London.

Moore, R. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, *27*, 249-266.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning.* New York: Macmillan.

Stewart, J. (1998). *Calculus: Concepts and contexts*. Pacific Grove, CA: Brooks/ Cole Publishing.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking*. Dordrecht: Kluwer.

Note

[1] Transcript conventions: [...] means break in the transcript, parenthetical remarks indicate gestures, mostly used to indicate which response the participant is referring to.

METAPHORS OF THE NOVICE: EMERGENT ASPECTS OF CALCULUS STUDENTS' REASONING

Mike Oehrtman

Abstract: The purpose of this poster is to report the findings of recent research on the refinement of students' metaphorical reasoning in a second semester undergraduate calculus class. We characterize changes observed in the nature of students' reasoning about concepts learned in previous courses and the features that allowed those changes to occur.

Theoretical Framework

Building on the extensive work of Eleanor Rosch on prototype effects and human categorization, George Lakoff has laid substantial foundations for a theoretical perspective on the role of metaphor in human understanding (Lakoff, 1987). Specifically, humans share a large domain of collective biological capacities and physical and social experiences. Abstract concepts get their meaning from preconceptual structure that is metaphorically projected from these experiential domains. Recently, George Lakoff and Rafael Núñez have begun developing a "mathematical idea analysis" of foundational concepts such as arithmetic, the real numbers, limits, and continuity (Lakoff & Núñez, 2000).

This idea analysis has focused on detailing sufficient embodied structure for the precise understanding of an expert and has not addressed the emerging understanding of someone who has only recently learned the concepts. For this purpose, we apply John Dewey's theory of technological inquiry (Hickman, 1990; Prawat & Floden, 1994) to our subjects' metaphorical reasoning. Systematic inquiry allows for a fortuitous coming together of structures emergent in the learner's prior knowledge and in the problem situation. The resulting dialectic interaction effects changes in each, resulting in the formation of new knowledge. Within this theoretical perspective, we focus on the learner's evaluation of their own metaphorical reasoning in a specific context and on the resulting changes in their understanding.

Methods and Data

Written data were collected from students in a second-semester calculus course at a large southwestern public university. Over 100 students completed surveys at both the beginning and end of the course asking in-depth conceptual questions about limits and rate of change (concepts covered in the previous semester). Changes in metaphor use were explored through responses from 20 to 30 students on weekly writing assignments and detailed follow-up correspondence. These questions dealt with subject matter being taught at the time but that also involved application or generalization of the concepts of limit or rate of change. Prompts for questions used on all instruments stated a non-routine fact and asked the subjects to provide a discussion of their understanding of the relevant concepts. Interviews with students in the first-semester course and with advanced mathematics undergraduate and graduate students were used to provide reference points for progression of our subjects' understanding.

Results and Conclusions

Analysis of the data indicates that situations revealing inconsistencies in students' understanding were likely to prompt them to adopt a wondering stance and engage in thought experiments about their application of metaphor. Strong students were then able to identify aspects that failed to provide appropriate structure for the mathematics, and they would look for other ways to understand the concept. The metaphors used by these second-semester students tended to be more sophisticated versions of metaphors used by first-semester students and were similar to (although less precise than) those used by advanced mathematics students. We conclude that students' active and productive use of metaphors is important in long-term concept development. This is facilitated by supporting a student in systematic inquiry of mathematically rich contexts that challenge their routine reasoning patterns.

References

Hickman, L. A. (1990). *John Dewey's pragmatic technology.* Bloomington: Indiana University Press.

Lakoff, G. (1987). *Women, fire, and dangerous things: What categories reveal about the mind.* Chicago: University of Chicago Press.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* New York: Basic Books.

Prawat, R., & Floden, R. (1994). Philosophical perspectives on constructivist views of learning. *Educational Psychology, 29*, 37-48.

Socio-Cultural Issues

AN EXAMINATION OF THE INTERACTION PATTERNS OF A SINGLE-GENDER MATHEMATICS CLASS

Amy Burns Ellis
San Diego State University and University of California, San Diego
aellis@sunstroke.sdsu.edu

Abstract: This study presents a description of the nature of one single-gender mathematics class in a public-school setting. Classroom observations of an all-female algebra class recorded the nature and frequency of student-teacher interactions. Two surveys explored attitudes and opinions about mathematics and the all-girls setting, and nine students representing a variety of attitudes and ability levels were selected for in-depth interviews. The results indicated that the classroom climate differed from typical coeducational mathematics classrooms. The students regularly shared the teacher's attention, identified closely with their peer group, and demonstrated a strong preference for a cooperative working environment.

Numerous studies have suggested that the single-gender mathematics classroom may positively affect girls' achievement levels and attitudes towards mathematics (AAUW, 1998; Gwizdala & Steinback, 1990; Lee & Byrk, 1986; Lee, Marks, & Byrd, 1994; Rowe, 1988). However, little is known about why such benefits occur and which students benefit most strongly. More importantly, few have investigated what specific elements of a single-gender mathematics classroom result in benefits to females. This paper addresses one such element by examining the nature of student-teacher interactions in the all-girls' setting. Understanding the nature of single-gender classroom interactions could provide insight into why some students experience improved performance and attitudes towards mathematics. The specific questions addressed here are: How does the all-girls' setting differ from a coeducational classroom, if at all? What sorts of interactions occur in the classroom among the students, and between the students and the teacher? Which students experience success in the single-gender setting? This study provides a description of the interaction patterns in one all-girls mathematics class and an analysis of which students most strongly benefited from the single-gender environment.

Background and Theoretical Perspective

Results from studies conducted over the past twenty years are not strongly conclusive as to whether single-gender settings are more effective than coeducational settings in promoting mathematics achievement for girls (Lee, 1998). One concern with characterizing the effects of single-gender education involves the difficulty in controlling for influences not directly related to the single-gender environment. The majority of studies about single-gender education have been conducted in private schools, and some question whether the effects found in parochial or independent institutions can be transferred to the public-school setting. In addition, the academic and developmental consequences of attending a single-gender school may vary for different groups of students (Riordan, 1998). However, despite the lack of conclusive evidence about the effects of single-gender schooling, many studies do reveal information regarding the relationship between the single-gender environment and girls' attitudes, participation levels, and achievement levels in mathematics.

Achievement and Participation

Studies examining the effect of the all-girls setting on mathematics achievement have shown mixed results. While some have revealed minimal improvements in mathematics achievement (Marsh, 1989; Smith, 1986), others have found that the single-gender setting can positively affect achievement levels (Lee, Marks, & Byrd, 1994; Marsh, 1989; Rowe, 1988). In addition, researchers hypothesize that the single-gender environment may encourage girls to participate more in mathematics. Leder and Forgasz (1994) examined single-gender mathematics classes in a coeducational public high school and found that females in these classes enrolled in mathematics courses for the following year at a rate equal to males, while their counterparts in coeducational classes enrolled at lower rates. Similarly, Lee and Byrk (1986) found that girls in single-gender Catholic schools enrolled in more mathematics classes than those in coeducational Catholic schools. Females in single-gender mathematics classes also participate at higher rates in class than girls in coeducational settings (Gwizdala & Steinback, 1990; Leder & Forgasz, 1994).

Affective Issues

While evidence for the effect of the single-gender setting on achievement levels is conflicted, the evidence for effects on affective issues is stronger. The single-gender setting has been found to provide positive benefits to girls' confidence levels in mathematics (Cairns, 1990; Rowe, 1988). Girls who attend single-gender schools have also been found to be less likely to consider science and mathematics a solely male domain (Colley, Comber, & Hargreaves, 1994; Lee, Marks, & Byrd, 1994). Studies also report that girls in single-gender schools are more interested in mathematics than their coeducational counterparts (Gwizdala & Steinback, 1990; Lee & Byrk, 1986). Interview studies reveal that girls in the single-gender environment report an improved working atmosphere, a close identification with peer groups, improved concentration, and fewer feelings of intimidation (Rowe, 1988).

Theoretical Framework

A feminist perspective derived from *Women's Ways of Knowing: The Development of Self, Voice, and Mind* (Belenky, Clinchy, Goldberger & Tarule, 1986) guided the development of this study. Belenky et. al explored how women come to know by describing five stages of knowing: silence, received knowing, subjective knowing, procedural knowing, and constructed knowing. This perspective helps focus research on the examination and exploration of the female experience through a female lens. One reason for considering a feminist orientation is that women have previously been studied primarily in relation to men. Taking the male experience as the universal standard, research focused on the characteristics females lacked that resulted in lower levels of participation and achievement in mathematics (Campbell, 1995). A feminist perspective instead considers girls' learning in its own right: rather than focusing on ways to change girls, researchers can explore ways to change mathematics instruction to better fit girls' needs.

Methods and Data Sources

Data collection occurred at a large urban public high school. The school created a program consisting of two single-gender academies, one for males and one for females. The academies were open to all ninth grade students, and participants joined on a voluntary basis. The female academy consisted of twenty-one ninth grade students who attended every class together as one group. The sole exception was for the mathematics class: twenty students attended an algebra class while one student studied geometry independently.

The study consisted of three major components. The first was a series of twenty-two classroom observations of the all-female algebra class over a period of twelve weeks. The data consisted of field notes in conjunction with student-teacher interaction codes. The interaction codes recorded the nature and the frequency of interactions that occurred between the students and the teacher (see Table 1). The last five codes were adapted from Stallings, Needels, and Sparks (1987), and allowed the researcher to study the quality of classroom discourse. The remaining codes emerged while observing this particular classroom, and were developed to better capture the full nature of interactions that occurred in this setting. The second component was a set of two surveys administered to the female algebra students at the beginning and the end of the observation period. The surveys gathered data on the students' attitudes towards mathematics and the single-gender setting. The third component of the study was a set of in-depth interviews addressing beliefs about mathematics with a subset of nine students. The subjects were chosen to represent a broad cross-section of attitudes towards mathematics, behaviors in the classroom, and academic achievement.

Results

Interaction Patterns

Table 2 displays the class interaction totals for twenty-two days. The students demonstrated a wide variety of participation levels. A pattern in which only one or two students dominate the conversation at the expense of the rest of the class did not emerge. Half of the students interacted more than fifty times during the observation

Table 1. Interaction Codes.

Code	Meaning
A	Student answers a teacher-generated question.
A+	Student answers a teacher-generated question and receives praise.
AC	Student answers incorrectly and is corrected.
*	Student initiates a comment to the teacher.
*A	Student initiates a question and the teacher provides an answer.
*I	Student initiates a question and receives no response.
?	Teacher asks a question requiring students to recall facts.
√	Teacher checks for understanding about mathematical procedures.
H	Teacher asks students to interpret, analyze, or evaluate.
-	Teacher reprimands behavior.
+	Teacher praises students' responses or behaviors.

period, an average of more than twice a day. The dramatic differences in participation rates between the students were mainly due to the fact that all participation in the class was voluntary.

An analysis of the data on a day-to-day basis revealed that ten out of the twenty students experienced at least one day in which they interacted most often with the instructor. For eighteen out of twenty-two class sessions, three to six students interacted in roughly similar amounts with the teacher. This group of students who interacted most often changed from day to day. Over the course of the semester, the data revealed a trend of students sharing the instructor's attention.

While Ana received the highest number of total interactions with the instructor, this is misleading. Ana did not dominate the discussion. The majority of her interactions (163 out of 247) consisted of her answering questions and receiving no feedback. Ana did not receive the greatest number of direct questions or comments from the instructor, and more than fifty percent of her responses occurred only after waiting for another student to respond.

The manner in which the students interacted with one another also differed from behavior in typical coeducational classes. The students' behaviors and interview responses revealed a strong preference for a cooperative work environment. Rosa's interview comment typified many of the students' reactions: "I like it best when we

Table 2. Interaction Totals for 22 Days.

Student	A, A+, AC	*, *A, *I	?	-	+	√	H	Total
Class	0, 0, 0	0, 0, 0	728	238	6	6	0	978
Adelle	59, 8, 6	0, 31, 0	3	2	0	0	0	109
Ana	163, 29, 11	1, 21, 9	3	8	2	0	0	247
Ariana	50, 5, 15	0, 48, 4	8	23	0	0	0	153
Betty	3, 0, 0	0, 3, 1	0	25	0	0	0	32
Brigitte	2, 0, 0	0, 1, 0	0	1	0	0	0	4
Estrella	4, 0, 2	0, 5, 2	1	19	0	0	0	33
Jamie	49, 11, 6	0, 20, 5	2	19	0	0	0	112
Jennifer	34, 7, 10	3, 30, 7	4	53	0	0	0	148
Luisa	88, 5, 4	0, 9, 0	0	1	0	0	0	107
Marba	0, 0, 0	0, 0, 0	0	9	0	0	0	9
Marcy	29, 6, 0	0, 2, 1	0	1	0	0	0	39
Mindy	66, 6, 6	0, 0, 0	2	2	1	0	0	83
Ming	30, 4, 3	2, 18, 2	8	10	0	0	0	77
Nadine	17, 1, 1	0, 16, 3	0	13	0	0	0	51
Peg	1, 1, 0	0, 0, 0	0	0	0	0	0	2
Rosa	2, 0, 0	0, 2, 0	0	1	0	0	0	5
Rupa	23, 0, 1	0, 3, 0	0	1	0	0	0	28
Stacy	5, 0, 1	1, 11, 0	1	7	1	0	0	27
Sally	9, 2, 1	1, 20, 7	6	17	2	0	0	65
Tamara	12, 0, 1	0, 3, 0	1	15	0	0	0	32
Total	646, 85, 68	8, 243, 41	768	464	12	6	0	2,341

work in groups". Mindy explained that "When working with other people, I can help the other people so they know how to do the problem." The students shared resources equally and respected their peers' autonomy in the learning process, and displayed a lack of appreciation for competitive instructional strategies. The instructor repeatedly tried to motivate the class through competitive measures which had met with success in the boys' class. For example, when attempting to convince the students to participate in a district-wide exam preparation, he issued challenges such as "Let's see if you can be more cooperative than the boys were today." The students did not respond to these tactics and did not display any interest in comparisons with the boys' class.

The students also displayed a willingness to provide assistance to one another, to ask questions, and to admit confusion. Rupa explained that "You are not just pressured, like if I say something, the boys will say that's a stupid question. In here, I just ask away." This response typifies the students' comments on the surveys and during the interviews. By the end of the term, the students had developed a strong group identity. They regularly communicated as an entire group rather than as a set of individuals, which was reflected in their language: the students used "we" more often than "I" when responding to the teacher. In addition, all nine students interviewed discussed the nature of the classroom using "we" rather than "I". For example, Ariana described a typical interaction with the instructor: "He goes over it too many times and then we already know it, and we try to tell him that we know it, and then he gets, like, really cold."

During the observation period, the instructor made a total of twelve positive comments, such as complimenting a student on her ability to explain a problem or praising her efforts. In contrast, he made 464 negative comments, such as disciplining students publicly or berating them for not trying harder. The supportive, safe atmosphere described by the students was not fostered by the instructor's behavior.

Who Benefited

Students who thrived in the class were those who left the semester with positive attitudes about mathematics and who demonstrated a growth of mathematical understanding. A subset of five students (Rosa, Rupa, Peg, Mindy, and Ana) both achieved at high levels and reported positive attitudes about mathematics on the exit survey. There are few similarities among the interaction patterns of these students. The most striking similarity is that none of the five students received a large number of negative interactions, coded as -, from the instructor. On average, negative comments from the teacher comprised 17% of a student's daily interactions. For this group of five, negative comments comprised only 3% of their total interactions with the instructor. These students also received a large number of positive comments from their peers.

Here the similarities end. Some of the five interacted often with the instructor and others did not. Some behaved enthusiastically in class while others were more reserved. Some students reported an enjoyment of the all-girls aspect of the class, while others reported that they preferred the coeducational setting. These five students formed a diverse group, with different experiences and learning preferences.

Discussion and Conclusions

Studies investigating interaction patterns in coeducational classrooms reveal a trend of unequal attention. The bulk of the teacher's attention typically falls on a small handful of students, usually male, who dominate the classroom throughout the entire year (Bailey, 1993; Becker, 1981; Campbell, 1995; Eccles, 1989). This pattern did not occur in the all-girls' class. The students shared the instructor's attention, and the group of students who interacted most often with the instructor changed from day to day. This more equitable distribution of attention was not facilitated by the teacher. He did not regularly call on individual students, instead preferring to speak to the entire group. When one student responded, he frequently followed up by addressing the remainder of his questions to that particular student. Thus it was the students and not the instructor who determined the pattern of interactions for any given day.

The students' readiness to participate often in class, ask questions of the teacher and their peers, and contribute to the class discussion parallels results from similar studies of student behavior in single-gender mathematics classes (Leder & Forgasz, 1994; Lee & Byrk, 1986). Their willingness to provide assistance to one another was also noticeably different from typical behaviors in a coeducational classroom (Bailey, 1993). The helpful and supportive attitude displayed by the students might have allowed some members of the class to feel comfortable enough to participate more as the term progressed. In general, a classroom atmosphere of caring behavior could be one reason why girls in single-gender schools typically experience higher levels of self-esteem, higher confidence levels, and more positive attitudes towards mathematics than their coeducational counterparts.

One of the most interesting features of this environment was the fact that the students demonstrated the above behaviors without explicit encouragement from the instructor. In several cases the teacher behaved in ways that could have discouraged students from taking risks in class. For example, he occasionally laughed at those who made mistakes, but the students did not repeat this behavior. He also ignored students' requests for assistance 41 times, an average of about twice per day, but the students themselves actively and willingly helped one another. The single-gender setting may have reinforced these positive behaviors, but the instructor's actions did not appear to encourage the development of a supportive environment.

The classroom atmosphere created a learning environment that might be more beneficial to some students than to others. Students who prefer a cooperative work environment, who identify closely with their peers, and who are content with supporting one another's learning would be most likely to appreciate this all-girls' setting. Students who prefer a competitive environment would probably not thrive in this particular setting. This single-gender mathematics class created an unusual opportunity for students who might have been unsure of their abilities or wary of speaking up in class. It allowed for a safe, caring environment in which students could begin to take risks in the classroom, and provided a more equitable distribution of attention and a uniquely collaborative and supportive atmosphere.

References

American Association of University Women. (1998). *Separated by sex: A critical look at single-sex education for girls*. Washington, DC: AAUW.

Bailey, S. M. (1993). The current status of gender equity research in American schools. *Educational Psychologist, 28* (4), 321-339.

Becker, J. R. (1981). Differential treatment of females and males in mathematics classes. *Journal for Research in Mathematics Education, 12* (1), 40-53.

Belenky, M. F., Clinchy, B. M., Goldgerger, N. R., & Tarule, J. M. (1986). *Women's ways of knowing: The development of self, voice, and mind.* New York: Basic Books.

Cairns, E. (1990). The relationship between adolescent perceived self-competence and attendance at single-sex secondary schools. *British Journal of Educational Psychology 60*, 207-211.

Campbell, P. B. (1995). Redefining the 'girl problem' in mathematics. In W. G. Secada, E. Fennema, & L. B. Adajian (Eds.), *New directions for equity in mathematics education* (pp. 225–241). New York: Cambridge University Press.

Colley, A., Comber, C. & Hargreaves, D. J. (1994). School subject preferences of pupils in single sex and co-educational secondary schools. *Educational Studies, 20* (3), 378-85.

Eccles, J. S. (1989). Bringing young women to math and science. In M. Crawford & M. Gentry (Eds.), *Gender and thought: Psychological perspectives* (pp. 36–58). New York: Springer-Verlag.

Gwizdala, J., & Steinback, M. (1990). High school females' mathematics attitudes: An interim report. *School Science and Mathematics,* 90 (3), 215-222.

Leder, G. C., & Forgasz, H. J. (1994). *Single-sex mathematics classes in a co-educational setting: A case study.* (Report No. SE-054-399). Paper presented at the annual meeting of the American Educational Research Association, Orlando, FL. ERIC Document Reproduction Service No. ED 372946.

Lee, V. (1998). Is single-sex secondary schooling a solution to the problem of gender inequity? *Separated by sex: A critical look at single-sex education for girls*, (pp. 41 – 52). Washington, DC: AAUW.

Lee, V. E., & Byrk, A. S. (1986). Effects of single-sex secondary schools on student achievement and attitudes. *Journal of Educational Psychology, 78* (5), 381-395.

Lee, V., Marks, H., & Byrd, T. (1994). Sexism in single-sex and coeducational independent secondary school classrooms. *Sociology of Education, 67* (April), 92-120.

Marsh, H. (1989). Effects of attending single-sex and coeducational high schools on achievement, attitudes, behaviors, and sex differences. *Journal of Educational Psychology, 81* (1), 70-85.

Riordan, C. (1998). The future of single-sex schools. *Separated by sex: A critical look at single-sex education for girls* (pp. 53-62). Washington, DC: AAUW.

Rowe, K. J. (1988). Single-sex and mixed-sex classes: The effects of class type on student achievement, confidence and participation in mathematics. *Australian Journal of Education, 32* (2), 180-202.

Smith, S. (1986). *Separate tables? An investigation into single-sex settings in mathematics.* London: Her Majesty's Stationary Office.

Stallings, J. A., Needels, M., & Sparks, G. M. (1987). Observation for the improvement of classroom learning. In D.C. Berliner & B.V. Rosenshine (Eds.), *Talks to teachers* (pp. 129–158). New York: Random House.

THE ROLE OF THE INTELLECTUAL CULTURE OF MATHEMATICS IN DOCTORAL STUDENT ATTRITION

Abbe H. Herzig
Rutgers University

Over the past several decades, the percentage of bachelors, masters, and doctoral degrees awarded in mathematics has been declining (National Center for Education Statistics, 1997). The number of freshmen expressing an interest in majoring in mathematics has decreased from 4.6% in 1966 to only 0.6% in 1988 (Green, 1989). Increasing numbers of students who begin studying science and mathematics switch to other majors (Seymour & Hewitt, 1997). The picture is even more bleak for women and for Latina/os, African Americans, and Native Americans, whose rate of participation in mathematics decreases at higher educational and professional levels (National Science Foundation, 1998).

There are several reasons why the loss of people from mathematics is a concern. These reasons include the effects on those individuals (notions of equity or justice), the discipline of mathematics (notions of diversity, efficiency, and power), and society at large (notions of economic survival).

First, as a democratic society, we are obligated to ensure that all citizens have equitable opportunities. Increased access to technical fields has many obvious benefits for individuals and groups who are currently underrepresented in those fields. Mathematics serves as an important gateway to many careers in our society, many of which are sites of affluence, influence, and power. If a more diverse range of people participates in mathematics, those people will have increased opportunities to avail themselves of those benefits. This view is based on a conception of equity as justice (Secada, 1989).

The second reason for concern is the inherent weakness of any system that is constructed on a narrow base. The strategy of using diversity to ensure long-term vitality has served well in a variety of natural, social, and economic systems; conversely, systems that fail to diversify are often unstable and vulnerable. Given the scientific challenges facing science in our society, a diversity of perspectives is essential to its continued success (Wilson, 1992). By broadening mathematics to include a more diverse range of thinkers, the discipline of mathematics will be enriched by an expanded range of mathematical thought, which will then help the profession of mathematics to flexibly meet the increasing challenges that are being imposed by society's increasing quantitative sophistication.

In an argument related to both the equity and diversity perspectives, we must consider the power of people who currently participate in mathematics. Mathematics has high prestige as a tool that help us understand and manipulate our natural world. Concentrating this prestige in the hands of an elite few confers substantial power on that elite, which allows them to maintain hegemony over mathematical development. As it counteracts (at least partially) the vulnerability described in the previous paragraph, this power also acts to inhibit equitable access to mathematics, and to limit the types of mathematical development that are valued and pursued. Alternatively, if more diverse individuals were to be included as full participants in mathematics, then the groups those individuals are seen to represent would share in that epistemological power, which might have equity implications reaching beyond the domain of mathematics.

A final argument is an economic one, which Secada (1989) calls enlightened self-interest. Society's needs for scientific leadership and a scientifically literate workforce are increasing. If we fail to educate large proportions of our increasingly-diverse society to participate in our increasingly-technological and quantitative economy, then we face the twin economic risks of a large pool of unemployable people and an undersupply of trained workers. In addition, if large numbers of people feel outcast from mathematics and science, then they are less likely to support critical societal investments in mathematical and scientific development. As our need for mathematically-literate workers and citizens increases, our economic security will depend on how well we have educated other individuals and incorporated them into full participation in the economy of the 21st century.

Research on attrition and persistence typically studies mathematics combined with the natural or physical sciences (exceptions include Becker, 1990; Stage & Maple, 1996). However, mathematics is a very different discipline by nature than the sciences (Biglan, 1973b). In some ways, this is appropriate—mathematics serves an important function as the language of science, and much mathematics has been developed to answer questions in science. Mathematics shares features with science such as high consensus on research paradigms and their location on the "pure-applied" continuum (Biglan, 1973a). This connection with the natural sciences is reflected in the emphasis on mathematics as an objective truth, which mathematicians work to discover (Maddy, 1990; Steen, 1999).

Mathematics is also one of the humanities. Mathematics is often appreciated for its aesthetic properties, and like other liberal arts, it represents a way of thinking that acts across fields of knowledge. In addition, graduate research in mathematics may be more similar to research in the humanities than to research in science. Science students generally begin research early in graduate school, which is rarely the case in humanities (Golde, 1999; Tinto, 1993) or in mathematics. Research in mathematics and the humanities is more likely to be individual and isolated, compared with a high degree of collaboration in the sciences (Becher, 1989; Golde, 1999; Nerad & Cerny, 1993).

Students' interests in mathematics, their experiences in the discipline, and the factors leading them to leave might be quite different than the reasons leading them into or out of the sciences. It is therefore compelling to study attrition specifically from mathematics.

Previous research on attrition has focused on characteristics of people who leave or on characteristics of the institutions of study (see Figure 1). Few studies have looked at the disciplines themselves, at the curricula, pedagogy, research projects, and other windows students have into the disciplines. Does the school experience encourage students in mathematics? What does it mean to do mathematics? What is mathematics? What are the dominant discourses of the discipline? Are multiple mathematical perspectives welcome? These questions inform what I call the Intellectual Culture of Mathematics (ICM).

This research project is a case study of one university-level mathematics department, focusing on the role of the ICM in the departure of graduate students. Graduate students are initially committed to their interest in mathematics, and have a demonstrated level of knowledge and mathematical ability. Based on the length of time and intensity with which they have participated in mathematics, their insights about their experiences may be particularly informative about the ICM. This study has two goals: to develop the construct of the ICM, and to investigate its role in doctoral student attrition. Hopefully, once this construct is better defined through this study, it will be useful in the future for studying more specific questions about attrition from and engagement with mathematics, particularly issues of gender, race, and class.

Methods

The results reported here are part of a larger study, based on open-ended interviews with 23 graduate students and 15 faculty members in one mathematics department, at a large, research-based public university.

Following a feminist framework, the interviews were open conversations. In addition to asking for specific information and facts, Anderson and Jack (1991) call for the need to invite participants to discuss "the web of feelings, attitudes, and values that give meaning to activities and events" (p. 12) and to give them "the space and the permission to explore some of the deeper, more conflicted parts of their stories" (p. 13). These subjective and personal aspects of participants' stories can give valuable insights about their experiences within mathematics. However, encouraging participants to talk about this subjective realm can be painful, given the nature of this research program (some participants were discussing what they perceive to be failure). To allow participants to avoid topics that were painful, while still leaving the interviews open to discussion of personal experiences and the meanings derived from them, each participant received an outline of discussion topics several days before her[1] scheduled interview, so that she had the opportunity to think about those topics, delete anything that she did not wish to discuss, and add topics that she considered relevant (Burton, 1999). Participants were also invited to return for as many meetings as they wished, depending on how much they had to say.

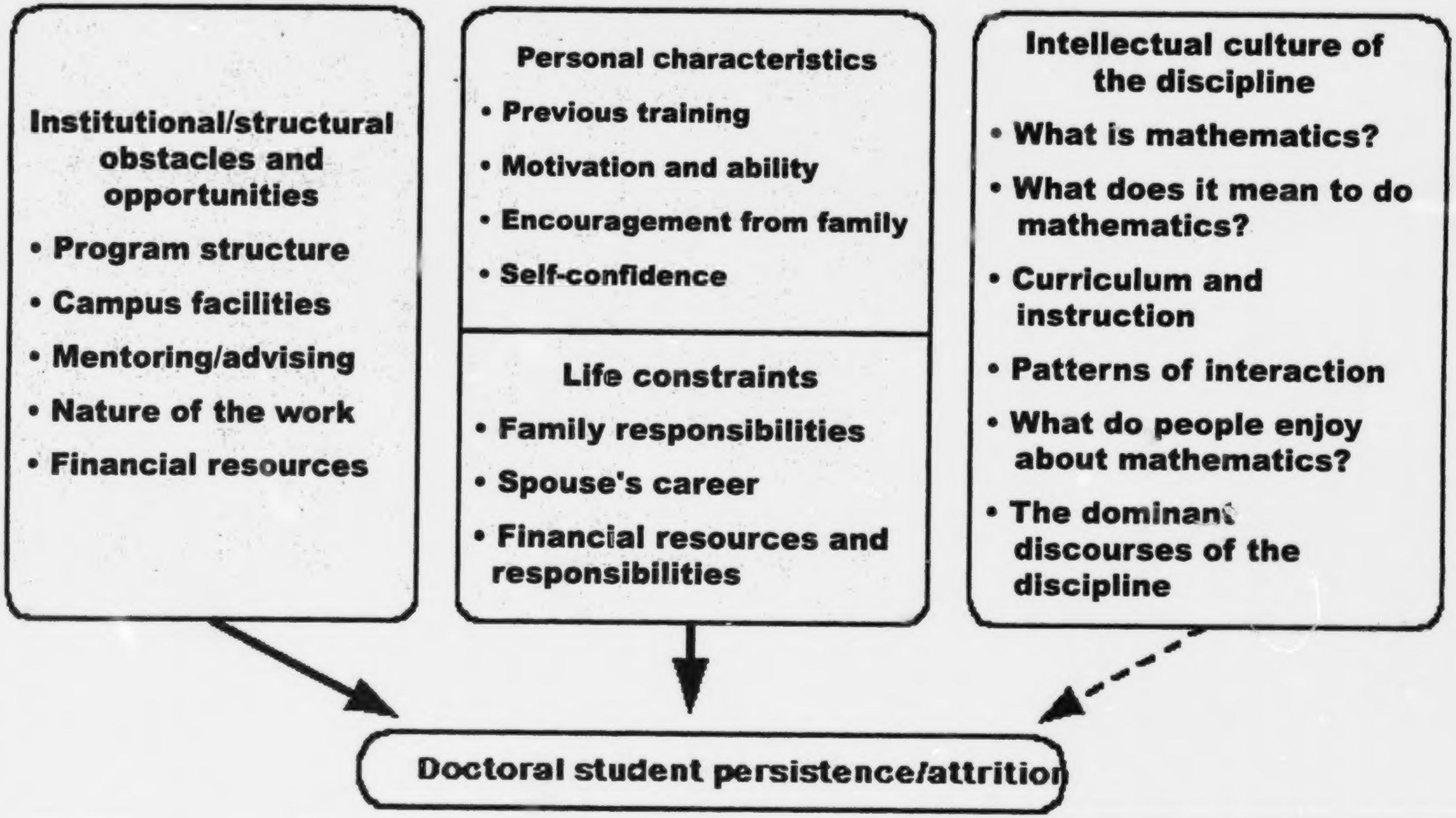

Institutional/structural obstacles and opportunities
• Program structure
• Campus facilities
• Mentoring/advising
• Nature of the work
• Financial resources
Personal characteristics
• Previous training
• Motivation and ability
• Encouragement from family
• Self-confidence
Life constraints
• Family responsibilities
• Spouse's career
• Financial resources and responsibilities
Intellectual culture of the discipline
• What is mathematics?
• What does it mean to do mathematics?
• Curriculum and instruction
• Patterns of interaction
• What do people enjoy about mathematics?
• The dominant discourses of the discipline
Doctoral student persistence/attritio

Figure 1. Factors affecting attrition and persistence of doctoral students in mathematics and science.

615

Interviews took place in a private office (all faculty were interviewed in their own offices). Interviews were audiotaped and transcribed.

The Participants

Graduate student participants had completed between one and seven years of study, including students who had left or were about to leave the doctoral program, and students who were planning to or about to complete their Ph.D.s. Interviews explored their initial experiences, interests, and goals in mathematics, reasons for choosing graduate study, expectations about their schooling, experiences in undergraduate and graduate school, conceptions of mathematics, and decisions about continuing or leaving. Their interviews ranged from 1.25 to almost 6.5 hours (in up to 3 sessions).

Faculty participants represented most major research areas in the department, at a range of professional levels, and included some who teach the introductory graduate courses which most graduate students take. Interviews explored their initial interests in mathematics, conceptions of mathematics, how they do mathematics, their interests in mathematics, experiences as mathematicians, and their roles as professors and advisors to graduate students. Faculty interviews ranged from 45 minutes to just over 2 hours.

This paper reports on 3 cases from the larger study, chosen because many of the issues they raise are representative of the sample overall, and because the contrast among the way they discuss these issues is illuminating. Lee was a graduate student who left the doctoral program after several years, following several unsuccessful attempts to pass her qualifying exams. She was interviewed about 6 months after leaving the program for a non-mathematical job. Chris had completed all of her coursework and exams, and was beginning her thesis research at the time of her interview. She reports that she came close to leaving the program several times, and that she is still not sure she will finish. Terry is a senior professor who has been in the department for over 30 years, and teaches an introductory graduate course that is taken by many graduate students in their first or second year.

Data Analysis

In the larger study, codes were developed based on a comprehensive review of the interview transcripts, to reflect things the participants discussed that were relevant to the intellectual culture of the mathematics department. These codes were then applied to the transcripts. A sample of data (one graduate student interview and one faculty interview) were coded by an independent coder to check for reliability. To confirm that participants' meanings are not distorted by my interpretations, all participants have been invited to review and comment on these findings.

For this paper, excerpts from the interview transcripts from Lee, Chris, and Terry were chosen to reflect common issues and to provide high contrast among different views on these issues.

The Intellectual Culture

Three major categories of issues raised by the participants will be discussed here: their conceptions of mathematics, their beliefs about teaching and learning mathematics, and the nature of the work of doing mathematics.

For Terry, mathematics is all about solving problems, from which he derives great pleasure. There is a common discourse among the faculty, and among many students, about the emphasis on abstraction and on deductive logic as the foundation of mathematics. It was alternatively described as a game, a quest, a puzzle, or something aesthetic.

> Terry: Well to me it's an intellectual exercise. I think it's something that's sort of fun to do, to think about.
>
> Chris: I think in some sense that it's a very pure sort of reasoning. You know you're not dealing with actual objects in the world which can be a little lopsided or, or fuzzy or whatever. It's very much just sort of well, let's define this set of this bunch of things and then let's see what happens when we switch them this way. But it's just very much sort of making up some structure and then just purely from what you've made up, reaching conclusions. It's not so much observational or just what you happen to think. It seems very much to me to be I don't know, in some sense very true, you know, because you're not actually dealing with anything real where truth doesn't really make sense a lot of the time.
>
> Lee: It was a game. It was my first real introduction to proofs and having had a logic course in the philosophy department the year before, this stuff was just you know just bang bang bang, you take the definition and you prove what you want to prove because they're all so easy.... And it was kind of cool.

For Chris, the idea that mathematics was theoretical, and not relevant to the "real" world was appealing.

> Chris: Maybe there's some attraction in being in a subject that most people don't care about. It's really pure math, there is pretty much zero application of it and maybe I find that sort of fun. It seems very fundamental.

Some students who left the program without completing their doctorates felt alienated from a mathematics that bore no relevance to the real world.

> Lee: The fact that the real numbers are an example of a system we study or the integers are an example of a ring, and we're abstracting back from that. And so first the abstraction is kind of cool. . . . Well it just comes to the point where I'm sitting in class, looking up at the professor and

> thinking this is all just nonsense. None of this has any correlation to the real world in any possible way. Sitting there in set theory class, talking about cardinals that we cannot prove exist, or it's consistent that they may not exist and all this stuff. Why would I possibly care? Why would anyone possibly care? And it just seems so removed from anything…and it just became really absurd that I would spend my life discussing how unsolvable a problem was.

Applied questions (or those that some students viewed as "relevant" to the real world) are considered to be low status, and not really mathematics at all.

> Lee: It's highly intellectualized and as highly abstracted as it could possibly be and they scoff at the real world and, and I just couldn't get into that. . . . Math is necessarily abstract, very abstract or it's not math. If you're talking about something concrete, then you're giving an example or you're dumbing it down.

Mathematics also has appeal because its results are clearly either right or wrong:

> Lee: It appealed to me then that things were right or wrong in math, whereas in the humanities, you had to listen to everybody's just wild and crazy theory about everything. I mean sitting in English classes and having people interpret Jane Eyre as some sort of statement about I don't know. People came up with some ridiculous stuff and we had to sit there and listen to it like it made sense and pretend to contemplate the idea that maybe this is the correct interpretation, whereas in math class you could just say, 'that's false.' And it appealed to me that in math you could say either that's correct or that's false, and that you didn't have to deal with this pseudo-intellectual b------t that goes on in the humanities.

Not only is mathematics appealing because results are either clearly correct or incorrect, but there is a correct way to *do* mathematics.

> Terry: Mathematics has a certain aesthetic quality to it and you should look for it. There are sort of grubby proofs and there are slick proofs in which, look for the slick ones. . I tell the students that I'm going to not only show them how to solve problems but how to solve them nicely . . . you know what's nice and what isn't nice when you see an argument.

Both Lee and Chris spoke of the importance of working with other students on both coursework and preparing for qualifying exams. This helped them learn about alternate approaches and help them when they were stuck.

Chris: Usually our method was to try to you know work them all out as much as we could separately first and then get together just sort of compare notes, see what we had. . . . It's sort of fun when you each come up with a different way of doing it, and you can say 'oh wow, I never thought of that.' . . . I've worked with her a lot over the years, and I think that helps me . . . having somebody I can count on who I know I can work with and can relate.

Lee: At first I worked alone but then as I got to know more people I worked in groups and that was a lot better and that was a lot easier and helped a lot more.

While students are sometimes encouraged to work together in coursework, Terry emphasizes that a thesis is a solitary project. While she worked on her thesis, she says that she spoke with her advisor a total of 3 or 4 times; she never got stuck or needed help. Despite reporting collaborations with other mathematicians, it seems that working alone is accorded higher status. When asked if she could think of a student that has stood out as being the most promising, Terry immediately described a particular individual.

Terry: Well, he was really much more independent than the others. I mean he worked different, different scenario. He came in with his own problem and he solved his problems and every day would come in with, with new results. I mean it was rather exciting.

Both Lee and Chris wanted more support and encouragement from their professors. They wanted teachers to answer their questions, to interact with them during class, rather than just "come in, lecture, and leave". They wanted more advising. Many students in the larger sample felt that the faculty would not show any interest in the students until they had passed their qualifying exams (usually in their second or third year). But the first years of the program were when the students felt most intimidated, and most needed interaction with their professors.

Chris: I think a lot of people have a similar experience maybe to my first year where I was just thrown in with not that much guidance, not really understanding my classes, just sort of falling behind and not knowing how to get help. I wasn't real comfortable going to my professors.

Lee: Terry was just mean in class. People would ask questions and [Terry] would put them down for asking the questions. [Terry] would make it very uncomfortable to show any confusion at all and it felt like a constant competition to show, to prove yourself somehow.

And more than that, they found graduate school to be an "unhappy place" and longed for the social climate to be warmer.

Lee: Um, it was a less friendly place than I needed it to be really. I was at this [summer undergraduate] program that was supposed to prepare us for what grad school was going to be like and the faculty was nice and supportive and helpful and the students were friendly and we worked together and everyone was very collegial. And get to grad school and the professors are just mean. . . . A lot of them are so out to prove that they're smarter, or just to show us that we don't know anything in our first year. The first year curriculum is designed to beat us down.

Like many of the other students in the larger study, Chris and Lee wished for their professors to help them make connections mathematical ideas, and to motivate the material that was presented in lectures.

Chris: I just wish some of the professors were better teachers I think. I mean it really helps so much when you get some sort of idea of the big picture and of what the motivation is and why one might be tempted to say that a particular theorem is true, rather than just saying 'here's a statement, it's true.' Well, what would make you think that? Which I think really does just boil down to being a better teacher, just giving more motivation, more background, more reasoning for why you're doing what you're doing, and where you're going with it.

Lee: It was just boring. It was just all let's make up some sort of definition about a graph and let's see what we can prove about it, and there was no point. . . . I once had to present a paper, which I did fine at. Once I actually sat down with it, I could understand what was going on in the paper but I didn't feel any reason to care. I went through and I proved such and such a result, but so what? It was a meaningless result to me.

But Terry believes that the student has to be motivated on her own:

Terry: I try to motivate. I love the material so much that it's hard for me to understand a student not loving it, and so perhaps I don't motivate it as well as I should . . . . I mean theoretical math is theoretical math. You have to love doing puzzles, but it's sometimes hard to motivate. I mean, why should you sit down and do a crossword puzzle?

Like many of the other faculty, Terry thought that the ability to do mathematics is a talent that cannot be taught:

Terry: I think you can teach people to go so far but the ones that really have the spark you can't teach that.

Q: How do you find those people?
Terry: I don't know. Hopefully they find you or they find the subject.

Since talent is what distinguishes a successful mathematics student from an unsuccessful one, Terry believes that successful student have the ability to do mathematics and to think mathematically on their own. When asked if she teaches her students how to do theoretical work in mathematics, she replied,

> Terry: How do you learn to read? Well I guess there are two schools, but one school is you just start reading and you pick up new words as you go along and in the same way that's how you should learn how to do research. . . . You should understand what a proof is and you should see that by example. I don't know that one can give a course. I mean you can give some hints or sometimes you could see a theorem and say well, it's clear that you should attack it this way because, but sometimes it's genius that tells you which way to go. That's why there are really smart mathematicians because there isn't an obvious approach to the subject.

The graduate students felt that their undergraduate training did not prepare them for graduate school:

> Lee: There doesn't seem to be a very good bridge between the undergraduate material and the graduate material. You come in, you take 700-level courses, they tell you to do it even if you're not sure you should. Maybe it would be good for a lot of us to take some undergraduate courses first, especially those of us who came from smaller schools and didn't get as rigorous a background, but it's just not mentioned. You come, you're hard core, you do it, and if you don't [snaps], too bad. No one cares. You're just a failure, you're just one of those people who couldn't cut it.

At the end of her first year, Chris recalled,

> Chris: I remember just feeling, wow, I just sort of got run over by a train and here I am standing at the end of it. It was definitely a very intense experience, and it was definitely way more than I thought it would be, way harder than I thought it would be. But there was also a sense of having survived something.

Yet to Terry, it was not an issue of undergraduate preparation, which could presumably be remedied by taking additional courses, but rather an issue of undergraduate courses serving as a filter for identifying students with talent.

> Terry: You have to have the talent, and I think the reason that we have the drop-out rates is because many American students maybe don't have the

> talent and don't know they don't have the talent because they've never been tested. They've never had a tough linear algebra course or a tough advanced calculus course. Without that you really don't know if you're a good mathematician. I think upper level undergraduate math should be a deciding point and in most schools it isn't.

This intense adjustment was a particular issue for American-educated students, compared with the more rigorous training (and hence weeding out) that many foreign-educated students received as undergraduates.

> Terry: My feeling about the Americans is that . . . they've never been really challenged, and without being challenged you really don't know whether you have any math talent or not, and I fear that many of them without math talent have come here under the impression that they had. . . . In [most foreign countries] when you go to college and you major in math, you only take math courses. Now here we give a broad education, but [that] means you don't go deep and if you don't go deep, then you don't know whether you have the talent to go deep and I think that's the difficulty. I don't think the difficulty is that you don't have the knowledge. Rather I think you don't know whether you even belong here because you haven't been tested.

Lee picked up on this difference between the American-educated and foreign-educated students, and the perception of competition that resulted.

> Lee: There's the usual pitting of international students against American students. We think they're overachievers who make us look bad. They think we're dumb. That's just kind of how it feels is that that's the atmosphere, is that they stick together and they're the smart ones and we're the ones who are just kind of playing catch-up and trying not to get our butts kicked too bad.

Another set of filters that help identify students that are capable of doing mathematics and succeeding in the program are the qualifying exams and other assessment instruments.

> Terry: [The qualifying exams serve] a function like many of these other hurdles, just to keep you from wasting your time here if you're not going to make it. I think that that's a reasonable thing to do. We don't want [to] waste people's time if they're here who are not going to make it.

In contrast, many students wished for exams and homework assignments to help them learn, to provide feedback to them about what they do and do not understand, and to guide their studying.

This need to prove oneself leads to a powerful sense of competition within the graduate program, with many students suffering blows to their self-esteem as mathematicians.

Summary

Most faculty interviewed felt that the high rate of attrition from the graduate program was natural and acceptable, as students learned that mathematics was not for them and moved on to more suitable occupations. In particular, Terry felt that the best way to help more students succeed is to bring in better students, and that the faculty were helping students learn whether mathematics was a good choice for them by setting up hurdles like qualifying exams. However, doctoral students who leave the educational pipeline after investing several years of their lives represent a range of inefficiently used resources, ranging from the waste in terms of their own lives and self-esteem, and the tremendous drain of resources to universities and their funding agencies (Golde, 1996).

Terry's belief that mathematical talent can't be taught must have had a profound effect on this professor's teaching. The students certainly did perceive, correctly, that they were being required to prove themselves before they received attention from their professors. Education researchers often focus on the impact on student learning of pedagogies and teacher characteristics; however, a teacher's philosophy of teaching is likely to have a profound impact on her teaching, and hence student outcomes, including both learning and affect. This aspect of the intellectual culture of mathematics might be an important factor to consider in studying the teaching of mathematics. In addition to beliefs about mathematical talent, these interviews have identified a number of other factors of the intellectual culture of mathematics: beliefs about teaching and learning, including beliefs about the purpose of assessment (that partially stem from this belief in talent), beliefs about the nature of mathematics and the nature of doing mathematics, and an admiration of solitary work.

In these and other ways, students who were continuing or completing the program, students who left before completing their doctorates, and faculty have different ways of talking about the *doing* of mathematics, and about the meanings that mathematical activity has for them. In this sense, mathematics can be conceptualized as a discursive community, in which the system works to remove people who don't share the dominant discourse. These factors point to a construct that may play an important role in the teaching and learning mathematics: the intellectual culture of the classroom and of other aspects of the educational experience.

Further investigations using these interviews will look at how the intellectual culture of the mathematics department matches (or mismatches) with students' initial interests and expectations; whether women and men experience the intellectual culture differently; and whether continuing students and those who have left without degrees experience the intellectual culture differently.

References

Anderson, K., & Jack, D. C. (1991). Learning to listen: Interview techniques and analyses. In S. B. Gluck & D. Patai (Eds.), *Women's words: The feminist practice of oral history*. New York: Routledge.

Becher, T. (1989). *Academic tribes and territories: Intellectual enquiry and the culture of disciplines*. Milton Keynes: The Society for Research into Higher Education and Open University Press.

Becker, J. R. (1990). Graduate education in the mathematical sciences: Factors influencing women and men. In L. Burton (Ed.), *Gender and mathematics: An international perspective*. Cassell Educational Limited.

Biglan, A. (1973a). The characteristics of subject matter in different subject areas. *Journal of Applied Psychology, 57* (3), 195-203.

Biglan, A. (1973b). Relationships between subject matter characteristics and the structure and output of university departments. *Journal of Applied Psychology, 57* (3), 204-213.

Burton, L. (1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics*, *37*, 121-143.

Golde, C. M. (1996). *How departmental contextual factors shape doctoral student attrition*. Unpublished dissertation. Stanford University, Stanford.

Golde, C. M. (1999). *The role of departmental context in doctoral student attrition: Lessons from four departments*. Submitted, unpublished dissertation.

Green, K. C. (1989). A profile of undergraduates in the sciences. *American Scientist, 78*, 475-480.

Maddy, P. (1990). *Realism in mathematics*. Oxford: Clarendon Press.

National Center for Education Statistics. (1997). *Digest of Education Statistics, 1997, NCES 98-015*. Washington, DC: National Center for Education Statistics.

National Science Foundation. (1998). *Women, minorities, and persons with disabilities in science and engineering: 1998*. Arlington, VA: National Science Foundation.

Nerad, M., & Cerny, J. (1993). From facts to action: Expanding the graduate division's educational role. *New Directions for Institutional Research* (80), 27-39.

Secada, W. (1989). Agenda setting, enlightened self-interest, and equity in mathematics education. *Peabody Journal of Education, 66* (2), 22-56.

Seymour, E., & Hewitt, N. M. (1997). *Talking about leaving: Why undergraduates leave the sciences*. Boulder, CO: Westview Press.

Stage, F. K., & Maple, S. A. (1996). Incompatible goals: Narratives of graduate women in the mathematics pipeline. *American Educational Research Journal, 33* (1), 23-51.

Steen, L. A. (1999). Theories that gyre and gimble in the wabe - A review of mathematics education as a research domain: A search for identity. *Journal for Research in Mathematics Education, 30* (2), 235-241.

Tinto, V. (1993). *Leaving college: Rethinking the causes and cures of student attrition* (2nd ed.). Chicago: University of Chicago Press.

Wilson, L. S. (1992). The benefits of diversity in the science and engineering workforce. In M. L. Matyas & L. S. Dix (Eds.), *Science and engineering programs: On target for women?* Washington, DC: National Academy Press.

Note

[1] In order to avoid the imposition of gender, race, and class stereotypes, and to protect the identities of my participants, I do not reveal these characteristics about them. I use the feminine pronoun to refer to all participants. While it can be argued that this is just as sexist a convention as the more customary use of the masculine pronoun, I expect that by using female pronouns, there is little risk that anyone will overlook the fact that some mathematics students and faculty are male (after Maddy, 1990).

GROUP TESTS AS A CONTEXT FOR LEARNING HOW TO FACILITATE SMALL GROUP DISCUSSION BETTER

Judith Kysh
San Francisco State University
Jkysh@sfsu.edu

Purpose

The purpose of this study was to examine differences in teacher-student discourse in two different small group contexts to see how practice in one setting might affect practice in another. The report will address the following questions:

> *How does teacher-student discourse on group tasks, which are considered to be a regular assignment, differ from their discourse during a group test?*
>
> *How might these differences be used to help the teacher improve his or her facilitation of small group work?*

Perspectives

The national reform effort in mathematics education calls for a constructivist approach to teaching mathematics. Recently developed curricula, including the program used in this study, are based on the theory that students construct their own understanding of mathematics and that teachers and materials can be prepared to better serve students in helping them to develop this understanding. These materials generally include problems designed for small group work so students can talk about their work as they are doing it. A major concern of teachers who are learning to use the newly developed curricula is learning to facilitate the efforts of students as they work on problems in small groups.

During the nineties, articles urged researchers to study classrooms in which teachers were attempting to help students develop their own understanding through the social negotiation of meaning as they worked together and talked with each other and the teacher (Cobb, 1996, Lerner 1996). There have been some studies focused on the discourse that occurs between students and teacher as the teacher circulates to work with students as they work on problems in their small groups (Brenner, 1995, Chiu, 1996, Kysh, 1999), but as a recent article in *Mathematics Teaching in the Middle School* illustrates (Knuth, 2001), there is a continuing need to find ways to support mathematics teachers as they are learning to facilitate small group discussion.

Methods

This study is based on data gathered in a year-long ethnographic study of an Algebra 1 classroom. I had arranged to teach an Algebra 1 class from October to June, in an inner-city school with a mixed population of Asian, African-American, Latino, and White students. The class was using Mathematics 1 materials, a replacement course for Algebra 1 designed to enhance students' problem solving, reasoning, and communication skills and to use the other areas of mathematics, geometry, graphing and functions, probability and statistics, as a basis for understanding algebra.

As the teacher, fully responsible for all aspects of the class, I could be an insider, but because I was only teaching one class I would still be an outsider in the some important respects. I gathered data in the four categories described by Eisenhart (1988): participant observation, interviews, collection of artifacts, and reflections including "emergent interpretations, insights, feelings, and the reactive effects that occur as the work proceeds." (p.106) To enhance my observations as a participant, I used an audio-tape recorder during class in order to record teacher-student exchanges. I wore a small tape recorder and an external microphone as I moved around the classroom from group to group. I also recorded groups of students by placing a tape recorder with an external microphone in the center of a group of three to four students on one of the student's desks.

As the researcher and the teacher in this study I listened to the tapes from the first semester during late January and early February and started the work of transcribing them. I was very disappointed to find myself doing such things as forcing students into directions I thought they should go as opposed to starting from where they were, giving step-by-step explanations thinly disguised as a sequences of questions, and talking just to the individual who asked the question without asking what the rest of the group thought. I heard myself on the tapes exhibiting all those behaviors I had been encouraging others to move away from in order to facilitate their groups. Then I listened to two of the tapes recorded on the days we had group tests. I thought the difference was stunning.

On the face of it there are good reasons for group test days to be different from regular group work days. Both student and teacher have different goals. On group test days the students are trying to show what they know, and the teacher is focusing on finding out what they know. While these should be goals on normal days, the goals of "getting the work done" for students and "covering the material" for teachers often interfere even for teachers who know that their covering of material has little to do with what students learn (Ball, 1996).

The study I was conducting was focused on student discourse, so at the time, I was not analyzing the transcripts of the tapes for changes in my own discourse; however, my reflections indicate that I thought that the way in which I worked with the groups on normal days had changed from fall to spring and that the change had been influenced by listening to the tapes. This study revisits the transcripts to analyze and compare the fall transcripts for the two different models, and then to analyze and compare transcripts of work done on regular group work days for the fall (November and December) with transcripts from the spring (April and May).

Data Sources and Evidence

This report is based on analysis of the transcripts of 24 tapes: 4 on group test days and 20 from regular group work days. The transcripts include discourse markings to show pitch, pauses, volume, breathing, drawn out words, openings, interruptions, and splices.

A part of the analysis in the larger study included comparing topically related sets, TRS's (Cazden, 1988) in the on-task talk and looking for discourse patterns in those sets. There was a very clear question-answer-verify (QAV) pattern in the teacher-student dialogue with control shifting back and forth between student and teacher depending on who was asking the question. Differences in length, organization, and in who controlled the TRS varied in relation to the strength, confidence, and current state of applicable knowledge of the student. A comparison of the transcripts from the two settings (group test and regular group work days) shows clear differences in lengths of the TRS's, in the control of the QAV patterns and, most significantly, in the questions both the teacher and the students asked.

The Topically Related Sets for group test days ranged from 1 to 36 turns with a mean length of 7 turns and a median length of 5. The higher mean was largely accounted for by one student, who was responsible for 33% of the longer TRS's. She generally attempted to play on normal days and tried to use the group test days to get answers to the questions she should have been asking on the other days. The average TRS length on normal days, both median and mean, was about 10 turns, and the range was from 2 to 77 turns.

Some of the differences can be seen in the appearance of two sample transcripts on a page. The teacher's turns are generally shorter and the student turns longer on group test days than on normal days. And, on group test days the student controls the questioning and the teacher responds, while in this example of a normal day the teacher took over the questioning and the student was responding.

Group Test

Latisha: Ms. Kysh, ah, right here, don't I times the three times five and then five tahmes two? [Latisha is multiplying by 5 to solve $\frac{3x}{5}+2=14$]

Teacher: What does, what does three/

Yolanda: Five times twelve right? [She has subtracted 2 to get: $\frac{3}{5}x=12$]

Teacher: Okay, yeah. What does this line mean … divide.

Latisha: Divide. So you wouldn't want to multiply these … [Showing $5(\frac{3x}{5}+2)=14(5)$].

Teacher: Oh, I see what you're saying! I see what you're saying. **Yes!** Okay, I think so.

On the group test day students more often started a problem and then asked a specific question, whereas early in the year, on normal days, they were more inclined to start by asking for help, as Latisha does in the example below. Notice also how the teacher does more explaining, and Latisha responds in phrases.

Normal Day

Latisha: I don't understand . these two.[Referring to parts c) and d)]

Teacher: Uhhm. It would **help** to draw another line up at seventy. So draw this line right h·e·re. . . N·o·w, we **know** because we graphed it that this is twenty-three right here, right. This length is twenty-three

Latisha: Um hmm.

Teacher: so why don't you **write** that down there, twenty-three. . . This we don't know exactly, we know approximately what it is, from your graph. But let's call that x right now. **Now**, can you write a ratio, that . relates x and twenty-three and fifty and seventy?

Latisha: x over fifty

Teacher: x over fifty equals/

Latisha: twenty-three over seventy.

Teacher: That's it.An' then you can solve that and figure out x exactly.

Latisha: Solve this, switch this around instead of this. I already solved it.

Even Carolina, who early in the year just asked for help, on group test days would ask a question about what she had already attempted.

Carolina: Do I have to do these on the calculator to just put four in here minus three and then, then times this one?

Teacher: Unhuh .... that's it ....

Early in the year,Tsaan Fou would often just mumble and point to his question in the text, and I would give a longer and more detailed explanation than was needed. The excerpt below is an illustration. Initially I thought he was hesitant to speak as an English language learner, but this turned out not to be the case.

Teacher: Umm. Be sure to write . the steps now . after this. Umkay . Okay, when you take away three, what do you have?

Tsaan: (R) umm.

Teacher: So show it with three missing. And . . okay, This is step one, right __

Tsaan: (R) *Right.*

Teacher: Tw o · o . . would be what? to take three away from both sides, okay. So you would have then **two** cups and just 1, 2, 3, 4, 5, 6, 7, 8. . An' two cups equals, equals eight right? So you want to *write* that step down. And write the last step down.

Tsaan: (Q) This step, write, right here?

Teacher: Yeah, you want to write each step until you get one cup. . . . Like we did up there . . [looking to overhead example] each step.

Tsaan: (R) *All right.*

Teacher: Okay

The first time Tsaan Fou responded to the teacher in sentences was during a group test, but by the end of the year he was regularly chiming in as in the example below.

Teacher: [to Tanya].. .because you said a lot of things. I want you to think about what it should be.. . .

Tanya: You subtract one-thousand, . um, two-seventy from one thousand?

Teacher: Okay, but **why** would that, why would that work?. . .

Tanya: 'Cause that's what it **e**quals. That in the middle

Tsaan: Oh, co, 'cause the whole thing is one thousand and this is, this is um two-thirty, two-thirty and you just take it out.

From the transcripts it is clear that on group test days the students' questions were more specific and based on work they had already begun in relation to a problem rather than questions about how to start. In turn, the teacher was more inclined follow the student's path, relate the student's idea to the group's work, and ask a question that would restart the group rather than to try to help the individual who had asked by explaining or guiding him or her through the problem. In other words, the discourse patterns on group test days appeared to present a better model of the kind of exchange that would encourage autonomy and reasoning and communication among students.

In the comparison of transcripts from fall to spring, the spring TRS's were often shorter and involved more interchanges with two or three students, rather than just one.

The dilemma of whether to lead a student in a particular direction remained, but there were clear examples where I followed the student's direction rather than impose my own. In the example below, Joel reverts back to pointing to the question, but he recovers quickly to send me off to another group.

Teacher:	What?
Joel:	The last one.
Teacher:	Oh, can you uhhmm, can you solve one of the equations for y? Either one, doesn't matter which one. Get y by itself.
Joel:	You'd have to subtract y and you'd get, end up with/ no wait, no I see, I see it, nevermind. Go, go, bye. Okay, I see it.

Another part of the original study included the classification of turns based on whether the on-task language was about mathematics, in mathematics, with mathematics, or beyond mathematics. A brief description of those categories follows:

About:	Student asked about or described a problem using standard English, as in, "I need help with number 4."
In:	Student asked a question or described his/her work using mathematical terminology. Often the student described a procedure.
With:	Student described a use of mathematics or used mathematical language to describe a situation such as writing an equation for a problem.
Beyond:	Student used standard English to make a connection with another mathematical idea or situation or related two situations.

On normal days 43.8% of the recorded student turns were about mathematics, 43.9% in mathematics, 10.1% with mathematics, and 2.3% beyond mathematics. On the day of the first group test these percentages were quite different: 30% about mathematics, 59% in mathematics, 7% with mathematics, and 3.5% beyond mathematics. If we assume that a higher percentage of student turns that are in, with, or beyond mathematics means more practice in articulating mathematical relationships then the group tests provided a better opportunity than normal work days.

Results, Conclusions, and a Point of View

Group test days provided an opportunity for this teacher, who was not pleased with her facilitation of student work in small groups, to practice better strategies. The tapes of the discourse on these days provided a model for practice of the strategies on regular class days. Also the tapes of the discourse on regular days provided counter examples, so I, as the teacher, could see what I needed to change.

Teachers need to view models of teaching practices that are new to them. Video tapes of others classrooms can help, but sometimes models closer to home are more useful. It is always difficult for teachers to visit the classrooms of colleagues, and audio taping is unobtrusive and is fairly easy to do in one's own classroom. Encouraging teachers to tape the discourse in these two situations could be a useful addition to staff development programs designed to develop teacher's skills in facilitating small groups.

The results of this study have implications for teachers' expectations in relation to working with small groups and for their roles in facilitating small group work. Clearly further work in examining the dynamics of small group work and talk is needed. This effort is just a small part of seeking a better understanding of the teaching and learning mathematics through work in small groups.

References

Ball, D., & Wilson, S., (1996). Recognizing the fusion of the moral and intellectual. *American Educational Research Journal, 33* (1), 155-192.

Brenner, M. (1995) *Development of mathematical communication in algebra problem solving groups: focus on language minority students.* Research report submitted to the Linguistic Minority Research Institute, University of California, Santa Barbara.

Cazden, C. (1988). *Classroom discourse: The language of teaching and learning.* Portsmouth, NH: Heinemann Press.

Chiu, M. (1996). *Building mathematical understanding during collaboration.* Ph.D. dissertation. School of Education, University of California, Berkeley.

Cobb, P. (1996). *Where is the mind? A coordination of socio-cultural and cognitive constructivist perspectives.* In C. Twomey Fosnot, (Ed.), Constructivism, theory, perspectives, and practice. New York: Teachers College Press.

Eisenhart, M. (1988). The ethnographic research tradition and mathematics education research. *Journal for Research in Mathematics Education*, *19* (2), 99-114.

Knuth, E. & Peressina, D. (2001). Unpacking the nature of discourse in mathematics classrooms. *Mathematics Teaching in the Middle School, 6* (5), 320-325.

Kysh, J., (1999). *Discourse in small groups in an algebra 1 class.* Ph.D. dissertation, Division of Education, University of California, Davis.

LEARNING PATHS TO 5- AND 10- STRUCTURED UNDERSTANDING OF QUANTITY: ADDITION AND SUBTRACTION SOLUTION STRATEGIES OF JAPANESE CHILDREN

Aki Murata
Northwestern University
amurata@northwestern.edu

Karen Fuson
Northwestern University
fuson@northwestern.edu

Abstract: This study examined the strategies used by Japanese first- and second-graders as they solved simple addition and subtraction numerical problems. For problems with totals ≤ 10, most children initially used recomposition strategies using five as a base or recall strategies, and then increasingly used recall. For problems with totals > ten, children increasingly used recomposition strategies involving 10. Count-based strategy use decreased. Overall, children's strategies relied on the flexible use of embedded numbers that supported their fluency in recomposition, and five and ten were used as bases for these recompositions.

There have been numerous studies that examined the addition and subtraction strategies of young children (e.g., Carpenter and Moser, 1984; Fuson, 1992a, 1992b; Fuson and Kwon, 1992; Siegler, and Shrager, 1984). Table 1 shows the well-documented developmental/experiential paths for conceptions of quantities and strategies of U.S./European and Korean/Taiwanese children. They typically move from Level I: perceptual unit items (single presentation of the addend or the sum) to Level II: sequence unit items (simultaneous presentation of each addend within the sum) and then to Level III: ideal chunkable unit items (simultaneous embedded mental representation of both addend and the sum). Recall strategies develop increasingly through the levels, varying with individual children. At the Level III, U.S. and European children usually use doubles $\pm$ 1 or known addition strategies, while Fuson and Kwon (1992) and Lee (2000) found that Korean and Taiwanese children used recomposition strategies using tens.

This study examined the strategies used by Japanese first- and second-graders as they solved simple addition and subtraction numerical problems. It contributes to an effort to describe the developmental sequences used around the world as children build their understanding of 5- and 10- structured numerical relationships. When examining different strategies of children, it is also necessary to consider how and why their strategies are developed. Although this paper focuses on the results of the strategy interviews of Japanese children, this is a part of a larger study that also explores how the strategies are supported by the culture as appropriate ways to consider numerical quantities by examining children's experiences in and out of school. This study also

Table 1. Conceptual Developmental Levels of Addition and Subtraction Solution Methods (Adapted from Fuson, 1992a, 1992b; Fuson and Kwon, 1992; Lee, 2000).

Levels	Conception of Quantities	Addition Solution Methods	Subtraction Solution Methods
I	Perceptual Unit Items: count things	Count All	Count Take Away
II	Sequence Unit Items: number words are the things	Count On	Count Down
III	Ideal Chunkable Unit Items: numbers are things	*Derived Facts/Recomposing: U.S.*	
		Doubles ± 1	Doubles ± 1
		Make a ten from one number: Korean and Taiwanese	
		Make a Ten	Take From Ten
		Up Over Ten	Down Over Ten
		Add Fives and Then Add Amounts Over Five	

relates students' strategy sequence to their learning opportunities in the classroom. The culture supports and influences the way school mathematics is taught and also the ways children experience numbers outside of school. This view was adapted from the Vygotsky's sociocultural perspective (1978, 1986) that asserts that understanding the formation of minds requires the study of sociocultural contexts and of particular semiotic cultural supports and practices as reflections of the culture itself.

Methods

Fifty (*n* = 50) Japanese first- and second-graders participated in the study. The setting of our study is an official full-day Japanese School in the U.S. It follows the official Japanese national course of study, and all the teachers at the school are certified Japanese teachers who are sent from Japan directly through the Ministry of Education. All of the participating children and their families are temporary residents in the U.S., living here for business purposes for a few years. Because they planned to return to Japan, the parents tried to preserve Japanese cultural ways in children's lives, and the members of the Japanese community in the area generally interacted among themselves rather than with Americans.

Out of the fifty children, twenty-eight ($n = 28$) children were first-graders and twenty-two ($n = 22$) children were second-graders at the time of the study. The 28 first-graders were from two different classrooms, and the second-graders were from a single classroom. Two different strategy-interview sessions were held: one in the middle of the school year in October, and another at the end of the school year in March. Because the second-grade classroom was fairly large (38 children), randomly-selected children were interviewed from the classroom; therefore the samples for the October interview differed from the one from March. For the first-graders, all children were interviewed for both sessions from both first-grade classrooms.

The individual interviews were conducted in Japanese. At the beginning of the interviews, it was explained to the children that the interviewer was interested in studying their thinking as they solved the problems and not just the answers. Counters, papers, and pencils/markers were placed within children' reach, and it was also explained that they were allowed to use them if they wanted to while solving and describing their thinking. Children were then presented the following ten problems: 5 + 2 or 6 + 2, 4 + 4, 7 + 5 or 8 + 9, 6 + 9, 7 + 7, 7 – 2 or 8 – 2, 8 – 4, 12 – 5 or 17 – 8, 15 – 9, 14 – 7, one at a time written on a 7" x 5" index card. When the strategy the child used was not observable (> 96%), the interviewer asked the child to explain how he/she solved the problem after he/she stated the answer. Because Japanese mathematics lessons often focus on problem solving with multiple solutions, the participating children sometimes attempted to explain the solution in more than one way. In that case, a standard follow-up probe was used to identify the strategy the child had just used. The interviews were tape-recorded, transcribed, and translated. The interviewer also took interview notes during interviews.

Response data were first differentiated as immediate (≤ 2 sec. after Siegler and Shrager, 1984)) or non-immediate responses. Observable counting strategies were then differentiated according to the coding framework (Table 1) of Fuson and Kwon (1992). For non-observable strategies, children' explanations were analyzed and differentiated according to the Conceptual of Quantities Levels and then to the different strategies. When a strategy did not fit the previously identified methods, new categories were created. Strategies were then further differentiated according to the kinds of quantities used with explanations. Follow-up interviews were conducted for unclear responses. The data were then separated into the problems whose totals were smaller than ten (≤ 10) and larger than ten (> 10) for analysis.

Results and Discussion

Japanese first- and second-graders who participated in the study used all of the strategies identified previously by other researchers in Table 1, and they used five new strategies: *Make-five* strategy for addition, *take-from-five* strategy for subtraction, *break-apart-to-make-ten* strategy for addition, *count-on-to-ten-and-add-over-ten* strategy for addition, and *count-up-to-ten-and-add-over-ten* strategy for subtraction (see Table 2 and the later text).

The percentage of correct answers overall was 97%. Addition and subtraction answers were both 100% correct at mid Grade 1 for ≤ 10, at mid Grade 2 for ≤ 10, and end Grade 2 for ≤ 10 and > 10. The only cases with less than 95% correct answers were mid Grade 1 addition > 10 (94%) and end Grade 1 subtraction > 10 (92%). The percentage of strategy use for the correct answers is shown in Table 3.

For the addition and subtraction problems with totals smaller than ten (≤ 10), almost half of the children (46% overall; 31% for addition and 65% for subtraction) initially used a *make-five* recomposition strategy involving fives (see Table 3). This *make-five* strategy is an addition strategy similar to ten-based recomposition strategies, but uses five as a unit instead of ten. For example, for the problem 4 + 4, a child explained;

> "4 + 4 is 8, because 4 and 1 more is 5, so bring 1 from the other 4, and 3 is left. Then, 5 and 3 is 8."

Children used such 5- structured thinking for subtraction also in the *take-from-five* strategy. For example, one child explained:

> "8 – 4 is 4. Because 8 is 5 and 3, take 4 from 5, then 1 left, plus 3 would be 4."

The children's understanding of numbers in relation to five is reflected by these strategies. At the beginning of the first-grade, children are provided much experience in their mathematics lessons breaking-apart and putting-together numbers between 5 and 10 using five as a base (e.g., 6 is 5 + 1, 7 is 5 + 2). With repeated exposure to seeing and experiencing numbers in relation to five, many children used 5- structured understanding of quantities by the middle of the first-grade when the interviews were conducted.

As the children gained more experience with the problems with totals ≤ 10, they increasingly used recall strategies (by the end of the second grade, 95% for addition and 100% for subtraction). Children typically explained that they "remembered" and "knew" the answer or "did not think" while solved the problem. In all three classrooms, children regularly practiced their "calculation" by using calculation cards (small flash cards put together by a ring). Fluency and speed were emphasized in the classrooms, and by the end of the second grade, children quickly recalled answers.

Table 2. New Strategies Japanese Children Used.

Levels	Conception of Quantities	Addition Solution Methods	Subtraction Solution Methods
II	Sequence Unit Items		
II-III	(Transitional Level)	Count On to Ten and Add Over Ten	Count Up to Ten and Add Over Ten
III	Ideal Chunkable Unit Items	Make-Five (total < 10)	Take-From-Five (total < 10)
		Break Apart to Make Ten	

Table 3. Percent Correct Responses by Strategy for Addition and Subtraction.

Strategy	Mid Grade 1	End Grade 1	Mid Grade 2	End Grade 2
Totals ≤10				
Count-based strategies:				
Addition	12	14	5	5
Subtraction	12	14	5	0
Recomposition strategies:				
Addition	31	4	10	0
Subtraction	62	30	35	0
Recall strategies:				
Addition	57	82	85	95
Subtraction	26	55	60	100
Totals in Teens				
Count-based strategies:				
Addition	24	20	3	0
Subtraction	40	18	10	0
Recomposition strategies:				
Addition	65	58	73	70
Subtraction	54	65	80	97
Recall strategies:				
Addition	11	21	23	30
Subtraction	6	15	10	3

Note: The problems for totals ≤ 10 were 5 + 2 or 6 + 2, 4 + 4, 7 – 2 or 8 – 2, 8 – 4), for totals in teens were 7 + 7, 6 + 9, 5 + 7 or 8 + 9, 14 – 7, 15 – 9, 12 – 7 or 17 – 8.

For problems with totals >10, Japanese children predominantly used recomposition strategies involving tens, and they did so increasingly as they gained more experience (at the middle of the first-grade, 65% for addition and 54% for subtraction; at the end of the second-grade, 70% for addition and 97% for subtraction). Unlike the ten-based recomposition strategy *up-over-ten* identified by Fuson and Kwon (1992) for Korean children, Japanese children's explanation for addition solutions using ten lacked directionality but indicated instead that they were breaking a number into two different parts. Therefore we called the strategy *break-apart-to-make-ten* strategy. A typical explanation is as follows:

> "7 + 7 is 14. 7 needs 3 more to make 10, so I *split* the other 7 into 3 and 4, then 10 and 4 is 14."

For subtraction, children also did not indicate directionality but used a more neutral *Take-from-Ten* solution. For example, one child explained her thinking for 15 – 9;

> "15 – 9 is 6. That's because 15 can be separated into 10 and 5, then 10 minus 9 is 1, then add the left-over 5 will be 6."

For all problems, children used fewer count-based strategies as they gained more experience. Relatively small percentages of children used count-based strategies in the middle of the first-grade (addition ≤ 10, 12 %; subtraction ≤ 10; 12%; addition >10, 24%; subtraction >10, 40%), and almost none used them at the end of the second-grade (See Table 3).

However, a group of children (first-graders, 5% in first interview, 7% in spring interview) did combine counting with a recomposition strategy to form transitional strategies. For an addition problem, they counted from one addend up to ten, recognized the number of times counted, and then separated the other addend into that number and a second number, and added the second number to ten. As one child explained this *count-on-to-ten-and-add-over-ten* strategy:

> "5 + 7 is 12. From 7 . . 8, 9, 10 . . so 3 take from 5 is 2, then 10 plus 2 is 12."

Children used this hybrid of counting and recomposition strategies for subtraction also. As one child explained the *count-up-to-ten-and-add-over-ten* strategy::

> "14 – 7 is 7. 7, and, 8, 9, 10 . . . is 3 more, then 4 will be 7."

These strategies are useful transitional strategies for children who are moving from seeing quantities as embedded sequences (Level II) to embedded chunks (Level III).

This paper does not allow space to present the rich variety of children's own words, terms, and references to describe their strategies. This considerable variation in children's descriptions of their thinking suggest that their understanding was developed through individual learning experiences and was not due to memorization.

At the beginning of the first-grade, children spent many classroom periods practicing decomposing and recomposing the numbers smaller than 10 before they learned addition and subtraction. Once addition and subtraction concepts were introduced, children were encouraged to use what they knew about embedded numbers within a number and discouraged to count unitarily. Classroom discussions often focused on seeing relationships between different numbers using 5 and 10 as bases. As they gained more experience, children practiced calculations for speed, fluency, and accuracy. This calculation practice always occurred in connection to what they knew about numbers conceptually, and teachers combined the fluency practices with check-ups of answers using semi-concrete manipulatives, such as small flower-shaped counters or counting sticks. Various calculation practices of this sort continued in to their second-grade year as they built on to what they knew with multi-digit addition and subtraction, and later learned about multiplication and its relationship to addition and subtraction.

Overall, children's strategies seemed to rely on the flexible use of embedded numbers that supported their fluency in recomposition. With the curricular emphasis on chunking of numbers in relation to five and ten from early on and de-emphasis on unitary counting, Japanese children seemed to have developed the 5- and 10- structured understanding of quantities well by the end of the second-grade and even very substantially by the end of first-grade. This 10-structured ways of looking at numbers should be of advantage as they work with multi-digit numbers. Future studies should address broader cultural influences upon children's thinking of quantities and identify interacting factors in their lives.

References

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15* (3), 179-202.

Fuson, K. (1992a). Relationships between counting and cardinality from age 2 to age 8. In J. Bideau, C. Meljac, & J. P. Fischer (Eds.), *Pathways to number* (pp. 127-149). Hillsdale, NJ: Lawrence Erlbaum.

Fuson, K. (1992b). Research on whole number addition and subtraction. In D. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 243-275). New York: Macmillan.

Fuson, K., & Kwon Y. (1992). Korean children's single-digit addition and subtraction numbers structured by ten. *Journal for Research in Mathematics Education, 22,* 148-165.

Lee, H-F. (2000). Supports for learning multi-digit addition and subtraction: A study of Taiwanese second-grade low-math achievers. In M. L. Fernandez (Ed.), *Proceedings of the Twenty-Second Annual Meeting of North American Chapter* *of the International Group for the Psychology of Mathematics Education* (pp. 723-728). Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Siegler, R. S., & Shrager, J. (1984). Strategy choices in addition and subtraction: How do children know what to do? In C. Sophian (Ed.), *Origin of cognitive skills*. (pp. 229-293). Hillsdale, NJ: Lawrence Erlbaum.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1986). *Thought and language*. Cambridge, MA: MIT Press.

AN ATTEMPT TO CLARIFY DEFINITIONS OF THE BASIC CONCEPTS: BELIEF, CONCEPTION, KNOWLEDGE

Erkki Pehkonen
University of Turku, Finland

Fulvia Furinghetti
University of Genoa, Italy

Abstract. In this paper we consider beliefs and the related concepts of conceptions and knowledge. Analyzing the literature in different fields, we observe that there are different views and different approaches in research about these subjects. Therefore, we have organized a panel that we have termed "virtual", since the participants communicated with us only by e-mail. We sent to the panelists nine characterizations related to beliefs taken from the recent literature, and asked them to express their agreement or disagreement with our statements and to give personal characterizations. The answers were analyzed and, as a final step, we outlined some common factors and relationships that may be taken as a background for studies in the field of beliefs.

Some twenty years ago, it was first time expressed that teachers' philosophy of mathematics is related to their way of teaching. Among other authors, Lerman (1983) pointed this out in the case of the relationship between philosophies of mathematics and styles of teaching. Thompson (1984) analyzed teachers' conceptions of mathematics and mathematics teaching on instructional practice. Similar results are achieved with research again and again, e.g., Lloyd & Wilson (1998). A number of studies show that also students have a particular view of mathematics (e.g., Lester al., 1989).

The purpose of this paper is to draw attention to theoretical deficiencies of belief research. Firstly, the concept of belief (and other related concepts) is often left undefined (e.g., Cooney al.,1998) or researchers give their own definitions that might be even in contradiction with each other (e.g., Bassarear, 1989, and Underhill, 1988). The second important problem is the inability to clarify the relations between belief and knowledge. We point out that, in carrying out our study, as far as it is possible we tried to keep a broad point of view, in order to reach quite general conclusions. Thus we do not refer to beliefs about a particular object (e.g., about mathematics, about teaching mathematics, about understanding) or a particular group of individuals (e.g., teachers, students).

Theoretical Background

Although beliefs are popular as a topic of study, the theoretical concept of "belief" has not yet been dealt with thoroughly. The main difficulty has been the inability to state the relation between beliefs and knowledge, and the question is still not clarified (e.g., Abelson 1979; Thompson 1992).

There are many variations of the concepts "belief" and "belief system" used in studies in the field of mathematics education. As a consequence of the vague characterization of the concept, researchers often have formulated their own characterizations for belief, which might even be in contradiction with others. The Table 1, which refers to the questionnaire we used in our study, presents a range of characterizations. Other characterizations could be mentioned. For example, Schoenfeld (1985, 44) offered a characterization different from that appearing in Table 1, stating that, in order to give a first rough impression, "*belief systems are one's mathematical world view*". Other researchers, Underhill (1988) for one, describe beliefs as some kind of attitudes. Yet Bassarear (1989), who sees attitudes and beliefs on the opposite extremes of a bipolar dimension, gives another different explanation.

Conceptions belong to the same group of concepts as beliefs. They are also used in different ways in mathematics education (and wider) literature. For example, Thompson (1992) understands beliefs as a sub-class of conceptions. But she claims: "*the distinction [between beliefs and conceptions] may not be a terribly important one*" (ibid. 130). Furinghetti (1996) who explains an individual's conception of mathematics as a set of certain beliefs follows Thompson's idea. Pehkonen (1994) who characterizes conceptions as conscious beliefs gives a different understanding.

In (Sfard, 1991), conceptions may be considered as the subjective /private side of the term 'concept' defined as follows: "*The word 'concept' (sometimes replaced by 'notion') will be mentioned whenever a mathematical idea is concerned in its 'official' form as a theoretical construct within 'the formal universe of ideal knowledge'* ". Whereas she explains that "*the whole cluster of internal representations and associations evoked by the concept - the concept's counterpart in the internal, subjective 'universe of human knowing' - will be referred to as 'conception'* " (p.3). The distinction between conception and knowledge is complicated by the fact that an individual's conception of a certain concept can be considered as a "picture" of that concept. Like a picture and its object are not the same, and usually the picture shows only one view on the object, similarly a conception represents only partly its object (concept). In general, this author does not use the word beliefs.

In order to face the problem of distinguishing between knowledge and beliefs, some structural differences between belief systems and knowledge systems have been noticed. For example, Rokeach (1968) organized beliefs along a dimension of centrality to the individual. The beliefs that are most central are those on which the individual has a complete consensus; such beliefs on which there are some disagreement would be less central. Whereas, Green (1971) introduces three dimensions, which are characteristic for belief systems: quasi-logicalness, psychological centrality, and cluster structure. Also Thompson (1992) emphasizes two of the Green's dimensions as characteristics of beliefs: the degree of conviction (psychological centrality), and the clustering aspect.

Implementation of the Study

The focus of the study at hand was to clarify some core elements in studies related to beliefs, conceptions, and knowledge which almost all specialists could accept or, if the different assumptions of researchers make it impossible to reach a complete agreement, to stress the existence of different positions. In the case of this second circumstance, we felt that our study would have contributed to convince researchers about the necessity of making explicit their assumptions. It was not our aim to introduce a "democratic" pattern according to which definitions are good if the majority of the researchers in the field of beliefs accept them. We simply want to stress the pitfalls generated by not clarified or ambiguous assumptions in research.

On the ground of the previous considerations we have worked out a questionnaire in which we listed nine belief characterizations (see Table 1) present in the recent literature (1987–98). They focus on one or more terms of the triad in question (beliefs-conceptions-knowledge). In the questionnaire the authors of the characterizations were not indicated. Each characterization was accompanied by the sentences: "Do you consider the characterization to be a proper one? Please, give the reasons for your decision!" Some empty lines followed each characterization. Additionally, there was the following final item: "Your characterization: Please, write your own characterization for the concept of 'belief'?"

In March 1999, we sent via e-mail our questionnaire to the 22 specialists invited to the international meeting, "Mathematical Beliefs and their Impact on Teaching and Learning of Mathematics", held in November, 1999 in Oberwolfach, (see Pehkonen & Törner, 1999). The specialists were asked to respond within two weeks. Altogether, 18 researchers (82 %) send us their responses in due time, commenting on all characterizations. Only half of them did give us their own characterization. The specialists responding to our e-mail questionnaire were from seven different countries: Australia, Canada, Cyprus, Germany, Israel, UK, and USA. The panel was virtual in the sense that only e-mail was used to communicate. We expected to collect data on the following points: agreement or disagreement with the given characterizations, possible improvements, reasons for agreement or disagreement, and personal characterizations.

Some Results

When reading the specialists' reactions to the nine belief characterizations, we confronted a big variety of ideas, and had difficulties to find patterns in it. Therefore, our first task was to group the responses somehow, in order to have an overview. Thus, we classified together all the answers into a five-step scale, discussing as long as we reached consensus: Y (= fully agreement), P+ (= partial agreement with a positive orientation), P (= partial agreement), P- (= partial agreement with a negative orientation), N (= fully disagreement). In Table 2 we report the summary of results obtained after

Table 1. The nine characterizations of the questionnaire.

Characterization #1 (Hart, 1989, 44)	"we use the word belief to reflect certain types of judgments about a set of objects"
Characterization #2 (Lester & al., 1989, 77)	"beliefs constitute the individual's subjective knowledge about self, mathematics, problem solving, and the topics dealt with in problem statements"
Characterization #3: (Lloyd & Wilson, 1998, 249)	"we use the word conceptions to refer to a person's general mental structures that encompass knowledge, beliefs, understandings, prefe-rences, and views"
Characterization #4 (Nespor 1987, 321)	"Belief systems often include affective feelings and evaluations, vivid memories of personal experiences, and assumptions about the existence of entities and alternative worlds, all of which are simply not open to outside evaluation or critical examination in the same sense that the components of knowledge systems are"
Characterization #5 (Ponte, 1994, 169)	"Beliefs and conceptions are regarded as part of knowledge. Beliefs are the incontrovertible personal 'truths' held by everyone, deriving from experience or from fantasy, with a strong affective and evaluative com-ponent."
Characterization #6 (Pehkonen, 1998, 44)	"we understand beliefs as one's stable subjective knowledge (which also includes his feelings) of a certain object or concern to which tenable grounds may not always be found in objective considerations"
Characterization #7 (Schoenfeld, 1992, 358)	"beliefs - to be interpreted as an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior"
Characterization #8 (Thompson, 1992, 132)	"A teacher's conceptions of the nature of mathematics may be viewed as that teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics."
Characterization #9 (Törner & Grigutsch, 1994, 213).	"Attitude is a stable, long-lasting, learned predisposition to respond to certain things in a certain way. The concept has a cognitive (belief) aspect, an affective (feeling) aspect, and a conative (action) aspect."

our classification. In order to get a better overview of the situation; different types of answers are summed up. Our first observation was that in the responses of the specialists, there was no clear pattern to be observed. But in some points, one can find some regularity. The answers were most unified in characterization #5 (by Ponte, 1994) where 15 specialists (83 %) disagreed with the statement, and three (17 %) agreed with it. When looking for the largest frequencies in Table 2, we elaborated the following grouping of the characterizations.

- The response of the panel to characterization #5 (Ponte, 1994) was a clear "no". According to its author, this definition is inspired by (Pajares, 1992), that is it generates outside mathematics education community.
- The next largest frequencies were in characterizations #7 (Schoenfeld, 1992) and #8 (Thompson, 1992). In these cases, most of the panel members (about 70 %) were in agreement with the characterizations (i.e., the answer was "yes"). This is not surprising, since papers of Schoenfeld and Thompson are much used as reference literature in research on beliefs. But also these were not accepted in consensus, since there were 3–4 specialists who responded clearly "no", and 2–3 others who agreed with them only partly.
- In three cases, we estimated the orientation of the panel to be positive, since the sum of "Yes" and "Partly yes" answers was larger than the negative ones: characterization #1 (Hart, 1989), characterization #3 (Lloyd & Wilson, 1998), characterization #9 (Törner & Grigutsch, 1994). For the high level of agreement here, we can find easily reasons: One can say that Hart follows the ideas of Schoenfeld, and Lloyd & Wilson those of Thompson.
- In characterization #6 (Pehkonen, 1998) agreements and disagreements were divided almost fifty-fifty (yes–no). In this characterization, the word "stable" has caused confusion, since in the case of beliefs it can be understood in different ways.
- In characterization #4 (Nespor, 1987), the majority of the responses were negative. Therefore, we can say that the answer was an almost no. This

Table 2. Agreement and disagreement of the respondents with the nine characterizations.

	1	2	3	4	5	6	7	8	9
Y = YES	7	7	9	4	2	7	11	11	7
P+ = PARTLY YES	4	1	3	-	1	1	1	1	4
P= PARTLY	2	7	4	4	-	1	3	2	2
P- = PARTLY NO	1	-	-	2	-	-	-	-	2
N = NO	4	3	2	8	15	9	3	4	3

characterization comes from outside mathematics education community.

- Additionally, it is interesting that in one case, characterization #2 (Lester & al., 1989), there was the highest number of "partly" answers.

In order to arrive at basic ideas on beliefs, which encompass as much as possible the feedback from the responses, we focus on the results of some items, which most clearly give us an orientation.

In the 15 negative answers to characterization #5, we have singled out quite clearly two central features determining the disagreement: the adjective *incontrovertible* and the *relation between beliefs and knowledge*. Another ambiguity we have observed originates from the use of the term *conception*, present in items #3, #5 and #8.

When reading the responses, the following points emerged which are of special interest for researchers: the origin of beliefs, affective component of beliefs, and the effect of beliefs on an individual's behavior, reaction, etc. The issues raised here, which we will discuss, are, of course, somewhat overlapping.

Beliefs might have their origin in many ways, as expressed by the following three quotations: "*Beliefs, in a particular context/situation, are part of a person's identity (or better, identities), which is (are) formed through learning, interacting, goals, needs and desires, and therefore also affective.*" (R.6)[1], "*Beliefs can also be adapted from others, especially from those in authority.*" (R.9), and "*Finally sometimes people believe things because they have noticed them in personal experience, but very often they believe "propaganda" instead (mathematics is hard, useful, dull, fun, etc.).*" (R.13)

In two responses (R.9, R.17), the affective component of beliefs was mentioned. As an example, we give the following: "*I think of beliefs as primarily cognitive with a significant affective component [...] and especially related to values [...] I also try to separate beliefs from more affective or attitudinal responses to mathematics (enjoyment of problem solving or preferences for certain mathematical topics).*" (R.9) This position is close to the spirit of the chapter (McLeod, 1992) on affective factors.

The fact that beliefs have an effect on an individual's behavior, reaction, etc. is a quite common assumption for researchers on beliefs (e.g., Schoenfeld, 1992). This is expressed, for example, by the following quotation: "*A belief is an attempt, often deeply felt, to make sense of and give meaning to some phenomenon. It involves cognition and affect, and guides action.*" (R.11) Some researchers (e.g., R.4 and R.17) stressed the importance of context in shaping beliefs or behavior.

Discussion

In gathering the criticisms and the constructive parts of the answers that we had at disposal we realized that there are points on which future research may be based. In the following we comment on some of these points.

Firstly we drop the idea of a multipurpose characterization suitable for all the possible fields of application (mathematics education, philosophy, general education, psychology, sociology) and refer our considerations to a given context, a specific situation and population. Also it is useful to link a given characterization to the goals that we have in mind when using the concept we are characterizing. *Contextualization and goal-orientation* make the characterization an *efficient* one.

There is also the need to specify concepts used in research. It seems to us that part of the previous discussion could be avoided, if we distinguish in mathematics between objective (official) knowledge (which is accepted by the mathematical community), and subjective (personal) knowledge (which is constructed by an individual). Individuals have access to objective knowledge, and construct (in the language of Sfard, 1991) their own conceptions on mathematical concepts and procedures, i.e., they construct some pieces of their subjective knowledge. In an ideal case, the conceptions and mathematical concepts in question correspond isomorphically to each other. In such a sense the two domains may be overlapping, but not coincident. In the domain of objective knowledge, there are parts that may not be accessible to individuals, or to which individuals have no interest. Conceptions individuals generate from objective knowledge become part of their subjective knowledge. This happens after an operation of processing information, in which the existing knowledge and his earlier beliefs intervene. In the domain of his subjective knowledge, there are elements that are strictly linked to the individual: they are beliefs intended in a broad sense that includes affective factors. Beliefs belong to individuals' subjective knowledge, and when expressed by a sentence they may be logically true or not. Knowledge always has a truth-property (cf. Lester et al., 1989). We can describe this property with probabilities: knowledge is valid with a probability of 100%, whereas the corresponding probability for belief is usually less than 100%.

Not always individuals are conscious of their beliefs. Thus we have to consider conscious and unconscious beliefs. Also individuals may hide beliefs to external scrutinizing, because in their opinion they are not satisfying someone's expectations. In Furinghetti's paper (1996) the phenomenon of the 'ghosts' in classroom is discussed: ghosts are the hidden or unconscious beliefs in action in classroom.

In the results, the terms incontrovertible and stable were disputed as attributes for beliefs. We suppose that this depends on the fact that those working in education need to trust in the possibility to act on beliefs, because otherwise the didactic action would not have sense. The intermediate solution of considering central and peripheral beliefs seems more flexible for describing how beliefs are modified.

In summarizing the results, we propose for studying beliefs and the related terms a list of basic recommendations, which should be used flexibly according to the situation, analyzed. They are:

- to consider two types of knowledge (objective and subjective)

- to consider that beliefs belong to subjective knowledge
- to include affective factors in the belief systems, and distinguish affective and cognitive beliefs, if needed
- to consider degrees of stability, and to leave beliefs open to change
- to take care of the context (e.g., population, subject, etc.) and the research goal in which beliefs are considered.

References

Abelson, R. (1979). Differences between belief systems and knowledge systems. *Cognitive Science, 3*, 355–366.

Bassarear, T.J. (1989). The interactive nature of cognition and affect in the learning of mathematics: two case studies. In C. A. Maher, G. A. Goldin & R. B. Davis, (Eds.), *Proceedings of the Eighth Annual Meeting of the Psychology of Mathematics Education-North America.* (Vol. 1, pp.3–10). Piscataway, NJ.

Cooney, T. J., Shealy, B. E. & Arvold, B. (1998). Conceptualizing belief structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education 29* (3), 306–333.

Furinghetti, F. (1996). A theoretical framework for teachers' conceptions. In E. Pehkonen (Ed.), *Current state of research on mathematical beliefs III* (pp. 19–25). Proceedings of the MAVI-3 Workshop, University of Helsinki. Department of Teacher Education. Research Report 170.

Green, T. F. (1971). *The activities of teaching*. Tokyo: McGraw-Hill Kogakusha.

Hart, L. E. (1989). Describing the affective domain: Saying what we mean. In McLeod & Adams (Eds.), *Affect and mathematical problem solving*, (pp. 37-45). New York: Springer-Verlag.

Lerman, S. (1983). Problem-solving or knowledge-centered: The influence of philosophy on mathematics teaching. *International Journal of Mathematical Education in Science and Technology. 14* (1), 59–66.

Lester, F. K., Garofalo, J., & Kroll, D. L. (1989). Self-confidence, interest, beliefs, and metacognition: Key influences on problem-solving behavior. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving* (pp. 75-88). New York: Springer-Verlag.

Lloyd, G. M., & Wilson, M. (1998). Supporting innovation: The impact of a teacher's conceptions on functions on his implementation of a reform curriculum. *Journal for Research in Mathematics Education 29*, (3), 248–274.

McLeod. D. B. (1992). 'Research on affect in mathematics education: a reconceptualization'. In D. A. Grouws (Ed.), *Handbook of research on mathematics learning and teaching* (pp. 127-146). New York: Macmillan.

Nespor, J. (1987). The role of beliefs in the practice of teaching. *Journal of Curriculum Studies, 19* (4), 317–328.

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*, 307-332.

Pehkonen, E. (1994). On teachers' beliefs and changing mathematics teaching. *Journal für Mathematik-Didaktik, 15* (3/4), 177–209.

Pehkonen, E. (1998). On the concept "mathematical belief". In E. Pehkonen & G. Törner (Eds.), *The state-of-art in mathematics-related belief research. Results of the MAVI activities* (pp. 37–72). Research Report 195. University of Helsinki. Department of Teacher Education..

Pehkonen, E., & Törner, G. (Eds.) (1999). *Mathematical beliefs and their impact on teaching and learning of mathematics*. Schriftenreihe des Fachbereichs Mathematik. Gerhard-Mercator-Universität Duisburg Gesamthochschule. Preprint Nr. 457.

Ponte, J. P. (1994). Knowledge, beliefs, and conceptions in mathematics teaching and learning. In L. Bazzini (Ed.), *Proceedings of the Fifth SCTP* (pp. 169-177). Pavia, ISDAF.

Rokeach, M. (1968). *Beliefs, attitudes, and values*. San Francisco: Jossey-Bass.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. 1992. Learning to think mathematically: problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics learning and teaching* (pp. 334–370). New York: Macmillan.

Sfard, A. (1991). On the dual nature of the mathematical objects. *Educational Studies in Mathematics*, *22*, 1-36.

Thompson, A. G. (1984). Teachers' conception of mathematics and mathematics teaching on instructional practice. *Educational Studies in Mathematics*, *15*, 105-127.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In A. D. Grouws (Ed.), *Handbook of research on mathematics learning and teaching* (pp. 127-146). New York: Macmillan.

Törner, G., & Grigutsch, S. (1994). Mathematische weltbilder bei studienanfängern – eine erhebung. *Journal für Mathematik-Didaktik, 15* (3/4), 211–251.

Underhill, R. G. (1988). Mathematics learners' beliefs: A review. *Focus on Learning Problems in Mathematics 10* (1), 55-69.

Note

[1] R.n stands for "respondent n".

USING TECHNOLOGY IN THE MATHEMATICS CLASSROOM AND ITS IMPACT ON GIRLS AND BOYS: TEACHER'S VIEW

Sonia Ursini
CINVESTAV, México

Gabriel Sanchez
UNAM, México

Mónica Orendain
ILCE, México

Cristianne Butto
CINVESTAV, México

Abstract: Since 1997 the Ministry of Education in Mexico has been sponsoring a national project in which technology (computers and TI-92 calculators) is used to support the teaching of mathematics at secondary school level (children aged 12 to 15 years old). One of our concerns during this project was to investigate if its implementation affects some of the aspects of girl's and boy's behavior in the classroom. A first approach consisted in investigating teachers' views concerning nine behavioral aspects (participation; capability to analyze a problem; capability to interpret the worksheets; initiative; requirement for help; dedication; defense of their own ideas; creativity; preference for working in teams or individually). The results show that teachers consider that to use technology in the mathematics classroom impacts the majority of these aspects and this impact is different for girls and boys. We found as well that male and female teachers have different perceptions of the observed behavioral changes.

Introduction

In the last decade there has been a strong tendency to introduce technology in the mathematics classroom. The intention in using it is to support students' mathematics learning. However, does the presence of technology in the mathematics classroom have some impact on gender differences? Does it affect, for example, boys' and girls' behavior in the mathematics classroom? There has not been much research yet investigating these issues. In the late 80's Noss (1987), using Logo for developing children geometrical concepts, found that the Logo experience consistently favored girls. Apple (1989), on the other hand, considered that mathematics and science curricula were strongly contributing to the reproduction of gender differences (in spite of the efforts of the experts to change this). He stressed that this could get worst with the introduction of technology to support the learning of these subjects. A finding of the IEA Computers in Education Study is that no gender gap appears among U.S. students in either grade 8 or 11 in computer performance. In a Mexican study (Morales et al., 1998) it was found that to work with computers promotes the development of creativity both in boys and in girls. In another study Dinkheller (1994) (cited in Penglase and Arnold, 1996) found that to work with calculators produced anxiety in both male and female starting college students but that after getting used to it, anxiety disappeared in both of them. These short comments and findings put forward the necessity for more investigation in order to find out the ways in which using technology in the mathematics classroom impacts girls and boys contributing or not to reinforce gender differences.

During the last three decades there has been a considerable amount of research studies investigating gender differences in mathematics learning. Studies on achievement, performance, attitudes, attributions were developed (Fennema, Leder, 1996; Hanna, 1989; Figueiras et al., 1998). Consistently, some differences in performance favoring males have been found, in particular when high cognitive level skills were required (Leder, 1992). There is evidence stressing that cultural and social pressure on male and female lead to different behavior that is internalized by individuals and this leads to different beliefs and attitudes that strongly affect the learning of mathematics. These different behaviors can be observed for example in relation to active participation, call for attention, requirement for help, involvement and dedication, creativity, discipline. Several research reports developed in different countries show that boys tend to be more active in the classroom and they participate more than girls; they ask for attention and for help more than girls and they receive more attention and help (Koehler, 1990; Subirat & Bruller, 1999). Some of them stress that not to help girls to develop a more active attitude can be detrimental for their learning because an active attitude is fundamental for the learning process and the acquisition of new knowledge (Subirat & Bruller, 1999; Cooper, Marquis & Ayers-Lopez, 1982). Girls, on the other hand, are usually considered to be more dedicated and constant (Figueiras et al., 1998). Although it is not easy to find a creative behavior in students, there are some characteristics (for example, paying more attention to the process than to the result, discipline and hard work, self-criticism) that can favor its development and these characteristics are found more often in women than in men (Maslow,1983).

EMAT: A Mexican Project

Since 1997 the Ministry of Education in Mexico has been sponsoring a national project, EMAT (Teaching Mathematics with Technology), in which technology (computers and TI-92 calculators) is used to support the teaching of mathematics at secondary school level. The software used in this project are: Spreadsheets (for arithmetic, pre-algebra and algebra); Cabri Géomètre (for geometry); SimCalc MathWorlds (to approach the idea of variation and its different representations); Stella (for modeling simple situations). A series of activities having these software as support and linked to Mexican curriculum were designed by local and external experts and presented through worksheets. Activities in worksheets were provided to the teachers involved in the project (16 teachers for the first year of the project and 73 teachers more for the second and third year; in total they attended about 10,000 children) after they attended the workshops in which they learned to use the TI-92 and one of the software mentioned above. During the workshops the didactical approach, from practice and particular examples to general theoretical principles, characterizing EMAT was stressed. The dynamic of the classroom organization was stressed as well: students work in pairs or triads; after a short introduction they are given the worksheets corresponding to the programmed activities; the role of the teacher is to observe pairs' or triads' work and to support them by answering their questions, providing hints and, when necessary, suggesting approaches; periodically the teacher organizes group discussions in order to arrive to a common understanding of the mathematical concepts involved in the activities. The proposed approach and the class dynamics are opposed to the one usually followed in Mexican schools in which general theoretical statements precede exercises and students tend to conform a passive audience: after listening to the teacher, they solve the proposed exercises individually.

One of our concerns during this project was to investigate teachers' point of view about behavioral changes in girls and boys when technology is used in the mathematics classroom. Some of the aspects we decided to investigate have been already analyzed by other researchers observing mathematics classrooms without technology. Other aspects concerned directly the EMAT project (for example, aspects 2 and 3). The aspects observed were:

1. **participation** (he/she comments about the proposed tasks with the teacher and/or mates; he/she intervenes in group discussions);
2. **capability to analyze a problem** (he/she understands the posed problem; is able to analyze the outcomes obtained on the screen in order to answer the questions of the worksheet);
3. **capability to correctly interpret the worksheets** (he/she is able to follow the indications of the worksheet and understands the purpose of the questions posed);
4. **initiative** (he/she proposes possible solutions and activities without consulting with the teacher; is able to take decisions autonomously);
5. **requirement for help** (he/she asks the teacher or a mate for help in order to develop the proposed task);
6. **dedication** (he/she gets involved in the task and persists in it);
7. **defense of their own ideas** (he/she is able to sustain his/her points of view with the teacher and mates);
8. **creativity** (he/she approaches the solution of the task in an original way and, on occasions, develops activities not indicated explicitly in the worksheet);
9. **preference for working in teams or individually** (he/she prefers working in pairs/triads and co-operates in the solution of the task or he/she prefers working on him/her own, independently of others).

Methodology

In order to obtain the required information we worked with 24 teachers (15 men and 9 women) that were attending EMAT groups and that voluntary decided to participate with us. Their professional background was heterogeneous (professional teachers, engineers, agronomists and chemists). Their experience as mathematics teachers was 17.13 years in the mean with a range of 26 years (31-5). The great majority (18) had two years experience in working in EMAT and 6 of them have been working in EMAT for already three years. Their mean age was 41.5 years with a range of 33 years (58-25).

As said above 9 female teachers participated in the study. Two of them had three years experience in EMAT and 7 had two years experience in EMAT. Female teachers' mean age was 40 years with a range of 32 years (57-25). Their experience as mathematics teachers was 18.12 years in the mean with a range of 20 (28-8). Four out of the 15 male teachers had three years experience in EMAT and 11 had two years experience in EMAT. Male teachers' mean age was 42.4 with a range of 28 years (30-58). Their experience as mathematics teachers was 16.6 in the mean with a range of 26 (31-5).

When starting the project none of them had experience in using computers nor in using them to support the mathematics classroom. We discussed with them the 9 aspects in order to share a common meaning for each one of them. Each teacher was asked to choose one, two or three (this was his/ her decision) EMAT groups he/she was attending and to give a mark to each student for each aspect. To qualify children behavior concerning each one of the nine aspects they were asked to use marks: 1 –when the considered aspect was almost never observed; 2 – when it was observed on occasions; 3 – when it was almost always observed. Aspect 9 (preference for working in pairs/triads or individually) was marked 3 if the preference was for working in pairs/triads, 2 if the student did not show a specific preference and 1 if he/she preferred working alone.

The behavior of a total of 1113 students (568 boys and 545 girls) aged 12-15 years old was analyzed in this way. They were attending different school grades and their experience in working in EMAT was different: 576 has been working in EMAT only for one year, 459 for two years and 78 had a three years experience. The great majority belonged to medium and medium-low social class. Additionally to calculators, 597 were using spreadsheets and 516 were working with Cabri Géomètre. After analyzing the data, 4 teachers (2 with 2 years and 2 with 3 years experience in EMAT) were interviewed.

Results and Discussion

The scores obtained by students for each one of the aspects considered were initially grouped together and they were analyzed using the Kruskal-Wallis test (Table 1). The variables considered were: students' sex; and students' permanence in EMAT.

The purpose of this analysis was to find out if teachers' marks reflected significant differences when students' sex or when students' experience in working in EMAT were considered. The results presented in Table 1 show that when the variable sex is analyzed, significant differences between girls and boys are found for all the nine aspects considered. Significant difference at level .01 can be observed for the aspects numbered 1, 2, 3, 4, 6 and 9. That is, teachers consider that there are significant differences between girls and boys concerning: participation (1) in the classroom; capability to analyze a problem and the outcomes obtained (2); capability in interpreting a worksheet (3); initiative (4) and dedication (6) when solving a task; and preference for working in teams or individually (9). There are as well significant differences, but at level .05, concerning requirement for help (5); defense of their own ideas (7); and creativity (8). Significant differences for almost all the aspects (except aspect 9) are found as well when variable permanence in EMAT is analyzed. For aspect 5 (requirement for help) the difference is significant at level .05 while, for all the other aspects the difference is significant at level .01.

These results suggest that the experience in using technology for learning mathematics does affect the great majority of the aspects considered and its impact is different for boys and girls. The only aspect that does not seem to change substantially when technology is used, at least in the way in which it was used in EMAT, is students' preference for working in teams or individually. In EMAT we were trying to promote work in pairs or triads, however it must be taken into account that the great majority of teachers were not used to this way of working. Although two or three students were usually sharing one computer, teachers were not encouraging them to work as a team and this decision depended mainly upon the students themselves. Teachers confirmed this when interviewed. For example, a teacher commented "On occasions, when there is a computer free, a boy moves to it in order to work on his own .... often girls ask for permission to move to another team .... in general, I observed that there are more boys than girls like to work on their own. Girls tend to prefer working in teams."

In order to have more information about the differences found using the Kruskal-Wallis test, data were re-grouped and for each aspect the mean scores given to boys

Table 1
Analysis of variance using the Kruskal-Wallis test :
χ^2 values calculated for each one of the 9 aspects observed

Variable	Aspects observed								
	1	2	3	4	5	6	7	8	9
Per sex	27.1*	15.2*	11.49*	9.42*	6.0**	15.84*	4.0**	5.41**	8.68*
Per EMAT	29.28*	18.93*	37.5*	25.17*	7.20**	51.55*	44.31*	14.03*	5.25

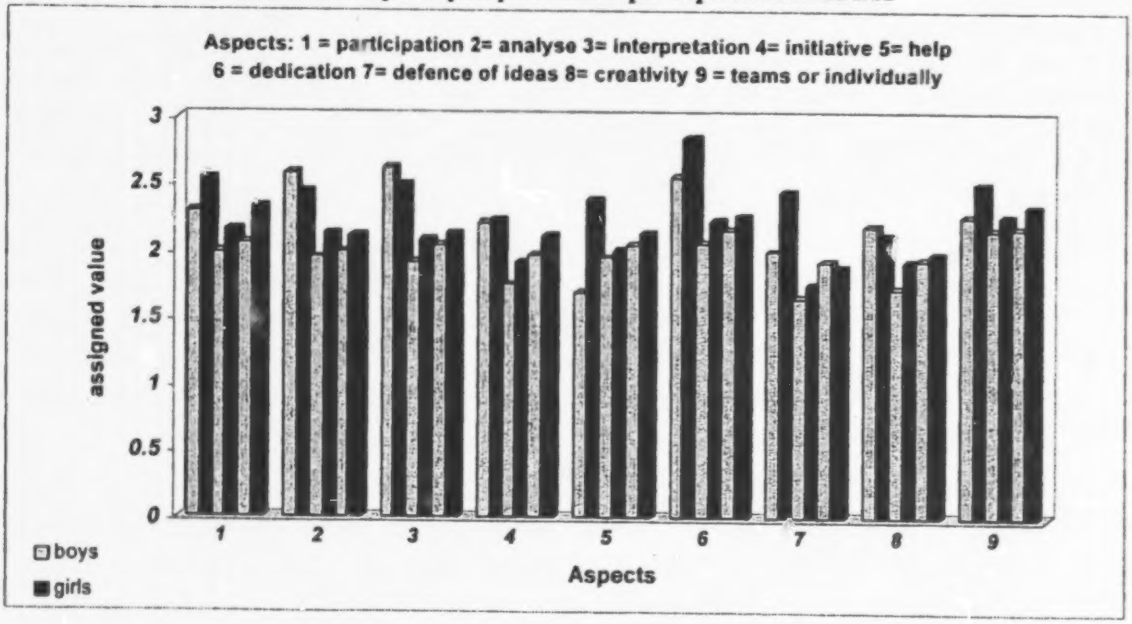
* $p < .01$ ** $p < .05$

and to girls with respectively one, two and three years experience in EMAT were analyzed. These scores are presented in Figure 1, where for each one of the 9 aspects considered there are three couple of bars. For each couple the first bar represents the mean of the scores given to boys and the second the mean of the scores given to girls. For each aspect, the first, second and third couple of bars refers, respectively, to students with three, two and one year experience in EMAT.

The mean scores presented in Figure 1 confirm the differences per sex for almost all the aspects examined. It can be observed that girls with one or two years experience in EMAT, obtain higher scores than boys in almost every aspect. The only exception is in the defense of their own ideas (7) for which the score boys with 1 year experience in EMAT have, is slightly higher than the one girls have. That is, when they start using technology boys are slightly better than girls in defending their point of view. For students with one-year experience in EMAT the highest difference between boys and girls corresponds to participation (1). From the point of view of teachers, girls starting to use technology tend to participate much more than boys. When interviewed, a teacher explained "Usually girls discuss the proposed task between themselves and with the teacher too, much more than boys. ... Girls tend to ask for help more than boys and they intervene more during group discussion."

Figure 1 shows as well that for almost all the aspects, the mean scores obtained by girls and boys with two years experience in EMAT is lower than the one obtained by students with less experience in using technology. That is, from the point of view of

Figure 1
Mean scores per aspect per sex and per experience in EMAT



teachers, both boys and girls who have worked in EMAT for two years participate less (1), they have less capability to correctly interpret the worksheets (3), they have less initiative (4), they show less dedication (6), they show less capability to defend their point of view (7) and they are less creative (8), than their mates who are just beginning to use technology. When interviewed all the teachers agreed that this outcome could not be attributed to EMAT but to different factors as: students' age and development, and students' own interests. It must be remarked that all the students with two years experience in EMAT were attending the second year of secondary school (children aged 13-14 years old). Teachers explained that the same happens for other subjects. A teacher said "I think that these results do not depend on the use of technology ... at that age students are not interested in what we are teaching them; they do not understand its usefulness. In general, when they start secondary school they are very enthusiastic, during the second year their interest slows down dramatically and in the third year they begin to understand the utility of what they are learning and their interest rises again. This does not depend on EMAT, it is due to their own development".

From Figure 1 it can be observed that the only exception concerns aspect 2 (capability to analyze a problem). Girls and boys with two years experience perform as well as their mates with one year experience. There is as well no change for boys in aspect 9 (preference for working in team or individually). Scores referring to aspect 5 (requirement for help) as well, slow down indicating that students with two years experience tend to ask less for help than their mates with one year in EMAT.

The panorama just depicted above changes substantially when the scores given to students with three years experience in EMAT are examined (first couple of bars for each category). For all the aspects studied the scores obtained by students with three years experience in EMAT are substantially different from those their mates with less experience have. Except for boys in aspect 5 (requirement for help), these scores are clearly higher. In particular, there are three aspects (2, 3 and 6) for which the difference with previous scores is remarkable. An improvement in students' capability to analyze a problem (2) and to interpret the worksheets (3) is not surprising after three years in EMAT where all the work with technology is supported by worksheets and students are asked to solve problems. More interesting is the result concerning dedication (6). Observe that for both, boys and girls with three years experience with technology, the score for this aspect increases substantially. However, it increases much more for girls than for boys. A possible explanation is that, in general, girls in Mexico have less opportunities than boys to be in contact with technology, therefore its presence in the mathematics classroom stimulates much more girls' curiosity and interest than boys'. This leads girls, more than boys, to increase their involvement in the proposed tasks and their determination to solve them. Teachers explained "Girls are more dedicated than boys, girls focus on the worksheet and they try to solve it. Boys do not pay too much attention to the worksheet and they focus their attention elsewhere, they explore more the other possibilities offered by the computer. Girls tend to follow instructions, boys feel more free to explore." Another teacher commented "Boys and girls are dedicated but boys less that girls, they are not so persistent, they feel they do not need to work too much, they feel 'culturally superior' (sic)".

It is as well interesting to notice that there are three aspects (2, 3 and 8) for which boys obtain scores higher than girls. That is, although both, boys and girls have improved in these aspects, teachers consider that boys with three years experience in EMAT are better than girls in analyzing problems and their outcomes (2), in interpreting the worksheets (3). Moreover, boys are considered to be more creative (8) than girls. Creativity is a characteristic difficult to find in students (Maslow, 1983), however, from the point of view of teachers EMAT has a positive impact on this aspect. When interviewed a teacher explained "I think that students develop their creativity when they look for a possible solution to the problem they are facing. Although they follow the worksheets, when they realize that something is wrong they look for another approach. This is evident to me in particular when they work with computers, using Cabri.". Another teacher commented "Students' creativity develops a lot. They develop different strategies, different ways to solve a problem and they find general solutions, they learn how to generalize".

For aspect 5 (requirement for help), girls' score is considerably higher than the one assigned to boys. That is, girls with three years experience in EMAT ask for help (5) much more than boys. Moreover, girls with three years in EMAT ask for help more than their mates with two or one year experience. This contrasts with boys who ask for help substantially less than their mates with less experience. A teacher explained "Girls ask more than boys for help, both about technical aspects and about mathematics. Boys ask less because when somebody ask others boys take mock of him."

Another interesting outcome concerns participation (1) and defense of their own ideas (7). Girls' scores for these two aspects are remarkably higher than those of the boys with the same experience in EMAT. A teacher's comment was the following "Boys are very passive, they accept others' point of view very easily. Girls are good in listening to others arguments but when they do not agree they argue back. They argue within the team too when they are convinced about an idea. ... Boys are obstinate, they think they are never wrong, they are stubborn; girls reflect and when they are convinced they defend their ideas."Finally, students' preference for working in teams or individually (9) does not change when their experience with technology increases. For both, boys and girls, there is a tendency to prefer working in a team and this is slightly stronger for girls.

In order to see if there were differences in the way in which male and female teachers were marking students in relation to the nine aspects, the mean scores were compared (Table 2). In Table 2 the first raw refers to the nine aspects. The index indicates the presence of a significant difference between male and female teachers' way of marking. The numbers in the second raw represent the mean of the scores given to students by male teachers for each aspect. Numbers in the third raw (in brackets) correspond to the mean of the scores given to students by female teachers. The forth and the fifth row indicate the corresponding standard deviation.

Table 2

The information presented in Table 2 indicates that there is a significant difference between male and female teachers in the way they mark students' behavior. Observe that the mean score given by female teachers is lower than the one given by men teachers for all the aspects except creativity (8). An explanation for these results might be that female teachers tend to be more demanding with their students than men teachers are. This can be deduced from the mean marks given to participation (1), capability to analyze a problem (2), interpretation of worksheets (3), initiative (4) and dedication (6). Moreover, concerning requirement for help (5) it is interesting to notice that female teachers consider that students in general do not ask too much for help. In contrast male teachers seem to consider that students require a lot of help. This might be explained considering the social role played in Mexico by women, for whom very often it is assumed that their "natural" role is to help, satisfy and take care of others. This is not assumed as a characteristic of male nature. These assumptions might originate the difference in the mean scores given to students for this aspect. Another interesting result refers to aspect 7 (defense of their own ideas). This aspect had the lower score by both male and female teachers. An explanation for this might be that given the traditional teaching approach usually employed in Mexican schools, students are not used to or encouraged to make a defense of their own ideas. However, when inter-

Table 2
Means scores and standard deviations per teachers' sex

	Aspects								
	1^1	2^1	3^2	4^2	5^3	6^1	7	8^3	9^1
μ	2.21 (2.08)	2.11 (2.02)	2.11 (2.05)	2.01 (1.90)	2.16 (1.84)	2.24 (2.17)	1.85 (1.84)	1.89 (1.96)	2.32 (2.12)
	.74 (.85)	.71 (.68)	.72 (.70)	.82 (.73)	.68 (.67)	.69 (.73)	.82 (.71)	.72 (.75)	.71 (.82)

$^1p < .001$ $^2p < .01$ $^3p < .05$

BEST COPY AVAILABLE

viewed both male and female teachers commented that EMAT was strongly promoting the development of this aspect and that girls, in particular, were developing this capability more than boys.

A comparison between the mean scores assigned by male and female teachers to girls and boys separately, was suggesting a general tendency to assign higher marks to girls than boys. However, an analysis of variance (ANOVA), although not totally appropriate for this kind of data, allowed us to re-examine the data differentiating teachers' sex and students' sex. The results obtained showed that there was not significant difference between the marks male and female teachers assigned to boys and girls. In all the cases we obtained $F<1$ (the highest value was $F = .69$) with $p >.05$.

Conclusions

The results presented in this paper provide evidence showing that the presence of technology in the mathematics classroom can have significant impact on several aspects of students' behavior. The way in which technology was used in EMAT helped both girls and boys to increase their involvement in the task and to persist in it. Although girls were more dedicated than boys, this behavioral aspect increased substantially for boys after working with technology for three years. Teachers considered that to work with technology helped students, in particular boys, to be more creative. A very important outcome refers to girls' participation and defense of their points of view. From the teachers' perspective, girls were participating more than boys, and they were defending their ideas with greater enthusiasm than boys. These outcomes contrast with results from other research where it is stressed that boys tend to participate more than girls. Girls' active behavior might be reflected as well in the increase of their requirement for help. Other researchers report that in general boys are more active and they ask for help more than girls. Another interesting result concerns the difference found between male and female teachers concerning the way they assign marks to students. More research should be developed in this direction.

References

Apple, M. W. (1989). *Maestros y textos: Una economía política de las relaciones de clase y de sexo en educación.* Buenos Aires: Paidós.

Cooper, C. R., Marquis, A., & Ayers-López, S. (1982). Peer learning in the classroom: Training developmental problems and consequences of children's spontaneous interactions. In L. C. Wilkinson (Ed.), *Communicating in the classroom*, (pp. 69-84). New York: Academic Press.

Figueiras, L., Molero M., Salvador, A., & Zuasti, N. (1998). *Género y Matemáticas.* Madrid: Síntesis.

Hanna, G. (1989). Mathematics achievement of girls and boys in grade eight: Results from twenty countries. *Educational Studies in Mathematics, 20*, 225-232.

Kloosterman, P. (1990). Attributions, performance following failure, and motivation in mathematics. In E. Fennema, & G. C. Leder (Eds.), *Mathematics and gender* (pp. 96-127). New York: Teachers College Press.

Koehler, M. S. (1990). Classrooms, teachers, and gender differences in mathematics. In E. Fennema, & G. C. Leder (Eds.), *Mathematics and gender* (pp.128-148). New York: Teachers College Press.

Kuyper, H., & Van der Werf, M. P. C. (1990). Math teachers and gender differences in math achievement: Math participation and attitudes towards math. In G. Booker, P. Cobb, T. Mendicuti (Eds.), *Proceedings of the Fourteenth International Conference of the Psychology of Mathematics Education* (pp. 143-150). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Leder, G. C. (1992). Mathematics and gender: Changing perspectives. In D. A. Grows (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 597-622). Reston, VA: National Council of Teachers of Mathematics.

Leder, G. C. (1996). Equity in the mathematics classroom: Beyond the rhetoric In L. H. Parker (Ed.), *Gender, science and mathematics* (pp. 95-104) Netherlands: Kluwer.

Leder, G. C., & Fennema, E. (1990). Gender differences in mathematics a synthesis. In E. Fennema & G. C. Leder (Eds.), *Mathematics and gender* (pp. 188-200). New York: Teachers College Press.

Maslow, A. (1985). *La personalidad creadora.* Barcelona: Kairos.

Middleton, J. A., & Spanias, P. A. (1999). Motivation for achievement in mathematics: Findings, generalizations and criticism of the research. *Journal for Research in Mathematics Education, 30,* 65-88.

Morales, C., Turcott, V., Campos, A., & Lignan, L. (1998). Actitudes de los escolares hacia la computadora y los medios para el aprendizaje. In *Tecnología y comunicaciones educativas*

Noss, R. (1987). Children's learning of geometrical concepts through Logo. *Journal for Research in Mathematics Education, 18*, 343-362.

Penglase, M., & Arnold, S. (1996). The graphics calculator in mathematics education: A critical review of recent research. *Mathematics Education Research Journal, 8*, 58-90.

Subirat, M., & Brullet, C. (1999). Rosa y Azul. In M. Belausteguigoitia & A. Mingo (Eds.), *Géneros prófugos: Feminismo y educación.* México: Paidos.

Note

This research was sponsored by the National Council for Science and Technology (CONACYT) Group Project G26338S.

NUMERAL FORMS IN EARLY ELEMENTARY GRADE LEVELS

M. Faaiz Gierdien
Michigan State University

This poster is about a comparison between the ways students in early elementary grade levels write numeral forms and a historical development of similar numerals. I observed that despite clear directions in commercial textbooks, many of my students made variant forms of our modern numerals. The objective of the poster is therefore to attempt to explain the ways my students write numerals.

According to Das (1927), Hill (1915) and Lemay (1977) among others, numeral forms as we know and write them today have a historical development. Lemay makes the argument that it is in Spain where our present numeral forms evolved. He gives the examples of scribes who copied lengthy manuscripts from Arabic into Latin or medieval Spanish. The numeral forms the scribes copied stabilized over a long period of time as ambiguities in different forms were addressed, making them acceptable in a community of practice. For example, Lemay refers to transitional forms and a possible "rotation theory" to explain how numerals evolved from Arabic and Latin into their present forms as we know them today. When early elementary grade level students copy numeral forms illustrated in commercial textbooks they are at a point where they enter into a community of practice. i.e., the classroom and the realm of formal mathematics itself. My argument is that variations in elementary students' numeral forms could be explained by looking at historical episodes that the scribes went through. There is thus a case of ontogeny recapitulating phylogeny.

The method involves a comparison in similarities between variations in the elementary student work and examples in literature on the historical development of numerals. Particular examples from zero through nine will be examined. The data sources are therefore the student work and numeral forms cited in Das (1927), Hill (1915) and Lemay (1977). Results show interesting similarities between historical variations and the student work.

This presentation has the goal of looking at the teaching and learning of mathematics differently. For example, there are moments when we can encounter historical episodes in the development of mathematics in the classroom. Students' struggles in copying and writing numeral forms are not unique. It has historical precedents and supports their understanding of number and thus their growth in mathematical understanding. The latter is a goal of the International Group on the Psychology of Mathematics Education (PME).

References

Das, R. S. (1927). The origin and development of numerals, *The Indian Historical Quarterly*, *3*, 97-120; 356-375.

Hill, G. F. (1915). *The development of arabic numerals in Europe.* Oxford.

Irani, R. A. K. (1955). Arabic numeral forms. *Centaurus*, *4*, (1), 1-12

Lemay, R. (1977). Hispanic origins of our present numeral forms. *Viator, 8*, 435-459.

STUDENTS' BELIEFS ABOUT MATHEMATICS AFTER MOVING OUT OF REFORM CURRICULAR EXPERIENCES INTO MORE TRADITIONAL CURRICULAR EXPERIENCES

Amanda Jansen Hoffmann
Michigan State University

This poster presents preliminary results of a study which examines students' beliefs about the nature of mathematics in a unique setting: after these high school students moved out of a "reform-oriented" middle school mathematics curriculum, the Connected Mathematics Project (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1997), into a high school with a fairly traditional curriculum. This project compares the beliefs that students express at this setting with results from research on students' beliefs in other traditional school mathematics settings (Boaler, 1997; Schoenfeld, 1988). I hypothesize that, in the case of this curricular shift, students' beliefs will vary from those expressed in other traditional school mathematics settings.

This project began out of a struggle to assess students' beliefs about the nature of mathematics in the context of a larger project (Jansen & Herbel-Eisenmann, 2001). Our participants' testimonies about their experiences with the shift between reform and traditional curricula in interview settings have focused on their learning experiences (e.g., teacher's approach, opportunity for discussing ideas) rather than their views on the nature of mathematics itself. In order to learn more about students' beliefs about mathematics, I extend existing research on beliefs by looking explicitly at the ways that our participants talk about mathematics as they discuss a set of problem tasks presented during a clinical interview, and by considering the extent to which students' experiences with curricular shifts and their beliefs interact.

Perspectives / Theoretical Framework

For the sake of this project, I utilize situated and cognitive perspectives in a complementary manner, as I consider the meanings students develop and their experiences to be mutually constitutive, similar to Cobb and Yackel's notion of reflexivity (1996). Neither beliefs nor experiences exist independently. Just as beliefs are the orienting ideas through which one interprets and constructs her view of the world (as beliefs make up the individual's current knowledge of the world and are used to create meaning (Cobb, 1986)), the particular contexts individuals experience guide the meanings they are able to create.

Research Questions

What beliefs about mathematics are expressed among high-school students who have moved from a reform-oriented to a more traditional mathematics program? In particular, what criteria do students use to determine what is mathematical about problem tasks?

Method

As part of a larger investigation of students' experiences as they moved from a reform-oriented middle school curriculum to a more traditional high school curriculum (Jansen & Herbel-Eisenmann, 2001), I designed an interview protocol in which I asked students to sort problem tasks from their current and past mathematics textbooks, and requested for them to comment whether the problems felt more or less like "math," and why. These interview transcripts were analyzed with particular emphasis on elucidating students' beliefs about mathematics, specifically looking for comparisons and contrasts to beliefs typically found in traditional school mathematics settings.

References

Boaler, J. (1997). *Experiencing school mathematics: Teaching styles, sex, and setting*. Buckingham: Open University Press.

Cobb, P. (1986). Contexts, goals, beliefs, and learning mathematics. *For the Learning of Mathematics*, *6* (2), 2-9.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31* (3/4), 175-190.

Jansen, A., & Herbel-Eisenmann, B. (2001). *Moving from a reform junior high to a traditional high school: Academic, adaptive, and affective transitions.* Paper presented at the American Educational Research Association, Seattle, WA.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (1997). *Connected mathematics project.* Palo Alto, CA: Dale Seymour.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of "well-taught" mathematics courses. *Educational Psychologist, 23* (2), 145-166.

ON-LINE MATH TEACHER CONVERSATIONS: GRAPHICAL, STATISTICAL AND SEMANTIC ANALYSES

Eric Hsu
Department of Mathematics
San Francisco State University

Purposes

The notion of community of practice is a powerful idea in teacher development research and practice. The essential idea is that learners learn and refine their craft through sustained structured interactions with a professional hierarchy of colleagues in the context of authentic work (Lave and Wenger, 1991). Since the rise of the Internet, it has become a crucial task to find effective extensions of this idea to the online realm (Pearson, 1999).

Indeed, there is reason to believe such extensions of off-line social structures are possible. The recent work of social network theorists such as Wellman and Gulia (1999) suggests that modern live social relations have a structure compatible with virtual social dynamics. There is also evidence that when online relations are combined with live ones, the combination can have surprising and powerful effects (e.g., Bos et al., 1995) on teacher development. It has thus become a crucial task to find intelligent and insightful ways of analyzing teacher interactions in electronic settings

Perspectives

We present a number of known techniques (Riel and Harasim, 1994) drawing from discourse analysis and social network theory along with new graphical approaches, and assess them as analytical tools in the theoretical framework of collective social exchange theory (Chen and Gaines, 1998).

Data Sources

Public an private electronic messages were archived from a distance-learning course run in spring 2000. The students were algebra teachers from various sites in Texas, who had regular, asynchronous class discussions over the Internet supplemented by two videoconferences. The subject matter was "contextual learning," a constructivist theory of learning based on modeling and applications. Hour-long interviews were held with ten students at course end.

Methods of Inquiry

We analyzed the electronic communications using methods of (in the language of Riel and Harasim) discourse analysis, message thread analysis and semantic trace analysis, comparing our results with emergent themes from a grounded theory analysis of the exit interviews. The thread and semantic trace analyses attempt to track the flow of teacher interaction and the development of pedagogical notions in the community discourse. We further develop these analyses using analogues of graphical tools from social network theory and computer-assisted text analysis.

Results

By analyzing the topology of the conversation networks and the geometry of graphical representations of the teacher interactions, we gain insight into a number of aspects of the social dynamics of the class. Robustness of conversations, the development of ideas, and their lack can sometimes be vividly seen in conversation maps. Our analysis may provide ways to distinguish between engaged and superficial reflective activity, by tracking the changing influence of teachers on each other's ideas and discourse.

We will argue that these insights combined with ideas from collective social exchange theory may explain the class dynamics and indeed the apparent lack of enduring effect of the class.

References

Bos, N., Krajcik, J., & Patrick, H. (1995). Telecommunications for teachers: Supporting reflection and collaboration among teaching professionals. *Journal of Computers in Mathematics and Science Teaching, 14* (1/2), 187-202.

Chen, L. L. J., & Gaines, B. R. (1998). Modeling and supporting virtual cooperative interaction through the world wide web. In F. Sudweeks , M. McLaughlin, & S. Rafaeli (Eds.), *Network and netplay: Virtual groups on the internet* (pp. 221-242). Cambridge, MA: MIT Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral practice.* New York: Cambridge University Press.

Pearson, J. (1999). Electronic networking in initial teacher education: Is a virtual faculty of education possible? *Computers & Education*, *32*, 221-238.

Riel, M., & Harasim, L. (1994). Research perspectives on network learning. *Machine-Mediated Learning*, *4* (2 & 3), 91-113.

Wellman, B., & Gulia, M. (1999). Virtual communities as communities. In M. A. Smith & P. Kollock (Eds.), *Communities in cyberspace* (pp. 167-194). New York: Routledge.

COOPERATIVE LEARNING IN A FIFTH GRADE ENGLISH AS A SECOND LANGUAGE MATHEMATICS CLASS

Theresa L. Jardine
Washington State University
Ltjardine@msn.com

Amy Roth McDuffie
Washington State University
mcduffie@tricity.wsu.edu

Purpose

In accordance with reform in mathematics education in the United States, in the Washington State Learning Standards for math, students are expected not only to give the right answer but to understand and explain how they arrived at their answer. "Explaining how" is a key element missing in many math lessons. I conducted an action research project to see if I could successfully implement cooperative groups where students are actively discussing math concepts and explaining their reasoning to peers (Hubbard & Power, 1993). By introducing cooperative groups in mathematics, I also wanted to motivate students to participate and increase their learning. Among the issues discussed in my research paper are: does a cooperative group plan engage the low-ability students while at the same time benefit all other students?; and, does the cooperative structure increase students' enjoyment and interest for mathematics?

Theoretical Framework

According to Leikin and Zaslavsky (1999), simply throwing together a group of students and calling it a "cooperative group" does not work. Four necessary conditions need to be met to constitute a cooperative-learning setting: (1) Students learn in small groups; (2) The learning tasks require that the students mutually and positively depend on one another and on the group's work as a whole; (3) The learning environment offers all members of the group an equal opportunity to interact with one another; and (4) Each member of the group has a responsibility to contribute to the group work and is accountable for the learning progress of the group.

In addition to the above criteria, Slavin (1987) strongly emphasizes that cooperative groups should be heterogeneous. The groups usually have four members, one high achiever, two average achievers, and one low achiever. The students in each group are responsible not only for learning the material being taught in class, but also for helping their groupmates learn (Slavin, 1987). In a study by Linchevski and Kutscher (1998), they found that average and weaker students' achievements in mathematics showed significant gains when cooperative groups were used, whereas the loss in achievements of the higher ability students was insignificant. Artzt (1999) also supports heterogeneous ability groups but for an additional reason: students of heterogeneous groups are more likely to produce different solutions to math problems and thus engage in more discussion that produces active learning.

Methods

The targeted population was a fifth grade English as a Second Language (ESL) Transition class at a southeastern Washington elementary school. Throughout the thirteen-week intervention, data was collected on seventeen students. Among the data that was collected to assist in forming a conclusion were videotapes of cooperative group work, scores on tests, a pre- and post-survey, and student interviews. I used a qualitative data analysis technique of analytic induction in searching for patterns and themes.

Results

Students did look forward to math lessons and working together to solve problems. Students talked about math to each other in their own terms and discussed how they arrived at a solution and correcting each other's errors. I frequently saw children formulating new concepts in their mind while they worked in their groups. Moreover, consistent with research advocating heterogeneous grouping, in my study: (1) All but one low achiever showed improvement; (2) four out of seven average achievers showed improvement; and (3) all the high achievers maintained their status.

Relationship to PME-NA Goals: In education, teachers are used to fads working their way into the curriculum just to be replaced later by a newer fad. I don't believe cooperative learning is a fad, it is a much-needed learning approach. However, I felt it needed to be studied in a math setting so that I and others could develop a deeper and better understanding of the psychological aspects of teaching and learning of mathematics from a classroom teacher's perspective.

References

Artzt, A. F. (1999). Cooperative learning in mathematics teacher education. *The Mathematics Teacher, 92* (1), 11-17.

Hubbard, R., & Power, B. (1993). *The art of classroom inquiry.* New Hampshire: Heinemann.

Leikin, R., & Zaslavsky, O. (1999). Cooperative learning in mathematics. *The Mathematics Teacher, 92* (3), 240-246.

Linchevski, L., & Kutscher, B. (1998). Tell me with whom you're learning, and I'll tell you how much you've learned: Mixed-ability versus same-ability grouping in mathematics. *Journal for Research in Mathematics Education, 29* (5), 553-554.

Slavin, R. E. (1987). Cooperative learning and the cooperative school. *Educational Leadership, 45* (3), 7-13.

FACTORS CONTRIBUTING TO MIDDLE SCHOOL GIRLS' SUCCESS IN MATHEMATICS: A STATISTICAL STUDY

Michelle P. Longest
North Carolina State University
mplonges@unity.ncsu.edu

Sarah B. Berenson
North Carolina State University
berenson@unity.ncsu.edu

Researchers have studied the dynamics of women's success in mathematics (for example, AAUW, 1992; Berenson, Vouk, Robinson, Longest, & Clark, 2001; Kerr, 1997; Smith, 2000). Studies reported that factors such as confidence play a crucial role in middle school girls' success (AAUW, 1992; Kerr, 1997). Sponsored in part by the National Science Foundation, *Girls on Track* is a year-round program at North Carolina State University and Meredith College. This program encourages increased interest in mathematics and awareness of careers in mathematics for girls selected to take Algebra I in seventh or eighth grade. Research conducted during the first two years of the project examined relationships between math achievement, proportional reasoning, and mathematics attitude factors as possible predictors of persistence in advanced math courses (Berenson, et al., 2001; Smith, 2000). This research answers several questions: 1. What are fast-track girls' attitudes towards advanced mathematics? 2. Do fast-track girls' attitudes change after taking Algebra I? In July 2000 and April 2001, we administered a 48-item math survey to 30 middle grade girls. The factors were: readiness, interest, support, confidence, gender bias, motivation, stereotypes, and priorities. The items were based on a survey conducted by the University of Minnesota Talented Youth Mathematics Program (UMTYMP) (Titus & Terwillinger, 1988). Statistical analyses of each of the eight factors produced varying results. Results of dependent difference of means tests for each factor indicated significant increases in *Girls on Track* girls' average levels of confidence (p-value < .01) and average levels of readiness (p-value < .01) for advanced math classes after having completed most of Algebra I. These confidence results differ from many research studies that have suggested a decrease in girls' levels of confidence as they move from middle school to high school (AAUW, 1992; Kerr, 1997). A significant decrease in the girls' average scores on belief in the presence of gender bias in math (p-value < .05) was found. Implications of this research might include the need to study why these changes are occurring and if certain interactions among the factors contribute to girls' persistence in advanced mathematics. These results generalize only to all of the fast-track girls who participated in *Girls on Track* 2000, and caution should be taken because we studied volunteer participants. This poster presentation relates closely to this year's PME-NA theme: *The Growth of Mathematical Understanding*, because this part of our longitudinal study examines the changes in the interaction of certain factors related to continued mathematical growth and success for middle school girls.

References

American Association of University Women Educational Foundation. (1992). *How schools shortchange girls: The AAUW report.* New York: American Association of University Women Educational Foundation.

Berenson, S., Vouk, M., Robinson, T., Longest, M., & Clark, M. (2001, April). *Girls on track: Developing a model to predict girls' success in advanced mathematics from middle school to college.* Paper presented at the Annual Meeting of the American Education Research Association [AERA], Seattle, WA.

Kerr, B. S. (1997). *Smart girls: A new psychology of girls, women and giftedness.* Scottsdale, AZ: Gifted Psychology.

Smith, N. H. (2000). *Confidence and achievement in fast track middle school girls.* Unpublished master's thesis, North Carolina State University, Raleigh.

Titus, J., & Terwillinger, J. (1988). *Gender differences in attitudes, aptitude, and achievement in a program for mathematically talented youth.* (Report No. TM 016 675). Silver Spring, MD: Association for Information and Image Management. (ERIC Document Reproduction Service No. ED 334 217)

Teacher Beliefs

THE BELIEFS AND INSTRUCTIONAL PRACTICES OF MATHEMATICS TEACHING ASSISTANTS PARTICIPATING IN A MATHEMATICS PEDAGOGY COURSE

Thomas C. DeFranco
University of Connecticut
defranco@uconnvm.uconn.edu

Jean McGivney-Burelle
University of Connecticut
mcgivney@uconnvm.uconn.edu

Abstract: A mathematics pedagogy course was developed and co-taught by mathematics and mathematics education faculty to 22 teaching assistants (TAs) in the mathematics department. Its purpose was to allow TAs to examine their beliefs about the teaching and learning of mathematics and to support them in changing their teaching practice. The course consisted of 5 seminar classes addressing issues surrounding pedagogy, epistemology, curriculum, and assessment. Throughout the course the TAs were asked to implement changes in their teaching based on class activities and discussion and to document their reflections on these changes, which served as the basis for subsequent class discussions. Journal entries, class assignments, interviews, and teaching observations served as sources for data and were analyzed using descriptive statistics and techniques from qualitative analysis. Results indicated that the TAs appeared to adopt a new set of beliefs regarding the teaching and learning of mathematics yet did not draw on these beliefs to inform their teaching practices. These results are presented and discussed in the context of research on beliefs and recommendations for the pedagogical preparation of TAs are offered.

Introduction

As early as the late 1800s, universities offered graduate teaching assistantships to attract students to graduate studies. However, by the end of World War II, colleges and universities expected graduate teaching assistants (TAs) to assume other responsibilities, such as grading and teaching (Hendrix, 1995). Today, TAs play a vital role in the day-to-day activities of university life and carry a considerable portion of the teaching load among two-and four-year colleges and universities. Further, it is projected that half a million new professors will be needed by the year 2014 (Baiocco & DeWaters, 1998) thus increasing the likelihood that TAs will continue to be an integral part of the teaching fabric of colleges and universities in the near future.

As the number of graduate students teaching college courses has increased, the adequacy of their preparation to teach such courses has come into question (Hammrich & Armstrong, 1995; Carroll, 1980). To remedy this situation, a number of universities have developed training programs or courses for TAs in disciplines, such as mathematics (National Advisory Group of Sigma Xi, 1989; Wagener, 1991; Hammrich & Armstrong, 1995; Damarin & West, 1979) and in areas such as, instructional strategies, learning styles, communication skills, undergraduate student needs, and issues specific to international teaching assistants (Enerson, 1996; Nyquist & Wulff, 1986; Travers, 1986; Wright, 1981; Damarin & West, 1979).

In spite of university efforts to prepare TAs to meet the challenges of teaching, research has found that TAs believe universities provide limited support in helping to prepare them to teach at the college or university level. Further, research (Monaghan, 1989) has indicated that many TAs have little or no prior instruction in pedagogical theory or experience in teaching and rely primarily on models of teaching they have experienced as students.

As a result of these issues, a one-credit mathematics pedagogy course was developed and co-taught (by mathematicians and mathematics educators) to the TAs in the mathematics department. Its goal was to provide an opportunity for the teaching assistants to examine their beliefs about teaching and learning mathematics and alter their teaching practice. The purpose of this study was to examine the nature of the TAs' beliefs as they progressed through this mathematics pedagogy course and to describe the constraints they faced in trying to make changes in their teaching practices.

Background of the Study

The study of teachers' beliefs and its impact on teaching and learning is relatively new (Thompson, 1992). However, a number of studies in mathematics education have found that teachers' beliefs about mathematics and the teaching and learning of mathematics play a significant role in shaping characteristic patterns of instructional behavior (Thompson, 1992). Ernest (1988) noted that among the factors that influence the practice of teaching mathematics, three are most notable: (1) the *social context* of the teaching situation, particularly the constraints and opportunities it provides; (2) the teacher's mental contents or schemas, particularly the *system of beliefs* concerning mathematics and its teaching and learning; and (3) the teacher's level of thought processes and *reflection*.

It is clear that the *social context* in which an individual teaches significantly shapes one's understanding of teaching and may provide constraints and opportunities for changing one's practice. As noted by Schoenfeld (1992), "the habits and dispositions of community members are culturally defined and have great weight in shaping individual behavior" (p. 340). The day-to-day routines and norms of classrooms and schools provide a cultural milieu in which individuals acquire a point of view with respect to the teaching and learning of mathematics. In a similar way, by virtue of their participation (i.e., as a student, teacher, and colleague) in the daily routines of university life and mathematics departments, TAs' view of the discipline and their instructional practices are shaped.

Some researchers (Green, 1971; Rokeach, 1964) have used the notion of a *belief system* as a metaphor, to describe the organizational structure of beliefs acquired by an individual. In this view, belief systems are dynamic and subject to change (Thompson, 1992) and may help explain certain behavior with respect to teaching and learning. Green (1971) has identified three dimensions of belief systems—a quasi-logical structure, the psychological strength between beliefs, and the clustering nature of beliefs. The quasi-logical structure of a belief system permits beliefs to be “primary’ (e.g., a belief that is used as a reason for other beliefs) or “derivative” (e.g., a belief derived from some other belief). The notion of which beliefs are most important, and thereby more resistant to change, has to do with the strength in which these beliefs are held. Psychologically “central”, or “core”, beliefs are held with greatest conviction and are least susceptible to change while “peripherally” held beliefs are more likely to be altered or changed. Finally, the third dimension of the belief system indicates that beliefs are held in “clusters” and generally in isolation from other clusters (Green, 1971).

In the mathematics education literature, there is an extensive body of research on teacher’s beliefs about mathematics and mathematics teaching and learning. In particular, research indicates that the beliefs held by teachers can have profound, though possibly subtle effects on their mathematics teaching (Thompson, 1984; Peterson, Fennema, Carpenter, & Loef , 1989). Researchers (Brown, Cooney, & Jones, 1990) have found that preservice teachers hold core beliefs regarding the teaching and learning of mathematics prior to formal teacher preparation coursework. Further, these beliefs may hinder one’s ability to align teaching practices with current reform efforts in mathematics education (Frykholm & Brendefur, 1997).

Together, the notion of the social context of the university and mathematics department, as well as the belief systems of the TAs, provided a foundation to examine the TAs’ understanding of the teaching and learning of mathematics and their classroom practice.

Research Questions

1. To what extent and in what ways did the mathematics pedagogy course lead to changes in TAs’ beliefs about mathematics and mathematics teaching and learning?
2. What factors help explain the nature of the TAs’ classroom practice?

Methods and Procedures

This study employed qualitative techniques to examine the teaching assistants’ beliefs and instructional practices. A description of the mathematics pedagogy course and participants in the study follows next.

Mathematics Pedagogy Course

The mathematics pedagogy course is offered through the mathematics department and is a requirement for new TAs in the department. The course is organized around five seminar-style class sessions, each two and one half-hours long, beginning in mid-September and ending in mid-November. This past year 2 faculty members from the mathematics department, 2 faculty members from the Neag School of Education, and 22 TAs attended class regularly. Classroom activities modeled a constructivist perspective of learning and assignments in-and out-of class encouraged the TAs to discuss and reflect on ideas about teaching and ways to change and improve their teaching practice. Typically, classroom sessions involved a cyclical process of class activities/discussion—classroom teaching—reflection that provided a support structure to help the TAs reflect on their beliefs about mathematics and teaching and implement changes in the classes they were teaching.

Participants

The Mathematics Department

At the time of the study, the mathematics department consisted of 30 full-time faculty and 3 adjuncts. With the help of the TAs, the department offers and teaches undergraduate and graduate mathematics courses and services a number of academic departments in the university. Being a Carnegie I Research institution, faculty believe their mission is to conduct research and publish their results within their respective mathematical fields. Normally, senior faculty members in the department teach upper division courses while junior faculty and TAs are assigned to teach the lower division courses. Also, senior faculty members are assigned to monitor and oversee the curriculum and the testing of remedial-level courses. Most instruction tends to model a transmission method of teaching. In recent years there has been shift in the culture of the department and an emphasis has been placed on curricula reform and improving classroom instruction.

The Teaching Assistants

There were 22 teaching assistants who participated in the mathematics pedagogy course. These students were either enrolled in Masters or Ph.D. programs in Mathematics and were supported by teaching assistantships. As part of their assistantship, TAs taught two remedial-level mathematics courses each semester. The TAs came from various backgrounds and cultures—for example, at the time of this study, there were TAs from a dozen countries in the mathematics department. In general, the TAs have acquired a view of the discipline and a view of teaching based on their own academic experiences. Further, they have had no formal training in teaching and lack an understanding of learning theory, curriculum, or assessment.

Data Collection and Analysis

Several sources of data were collected throughout the study, including interviews, journal entries, questionnaires, and classroom observations. For example, prior to the start of the mathematics pedagogy class each TA participated in an interview regarding their views about mathematics and the teaching and learning of mathematics. In addition, throughout the semester, the TAs kept a reflexive journal about their teaching experiences. During each pedagogy class the TAs were also asked to respond individually to a series of open-ended questions about issues related to topics discussed in class that day. Finally, during the months of November and December faculty members associated with the mathematics pedagogy course observed a lesson taught by the TAs. This lesson was videotaped and the TAs were asked to review the tape and respond to several questions about the lesson and their teaching performance. In addition to completing these questionnaires, several TAs participated in follow-up interviews. Qualitative techniques were employed to analyze the data. In particular, all of the interviews were transcribed and coded, and a cross-case analysis was used to identify themes and patterns with respect to TAs views about mathematics teaching and learning prior to the mathematics pedagogy course. In addition, data from journals, questionnaires, and observation notes, were reviewed for themes and organized in partially-ordered meta matrices (Miles & Huberman, 1994) to capture changes in the TAs' beliefs and practices during the course of the semester.

Results and Discussion

In order to answer research question 1, information regarding the TAs' beliefs prior to and after the mathematics pedagogy course was collected, analyzed, and reported next.

TAs' Initial Beliefs About Mathematics and the Teaching and Learning of Mathematics

It was apparent that prior to the mathematics pedagogy course the TAs had acquired a belief system about mathematics teaching and learning consistent with that of novice teachers and relying primarily on models of teaching they had experienced as students. In general, the TAs indicated they believed "being knowledgeable" was the principal attribute of effective teachers and that the act of teaching involved "giving knowledge to students". This view of teaching (i.e., transmission model) was also evident in the TAs' description of their instructional style—in every case the TAs described a teacher-directed approach that involved very little, if any, classroom discussion. Further, when asked whether they used small-group work in class, many TAs stated they encouraged their students to work together on assignments outside of class and as a means to review for a test, but generally believed that students learned mathematics by solving problems on their own. Finally, when asked to describe their understanding of how students learn mathematics the TAs provided a naïve perspective that included: "students learn in different ways", students learn by reading the textbook and reviewing their notes", and "students learn by memorizing information." What impact did the course have on their beliefs?

Impact of Mathematics Pedagogy Course

In general, the TAs indicated that the mathematics pedagogy course played a significant role in challenging their long-held beliefs about the teaching and learning of mathematics. For example several TAs commented they now understood that the goal of teaching was to promote an understanding of the material rather than "getting through the material and having students memorize and regurgitate" information. The TAs also mentioned that the course helped them to understand the difference between teaching and telling. Finally, several TAs described how class activities and discussions with other TAs and journal writing assignments caused them to be more reflective about their teaching practices and more willing to take risks in trying different approaches to teaching.

One activity that appeared to have a significant impact on the TAs' beliefs regarding the way students learn occurred after watching an excerpt of the video, *A Private Universe* (Schneps & Sadler, 1992). This video explores the nature of misconceptions that students bring and hold on to in a learning situation. After the video, the class discussed several theories of learning, including constructivism. As part of the class activities and subsequent journal entries, the TAs described their understanding of how students come to know mathematics and how this information might inform their teaching. For example, a number of the TAs indicated that students learn mathematics by "fitting in" or assimilating new information into pre-existing knowledge structures. They also recognized that students' prior knowledge might contain misconceptions, which may influence their learning. Teaching strategies outlined by the TAs to support a constructivist epistemology included allowing students to work on problems in class to uncover misconceptions about mathematics and using examples/counterexamples to challenge students' misconceptions. In addition, the TAs suggested that technology, visual aids, and varying the mode of instruction might provide for more effective instruction and increase student understanding of the material.

At the end of the mathematics pedagogy class it became evident that the TAs had acquired a different set of beliefs regarding the teaching and learning of mathematics. So what impact did this have on their classroom practice? In order to answer this question (i.e., research question 2) data from classroom observations, journal entries, and class assignments were analyzed and discussed below.

Factors that Explain the TAs' Classroom Practice

Classroom observations of the TAs at the end of the mathematics pedagogy course revealed classroom instruction that was largely teacher-directed (i.e., transmission model) and involved very little, if any, student-student or teacher-student interactions. Analysis of the data indicated that the background and experience of the TAs and the cultural norms of the university and the mathematics department provided a lens to examine and understand the TAs' classroom practice.

Over the years, the TAs have been enculturated into the field of mathematics (i.e., as students learning mathematics) and teaching (i.e., models of teaching they have observed) by virtue of their participation in the day-to-day routines of their past and present school experiences. It is clear that these experiences, in addition to their lack of pedagogical training, helped shape their teaching behavior. Further, since many of the TAs planned to pursue careers in actuarial science or at research institutions, teaching did not appear to be a high priority. In addition, the international teaching assistants' (ITA) perceived marked differences between the American educational system and schools in their native countries with respect to the role of students and teachers. For example, many ITAs' expressed difficulty in making eye contact with students and getting students to come to class and complete homework, while most TAs' expressed difficulty in questioning students and viewed the role of the teacher as the central authority figure in class.

The cultural norms of the mathematics department and the university provided a milieu, which further shaped the TAs' point of view with respect to the teaching and learning of mathematics. For example, all of the TAs were expected to follow demanding and rigid common course syllabi, which were designed by senior faculty members in the mathematics department. As a result, many of the TAs' viewed teaching as "covering the material" rather than promoting student understanding. Further, by virtue of being a Carnegie I Research institution, research is valued and rewarded. Faculty in the mathematics department viewed their primary role as publishing their research and these values were communicated to the TAs through daily interactions.

This study revealed that although the TAs adopted a new set of beliefs about the teaching and learning of mathematics their classroom practices remained the same. Research on belief systems (Green, 1971) provided a plausible explanation for why the TAs did not draw upon their newly acquired beliefs in changing their classroom instruction. In particular, the clustered nature of beliefs and the fact that individuals hold core and peripheral beliefs implies that beliefs may be held in conflict and that certain beliefs are more strongly held, and perhaps acted on, than other beliefs. As Thompson (1992) noted, a teacher may feel it is more important (i.e., a central belief) to answer student questions for reasons of maintaining authority and credibility than for clarifying the subject to students (i.e., a peripheral belief). In this study, it appeared that the newly acquired beliefs adopted by the TAs were peripheral and held in conflict to their core beliefs about the teaching and learning of mathematics. For example, although the TAs indicated a new understanding of how students learn mathematics (i.e., by actively constructing knowledge) this belief seemed to be held peripherally and in conflict with their views about the role of teachers (i.e., to deliver information or as the central authority figure in class). In the end, the TAs appeared to rely on their central beliefs about the teaching and learning of mathematics in defining their teaching practices.

In this study, it appeared that university and departmental norms validated the TAs' instructional behavior. For example, the TAs' ability to keep up with common course syllabi and prepare their students for the common exams was viewed by the TAs as evidence that they were being effective teachers. Further, as the cultural norms of the university and mathematics department helped validate the TAs' teaching practice, they may have also acted as barriers, preventing the TAs from becoming dissatisfied with their teaching, a prerequisite for initiating innovation as described in change process models (Edwards, 1994; Evans, 1996).

Final Remarks

This study sought to understand the nature of TAs' beliefs about mathematics and the teaching and learning of mathematics and their teaching practice. The results indicated although the TAs adopted new sets of beliefs regarding the teaching and learning of mathematics, their teaching practice remained unaltered. So what have we learned from this study regarding ways to help TAs become more effective teachers?

- Mathematics pedagogy courses must be viewed as ongoing professional development experiences that support TAs through the long and complex process of changing their teaching practice. It is important that such courses be collaborative efforts designed and taught by both mathematics and education faculty.
- Such courses should create opportunities for TAs to become dissatisfied with their practice by incorporating activities that challenge their firmly held beliefs about mathematics and the teaching and learning of mathematics.
- Further, mathematics pedagogy courses must help TAs acquire the skills necessary to carry out innovations in their teaching. To do so, effective models of teaching, critical reflection, and discussion must be central components of the course.
- Finally, there needs to be a shift in the cultural norms of universities and mathematics departments—institutionally, faculty must begin to see how research can inform one's teaching and engage TAs' in discussions about pedagogical matters.

We believe that through this process, mathematicians and mathematics educators can work collaboratively to improve the pedagogical preparation of TAs.

References

Baiocco, S., & DeWaters, J. (1998). *Successful college teaching: Problem-solving strategies of distinguished professors*. New Jersey: Allyn & Bacon, Prentice Hall.

Brown, S., Cooney, T., & Jones, D. (1990). Mathematics teacher education. In W. R. Houston (Ed.), *Handbook of research on teacher education* (pp. 639-656). New York: Macmillan.

Carroll, G. (1980). Effects of training programs for university teaching assistants: A review of empirical research. *Journal of Higher Education, 51* (2), 167-83.

Damarin, S., & West, G. (1979). Preparation of foreign graduate students to teach mathematics: An experimental course. *American Mathematical Monthly, 14* (3), 494-97.

Edwards, T. (1994). Looking for change in teaching practice in a mathematics curriculum innovation project: Three case studies. (Doctoral dissertation, The Ohio State University). *Dissertation Abstracts International, 55*, 6.

Enerson, D., Plank, K., & Johnson, R. (1996). *Creating a community of teachers: The Penn State course in college teaching*. Paper presented at a conference of the National Center on Postsecondary Teaching, San Francisco, CA.

Ernest, P. (1988, July). *The impact of beliefs on the teaching of mathematics.* Paper prepared for ICME VI, Budapest, Hungary.

Evans, R. (1996). *The human side of school change: Reform, resistance, and the real.* San Francisco: Jossey-Bass.

Frykholm, J. A., & Brendefur, J. (1997, March). Promoting mathematical communication in the classroom: Two pre-service teachers' conceptions and practices. Paper presented at the Annual Meeting of the American Educational Research Association. Chicago, IL.

Green, T. (1971). *The activities of teaching.* New York: McGraw-Hill.

Hammrich, P., & Armstrong, R. (1995). *A model program: Discipline-specific instruction for graduate teaching assistants*. In selected papers from the National Conference on College Teaching and Learning.

Hendrix, K. (1995). *Preparing graduate teaching assistants to effectively teach the basic course.* Paper presented at the Annual Meeting of the Southern States Communication Association, New Orleans, LA.

Miles, M., & Huberman, A. (1994). *Qualitative data analysis.* Thousand Oaks, California: Sage.

Monaghan, P. (1989). Feeling they are exploited, writing instructors seek better treatment and working conditions. *Chronicle of Higher Education, 35* (30), A13,15.

National Advisory Group of Sigma Xi. (1989). *An exploration of the nature and quality of undergraduate education in science, mathematics, and engineering.* A report of the National Advisory Group of Sigma Xi, The Scientific Research Society, Research Park Triangle, NC.

Nyquist, J., & Wulff, D. (1986). *The training of graduate teaching assistants at the University of Washington*. Paper identified by the Task Force on Establishing a National Clearinghouse of Materials Developed for Teaching Assistant Training, Seattle, WA.

Peterson, P., Fennema, E., Carpenter, T., & Loef, M. (1989). Teachers' pedagogical content beliefs in mathematics. *Cognition and Instruction, 6*. 1-40.

Rokeach, M. (1960). *The open and closed mind*. New York: Basic Books.

Schneps, M., & Sadler, P. (1992). A private universe. *The Annenberg/CPB Math and Science Project.* Smithsonian Center for Astrophysics, Harvard University, Boston, MA.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Thompson, A. (1984). The relationship of teachers' conceptions of mathematics teaching to instructional practice. *Educational Studies in Mathematics*, *15*, 105-127.

Thompson, A. (1992). Teacher's beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

Wagener, U. (1991). Changing the culture of teaching: Mathematics at Indiana, Chicago, and Harvard. *Change, 23* (4), 28-37.

Wright, D. (1981). *Integrating a GTA training program with faculty development.* Paper presented at the Annual Meeting of the American Educational Research Association, Los Angeles, CA.

WHAT IMPACTS TEACHERS AS THEY IMPLEMENT A REFORM CURRICULUM?: THE CASE OF ONE FIFTH GRADE TEACHER

Theresa J. Grant
Western Michigan University
terry.grant@wmich.edu

Kate Kline
Western Michigan University
kate.kline @wmich.edu

Abstract: This study was undertaken to better understand the ways teachers utilize a reform elementary mathematics curriculum in the beginning stages of implementation and the factors that influence the implementation. The teacher's beliefs about mathematics teaching, his understanding of students' reasoning, and the ways he engaged with his students' reasoning were analyzed in order to obtain a clearer picture of what teachers bring to the implementation of a new curriculum. While the teacher believed in reform and the curriculum he was using, he struggled to transform his teaching to focus on understanding. He was particularly challenged by eliciting and pursuing a variety of strategies rather than the ones he had in mind and using incorrect responses as a site for learning.

Theoretical Perspective

For the past decade, the Standards documents put forth by the National Council of Teachers of Mathematics (1989, 1991, 1995, 2000) have forced us to rethink *what* mathematics is taught, *how* that mathematics is taught, and the intricate interplay between content and pedagogy. Initial reactions to these documents were often simplistic, focusing on a single aspect of the vision such as simply adding certain content or using cooperative groups (Burrill, 1997). The development and publication of Standards-based curricula that make this vision more explicit provide a unique opportunity to avoid these superficial interpretations of the Standards. However, the challenge still exists as to how to support teachers as they implement reform curricula.

Although the research on teacher change has provided us with a better understanding of the complex process of changing beliefs and practice (e.g., Lloyd, 1999; Simon & Schifter, 1991; Wood, Cobb, & Yackel, 1991), we are just beginning to understand how to best facilitate change in the climate of reform (Ball, 1996). Recently researchers have suggested that there are essential characteristics of classrooms that support the growth of mathematical understanding, including the ability of teachers to elicit children's solution methods and enable them to extend their mathematical thinking (e.g., Fraivillig, Murphy, & Fuson,1999; Empson, 2000). Our research, studying a sample of elementary teachers from the more than 400 in our project as they implement a reform mathematics curriculum, has also led us to conclude that it is the teachers' ability to focus on students' reasoning and distinguish among those that are mathematically significant that impacts their ability to implement the curriculum (Grant & Kline, 2002). We undertook this intensive study of a teacher's day-to-day practice to more deeply analyze the factors that influence the in-the-moment decisions made while teaching a reform curriculum and develop further this theoretical stance about teachers' abilities to engage with students' ideas.

Setting the Stage

For the past three years the authors have been working with several local elementary school districts as they work to improve the mathematics instruction in their schools. These districts have all adopted one of the NSF-funded reform curricula, *Investigations in Number, Data and Space* (henceforth called *Investigations*). This study takes place in a fifth-grade mathematics class at the beginning of the final year of a three-year phased implementation of *Investigations* in a small rural school district. The teacher of this class, Doug, has been a teacher for almost 20 years and has just returned to teaching after being a middle school administrator for ten years. Doug taught two sections of mathematics to the fifth grade students in his building and one of those sections was used for the study. Doug volunteered to participate in the study and was anxious to receive feedback on his teaching.

As fifth grade was the last grade to implement the new curriculum, this was the first year Doug taught mathematics using these materials. The majority of his students had their first exposure to the *Investigations* curriculum in fourth grade. However that year was the first year for the fourth grade teachers to implement the curriculum and there was some indication that two of the three teachers in this particular school did not use the curriculum whole-heartedly. Prior to fourth grade Doug's students had received instruction from a traditional textbook.

Data Collection and Analysis

An ethnographic approach was used to observe and interact with the students and teacher in this study. The teacher was observed daily by the authors while the first unit of the reform curriculum was taught, lasting approximately eight weeks. The unit focused on whole number computation and number sense, developing such ideas as factors and multiples, using known multiplication and division problems to figure out unknown ones, and the relationship between multiplication and division.

A classroom observation instrument adapted from the QUASAR project (Stein, Grover & Silver, 1991) was used to capture such aspects of the implemented lesson as the general format (e.g., launch, students at work, and closure; use of whole-group, small-group or individuals at work); the ways students were and were not supported in their investigations and reasoning about mathematics (particular attention was paid to the ways the teacher probed student thinking); and the ways in which the lessons were altered. Brief interviews were conducted with the teacher before and after most lessons to ascertain what he was thinking going into the lesson and his reflections following the lesson. In addition, an extensive final interview was conducted at the end of the unit to investigate the teacher's reflections on the entire unit and what was learned about mathematics, teaching, and how students were thinking. The questions included: Have you learned anything new about number from teaching this unit?; What was the mathematical emphasis of this unit?; Overall, did your students develop an understanding of this mathematical emphasis?; Did any of the students' thinking surprise you during the unit?; What does the student work from the final assessment in the unit tell you about what they understand?; How do you feel your teaching has changed?; and What is the most challenging aspect of teaching Investigations? This interview was audio-taped and transcribed.

Field notes from observations and pre- and post-observation interviews were compiled and discussed by the researchers throughout the eight weeks. This allowed for the use of a "grounded theory" (Glaser & Strauss, 1967) approach to data collection and analysis, which was particularly instrumental in guiding conversations with the teacher and focusing his attention on particular aspects of his teaching that were at odds with the curriculum. The field notes were analyzed after completion of the study as well to identify patterns in the way Doug was interacting with his students.

Results and Discussion

In our preliminary interview with Doug, he indicated that his philosophy about teaching aligned with that of *Investigations*. Doug explained that he liked being flexible and allowing a lesson to go where the students took it. He believed it was important that students understand mathematics and not just memorize procedures taught to them. Doug had typical reservations about being new to the curriculum, but looked forward to using it. However as he taught this first unit, he clearly struggled to keep the focus on students' understanding. Our analysis of the data yielded several key issues that challenged Doug in his teaching; these are described in the following sections. Although many of the issues Doug faced are related, they will be discussed separately below to highlight their unique contributions to understanding the challenges of changing ones practice.

Terminology before Concepts

On our first observation, Doug decided not to use the curriculum, but rather gave the students a worksheet to work on in pairs and then a similar one to do independently as an assessment. The worksheet involved identifying factors and multiples of specific numbers, identifying prime numbers, and odd and even numbers. The mode of instruction was direct instruction with the teacher as the clear authority in the classroom. No questions were asked about how students arrived at solutions or about how they were thinking. Doug explained that he felt it was necessary to do this activity to give the students a chance to get a handle on the terminology they would be using in the unit. He believed that they needed to know this terminology before they could do any investigations about numbers and their characteristics. This belief that definitions and terminology should be learned *before* investigating ideas, rather than *while* investigating ideas, reappeared in Doug's teaching throughout the unit. This view that terminology must be learned first is in conflict with the curriculum's approach to learning and can undermine the students' ability to thoroughly understand the concept(s) represented by the terminology.

Focusing on Predetermined Strategies

While Doug expressed a belief in the importance of developing a variety of strategies, he tended to focus on one or two strategies that he thought were the most important, many times in neglect of other strategies. For example, students were asked to think about how they could figure out the answer to 6 x 8 if they did not have it memorized already. One student suggested starting with 5 x 8 = 40 and then adding on one more 8 to get 48. Doug accepted this along with one more strategy and then said, "Can't we use 3 x 8 = 24 and then double?" He suggested this doubling strategy with a tone that almost negated the others that were shared previously. In other instances, like the one described below, Doug chose to alter the lesson by either eliminating open-ended questions designed to probe thinking or replacing them with more focused questions designed to focus on particular strategies. This pattern reoccurred throughout the unit, often making it clear that he valued certain strategies over others.

It is certainly the case that some strategies should be highlighted for their efficiency and mathematical importance. However, Doug often singled out strategies early in a lesson which tended to circumvent the students' thinking and use of other equally important strategies. For example, in one lesson where students were asked to find as many factors as possible for 100, 1000, and 10,000, Doug focused on only two strategies. He explained to the students that they could skip count by a number and if they landed on 1000, it was a factor of 1000. An alternative strategy he suggested they use was to divide the number they were thinking of into 1000 on the calculator and use it if the answer was a whole number. The lesson plan does suggest that you begin with the idea of skip counting, since the students have been working on skip-counting patterns for various numbers. However, the lesson plan also reads, "Record only one or two suggestions for each number and ask: How can you use what you already know to find some more numbers without actually doing the counting all the way to 1000 or 10,000?" (Kliman, Tierney, Russell, Murray, & Akers, 1998, p. 51). Students could potentially bring a variety of perspectives to this investigation. They had been working with area models for multiplication and could think of rectangle dimensions. They could also use factors from one list (factors of 100) to help them think about factors of 1000 or 10,000. Because of the suggestions Doug made at the beginning of the lesson and continued to reinforce while students were working, the notion of searching for relationships among factors (a key mathematical emphasis of this lesson) was lost.

Authority on Correctness

Doug was the clear authority on the correctness of solutions in the classroom. He rarely pursued incorrect solutions to ascertain whether the reasoning was valid, but rather would inform students that they were incorrect and proceed to another student. A routine that was revisited throughout the unit was to skip count by a given number. On one day, students were working on skip counting by 400. On two different occasions, students provided an incorrect response. Rather than using that as an opportunity to let those students and the entire class think about the situation, Doug would simply tell the student he was wrong and tell him how to think about it. For example, one student was to add 400 to 1200 and said 1500. Doug told him he was wrong and said, "What is 2 + 4? Okay, so what is 1200 + 400?" On another occasion, students were working on factor pairs for 300. One student offered 50 x 5 and Doug would not write it on the board. Rather, he said, "50 is right," and wrote 50 x ___ on the board. The student then offered 5 x 60 and Doug replied, "You're almost there, but we're working with 50." Clearly, these exchanges established his authority in the classroom and undermined some of the thinking that the students could have done around these issues.

Selective Pursuit of Reasoning

When explanations of thinking are part of teaching, it is not uncommon for students to offer incomplete, unclear, and sometimes nonsensical responses. A large part of the challenge of orchestrating discussions around student thinking is to pursue these kinds of responses to figure out what makes sense. Doug would occasionally find himself confronted with such explanations and be unsure of how to proceed. For example, during a lesson early in the unit a student offered the following strategy to figure out 6 x 8: he said he would begin with 6 x 10 = 60, and then take off 4, and take off 4, and take off 4 to get 48. The student explained that he would take off 4's because they are easy to subtract. It was clear that the student knew the answer to 6 x 8 was 48, and was starting with 60 and subtracting 4's until he got to the correct answer. While Doug recognized there was something odd about this strategy, he decided to not pursue it and go on to another student.

It would have been beneficial to discuss this nonsensical approach for a variety of reasons, however. In the first place, the students need to understand that simply arriving at "the answer" does not legitimize any approach. The approach must always make sense in the context of the problem/situation. The second issue is that the use of a nonsensical approach can virtually bypass the mathematics of the context. One could begin the problem at hand by interpreting 6 x 8 as 6 groups of 8 (8 groups of 6) and using other known groups (i.e. 6 x 10 or 6 groups of 10) to help them figure out the solution. And it is the compensation done at this point (taking away 6 groups of 2, 6 x 10 - 6 x 2 = 6 x 8) that must utilize an understanding of these groupings in order to sensibly arrive at a solution.

By not addressing the issue of nonsensical approaches to problems nor the mathematics underlying invented approaches to multiplication, future lessons posed greater challenges. For example, as students began working on invented procedures for multiplying larger numbers, they struggled to develop appropriate ways to break down the problems into simpler problems they could use.

Attempting Changes

During the unit, Doug made several attempts to change his practice to focus more on student thinking. About half-way through the unit, he made a conscious effort to position himself differently in the classroom. He would often stand in the back of the room and let students go to the front as they shared their reasoning. He also attempted to pursue incorrect responses on occasion. In one lesson, students were provided with a set of related problems (4 x 25, 40 x 25, 6 x 25, 10 x 25, 50 x 25) with the goal being to solve 46 x 25 using the solutions to these problems. One student offered his solution at the overhead by saying he created his own related problems to use. He wrote 40 x 15 and 6 x 15 and explained that you just add the answers to those problems to get the solution to 46 x 25 because 40 + 6 = 46 and 15 + 15 = 25. When the students shouted that 15 + 15 = 30, he changed 6 x 15 to 6 x 10 and was convinced that the answer to 40 x 15 plus the answer to 6 x 10 would be the same as the solution to 46 x 25. Doug knew this was incorrect and chose a particular student to come up and explain why that strategy would not work. Unfortunately, the student did not actually address the incorrect strategy; rather, he decided to share his own correct strategy instead. When asked in the final interview how he felt about pursuing incorrect strategies with the rest of the class, Doug said, "I think it's real important. That's one of the ones I struggle with…how to help other kids see why that's wrong."

Conclusions

The factors that influenced this teacher's implementation of a mathematics reform curriculum have reaffirmed the research finding that one of the most important factors of successful implementation is the teacher's ability to engage with students' ideas. This case study along with related studies conducted by the authors extends the research by suggesting finer categories/characteristics influencing teachers' ability to make student thinking central. It is essential that teachers believe in the importance of developing a variety of ways of reasoning in mathematics. In addition, they need to have an understanding of the mathematical significance of different ways of reasoning and be able to distinguish among them while teaching. It is also important to realize that the authority for correctness should lie with the classroom as a community, rather than solely with the teacher.

Our work with the larger population of teachers in our project has led us to conjecture that what may be at the heart of these issues is understanding what it means to develop students who are powerful and independent problem solvers. One may view the major purpose of the *Investigations* curriculum as developing the ability in students to think on their own and to make judgments about the reasonableness of their thinking. If one thinks of the issues involved in teaching reform curricula through this lens, then the way you view the characteristics (described in the previous paragraph) change dramatically. If you view the teacher's role as moving students along a continuum toward being independent problem solvers, the teacher must encourage and elicit multiple strategies and create a classroom environment where explanations on the reasonableness of solutions is the norm. This suggests a framework for thinking about supporting teachers as they implement new curricula and for designing further research on what factors impact teachers' enactment of reform curricula.

References

Ball, D. L. (1996). Teacher learning and the mathematics reforms: What we think we know and what we need to learn. *Phi Delta Kappan, 77* (7), 500-508.

Burrill, G. (1997). President's message: Show me the math! *NCTM News Bulletin, 33* (9), 3.

Empson, S. (2000). A reconceptualization of teacher thinking in terms of mediating resources. In M. L. Fernandez (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 661-666). Columbus, OH: ERIC Clearninghouse for Science, Mathematics and Environmental Education.

Fraivillig, J., Murphy, L., & Fuson, K. (1999). Advancing children's mathematical thinking in Everyday Mathematics classrooms. *Journal for Research in Mathematics Education, 30* (2), 148-170.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research.* New York: Aldine De Gruyter.

Grant, T., & Kline, K. (2002). Developing elementary teachers' knowledge of content and pedagogy through implementation of a standards-based mathematics curriculum. In E. Guyton & J. Rainer (Eds.), *Teacher education yearbook x: Meeting and using standards in the preparation of teachers.* Kendall Hunt. In press.

Kliman, M., Tierney, C., Russell, S., Murray, M., & Akers, J. (1998). *Investigations in number, data, and space curriculum: Mathematical thinking in grade 5.* Menlo Park, CA: Dale Seymour.

Lloyd, G. M. (1999). Two teachers' conceptions of a reform-oriented curriculum: Implications for mathematics teacher development. *Journal of Mathematics Teacher Education, 2* (3), 227-252.

National Council of Teachers of Mathematics. (1989). *Curriculum & evaluation standards for school mathematics* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics* Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics..* Reston, VA: Author.

Simon, M. A., & Schifter, D. (1991). Towards a constructivist perspective: An intervention study of mathematics teacher development. *Educational Studies in Mathematics, 22,* 309-331.

Stein, M. K., Grover, B. W., & Silver, E.A. (1991). Changing instructional practice: A conceptual framework for capturing the details. In R. G. Underhill (Ed.), *Proceedings of the Thirteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 36-42). Blacksburg, VA: Virginia Polytechnic Institute and State University.

Wood, T., Cobb, P., & Yackel, E. (1991). Change in teaching mathematics: A case study. *American Educational Research Journal, 28* (3), 587-616.

THE STORY OF FIRST-YEAR TEACHERS' STRUGGLE TO MAINTAIN A REFORM PERSPECTIVE AFTER EXPERIENCING INTEGRATED MATHEMATICS CONTENT AND METHODS IN THEIR TEACHER PREPARATION PROGRAM

Lynn C. Hart
Georgia State University

Rationale and Theoretical Perspective

For the last two decades, constructivism (Confrey, 1987; Gergen, 1992; Phillips, 1995) has emerged as the epistemological foundation for mathematics learning. A vision of how to teach within such a theoretical framework has materialized (Simon, 1995; Steffe & D'Ambrosio, 1995) and the research community has explored teacher change from this perspective (Ball, 1994; Fennema & Nelson, 1997; Hart 1993, 1994; Schifter, 1995, 1998). One significant component of the research has been the role of beliefs in teacher change. Over time, a substantial body of literature has emerged providing evidence that teachers' beliefs drive their teaching of mathematics (Pajares, 1992; Richardson, 1996). In order to change teachers' practices, we need to consider teachers' beliefs. However, we know that beliefs are difficult to change (Lerman, 1997), that the beliefs teachers' espouse are not always consistent with the way they teach (Brown, 1985; Cooney, 1985), and that changing teachers' beliefs takes time (Richardson, 1996). Moreover, Pajares (1992) tells us that beliefs about teaching are well established by the time a student enters college. They are developed during what Lortie (1975) calls *the apprenticeship of observation* that occurs over their years as a student. They include ideas about what it takes to be an effective teacher and are brought to their teacher preparation program. Given this, it seems imperative that teacher education programs assess their effectiveness, at least in part, on how well they nurture beliefs that are consistent with the program's philosophy of learning and teaching. Also, it is important to study how consistent the beliefs teachers espouse are with their teaching practices, i.e., can teachers do more than *talk the talk?*

Since most beliefs are formed through experience over time, pedagogical practices that support constructivist theory can be nurtured if we engage novice teachers in constructivist experiences both in learning mathematics and in teaching mathematics. This does not insure change, but certainly facilitates it. However, change is limited when preservice teachers learn mathematics content differently than they learn mathematics methods. Given the limited amount of time preservice programs have to impact teacher development, if the mathematics is taught by lecture and the methods use a constructivist environment, the experience is diluted and the chance for change is significantly decreased.

The Intervention: The Urban Alternative Preparation Program

The Urban Alternative Preparation Program (UAPP) is housed in the Department of Early Childhood Education at a large, urban university. Students enter the program as a cohort, taking all coursework together. The students hold non-education undergraduate degrees and are preparing to become certified in a K-5 *urban* setting. The program is grounded in constructivist learning theory and promotes practices that support that perspective. In Phase I of the program, students obtain certification after four semesters. Coursework in Phase I includes four math-related courses: two courses in mathematics content and two courses in mathematics education. To participate in Phase II students must accept jobs teaching elementary school in an urban setting (defined for our purposes as schools with at least 75% free or reduced lunch). During this time students pursue a masters degree. Their coursework during Phase II includes receiving coaching in mathematics (content and methods) in their individual classrooms and a university course on Issues in Teaching Mathematics.

During the first year of the program, I taught the twelve semester hours of content and methods as an integrated, seamless course over three consecutive semesters. Instruction was consistent with a constructivist philosophy. Students learned the mathematics from the mathematics book using many of the methods suggested in the methods book. We used oral reflection, videotaping and written logs to examine the methods used to deliver the content. During Phase II (as part of their masters program) I mentored these same teachers through monthly visits to their classrooms. In addition, I taught the Issues in Teaching Mathematics course that teachers attended as a group to collaborate and reflect on their practice. Part of that coursework was completed on the web.

Participants

Fourteen of the original cohort of 20 teachers completed Phase I and obtained certification. Only 8 teachers actually took a teaching position in an *urban* school and are part of this study. They ranged in age from 25 to 41. There were two African-American females, one Asian male and five Caucasian females. Two teachers were in a first grade classroom, five teachers were in a second grade and one was in a third grade classroom.

Methods

In order to assess how successful the UAPP integrated math/methods sequence was at changing the beliefs and practice of the preservice teachers in the program, a study was conducted through Phase I – Teacher Certification. To assess change in beliefs and consistency of beliefs with practice, three data sources were used. A Mathematics Beliefs Instrument (MBI) was administered before and after Phase I. In addition the participants routinely completed reflection logs describing their experiences in teaching mathematics in their field placements, and I observed their practice through regular classroom visits to their placements. Results from this work suggested that teachers had in fact changed their beliefs in a direction more consistent with reform practices and their practice was somewhat consistent with their beliefs (Hart, in press). However, that research was only the first step in assessing the impact of the program.

As the mathematics coach for Phase II, I had the opportunity to follow these teachers into their first year in an urban school. I was able to observe in their classrooms and to continue to study their change. To study their practice and beliefs I collected data from three sources. First, at each of the whole-group class meetings teachers completed reflection logs about their teaching of mathematics. Second, teachers responded to web assignments that required them to write about other teachers' practice through case discussions (Stein, Smith, Henningsen & Silver, 2000; Schifter, 1996). Third, field notes were made during observations in teachers' classrooms. These data sources provided a triangulation of perspectives: the teachers' view of themselves, the teachers' view of others and my view of them. All of these data were analyzed using qualitative methods, looking for themes that emerged about their practice and beliefs. Finally, the MBI was administered again at the end of their first year of teaching and compared to the results from the previous year. Descriptive statistics were used to describe general trends in their change.

Results

Practice

Personal reflection logs. Perhaps the most vivid theme that emerged from the data at the beginning of the school year was the teachers' frustration at the deeply rooted, traditional mathematics culture of their schools and their incredible struggle at confronting this culture. In the first few months they describe their schools' mathematics curricula as very traditional, skill-driven programs. Many of the schools did grade level planning in which one person planned the mathematics lessons for the entire grade level for the week. Lessons frequently consisted of pages to cover in the book and worksheets. As new teachers, these students expressed frustration at this kind of work. One teacher wrote, "As the new kid on the block I am afraid to speak up and challenge what these teachers are saying. I want to be accepted and not appear to be a trouble maker." Another teacher stated "I think I am going to offer to be the planner for math, so I can try to suggest some new things." A third said, "I try to go along with what they suggest and then go in my room and do what I know my kids need, but I am always behind everyone else!" They were caught between the beliefs and values they learned in the UAPP program and norms of the environment in which they were working.

As the year progressed they began to turn their attention to their students' learning and made statements in their logs that suggest that their resolve was shaken but not altered. They communicated that they really believe the philosophy and values of the reform. They wrote, "my best math instruction often occurs spur of the moment from the kids ideas"; "these kids *need* to manipulate most of the time to work through the problems not just do workbook pages"; " my kids are explaining their strategies and loving it – they get so excited it can cause management problems"; "sometimes it is hard to get them to think about number sense, they want me to tell them"; and, "my children are stuck by 'work book' ways of problem solving. It's difficult to get them out of the box."

Their perspective seemed undaunted by the incredible roadblocks they encountered throughout the year, e.g., pressure from administrators to emphasize skill development in order to do well on standardized tests, colleagues who alienated them for their unorthodox methods, students who balked at 'sharing their thinking", limited resources, and limitations in their own understanding of the content. Yet in their reflections at the end of the year the teachers repeatedly articulated their continued support of a reform approach to teaching mathematics. They commented, "we were taught to make the children think, rationalize and to not just accept what is said to be true. I still believe in all of those things very deeply, but boy is it a difficult task within the schools and school systems where we are placed"; "I believe that it is not only possible but in the best interest of the child to teach mathematics as we have been taught in class"; "I totally think it is possible and vital to the students in an urban setting to teach the way we were taught"; and, "I absolutely believe that it is possible to teach mathematics the way you have promoted in an urban setting". One first grade teacher tells the story this way: "On some of those days when my energy has been drained to nothing and my optimism torn to shreds, I kind of look over into the 'greener' field of worksheets and books where the teachers are full of energy and flowing from topic to topic, and wonder if maybe I can linger there for a month or two to regain my energy. And then it happens . . .a cute, big-headed little boy looks at me and says 'Look Ms. Lowe, 2 tens and 17 ones makes 37 too . . . I just traded in one of my tens sticks!' Those days of working with the Unifix cubes, Popsicle sticks, and bean sticks have paid off. Then I look in *my* field and realize that their fundamental understanding will be the foundation of years of math learning. I will trade my weariness for *that* any day . . ."

Case discussions. The most striking theme in their written case discussions was the teachers' ability to think deeply about the methods and strategies a teacher was using and how little they were able to comment on the mathematics content of the case. For the most part their observations reflected a real awareness of methods that supported a reform philosophy and those that did not. For example, they noticed immediately that a teacher in one of the cases "was quick to cave in and give the algorithmic procedure for solving the problem when their students became frustrated." They also noticed when a teacher, " didn't give them an algorithm to plug into but redirected the students back to the diagram for understanding." They had more insights, giving more elaborate comments. For example, one teacher commented on a case in which the same lesson plan was presented to two different sections of students, "Ron's stressful reaction to *his* time limits for the class and his underlying need to rush the students through the lesson without allowing them any real time to reflect reduced the cognitive demands of the second period class. In the second period he resorted too quickly to providing the students with the algorithm due to time constraints and his fear of their frustration. In the 6th period he relaxed a bit and stuck to his original plan-prompting the students to use the diagram/visuals provided to encourage thinking." However, when asked to comment on the mathematics in a case, such as a statement when a student in the case says "Are you saying that 100% = 1? I thought that 100% = 100!", few of the teachers were able to talk about the *mathematics* very deeply and a few revealed some shocking misconceptions. For example one student said "We commonly think that the whole is always 100%, when it actually depends on what the given problem is." While others saw the potential richness for the mathematical discussion, their comments were quite brief, often limited to one sentence. "It is important to know what 100% is in relation to the problem." "It is important to know when switching from decimals to percents." "His statement will better help someone understand what is happening." They seemed to feel very comfortable analyzing the teacher's methods and his or her alignment with reform practice, yet there was little discussion of the mathematics itself and why it was difficult or where potential misconceptions might lie.

Classroom observations. Classroom observations at the beginning of the year mirrored the conflict teachers' expressed in their logs. Their practice showed a tremendous effort to implement reform practice but also frequently would revert back to what might be called traditional strategies. For example, in an observation of Mona during her third month of teaching her class was working on basic facts. Mona had developed an activity in which her children drew dots on eight Popsicle sticks (a stick with one dot, a stick with two dots, a stick with three dots and so on). Mona then held up her own stick with, say, five dots. She posed the problem. "Can you show me this many dots using other sticks that you have?" After a long silence in which Mona later declared she almost gave up, one very shy boy held up a stick with two dots and a stick with three dots. Others slowly caught on and Mona encouraged conversation from the students about how they had figured out their solution. After several more problems of this type she had the students open their workbook and do the page of 25 basic number facts that was assigned by the grade level for completion for that day.

By spring their convictions were holding. For example, in one observation a teacher was reviewing addition of one digit plus two digit addends by using a missing addend vertical format ($7 + X = 15$). She required each student to explain "how" he or she had solved the problem. One student had used doubles (7+7=14 and 1 more) and another student had counted on (7, 8, 9, . . .). The teacher next gave the problem ($9 + X = 15$). The class proceeded through another explanation and at the end a student commented that he could do $8 + X = 15$ using what he knew from the two previous problems. The teacher encouraged the student to explain and used that idea to build other related problems, like 6+X, 5+X, etc. Through discussion the students saw more patterns where they could use existing problems to help them solve more related problems, e.g., (3+X=15 and 12+X=15). Examples like this were seen in other classes. For the most part teachers had not given in to the pressures of traditional instructional norms and in many cases were feeling more comfortable in their role as a classroom teacher and showing more creative methods for traditional content.

Beliefs

In both the reflection logs and the case discussions their comments continued to show strong alignment with reform mathematics and with the beliefs they had expressed in the study at the end of Phase I. For example, at the end of student teaching Tiffany responded *more false than true* to the statement that "Math problems can be done correctly in only one way" (MBI, Item #23). In the case writings she commented: "Too often students are intimidated by math because of the fact that there is only one right answer. What students fail to realize is that the problem can be solved in many different ways." Clearly demonstrating a consistency in her responses. Another student, Jessica, responded *disagree* to item #10 on the MBI (In K-5 mathematics, skill in computation should precede word problems). In her log she wrote, "I think concepts should be taught before algorithms, because if we know the algorithm one doesn't tend to try and understand why it works just to do the problem and get a correct answer." Other comments consistent with reform philosophy were found. "It is kind of hard to get the students to think about number sense stuff. They always want a rule." "The students need to explain their strategies." "It's important to give the students enough time to explore the concept." It's hard to get the students to slow down and explain everything to a partner, but it is so important!"

Finally, the MBI results from the end of Phase I to the end of Phase II found that of the 223 possible response changes in self-reported beliefs (28 items x 8 teachers with one item omitted), 81.6% (182 items) remained unchanged, 9.0% (20 items) changed in a direction more consistent with reform philosophy, and 9.4% (21 items) changed in a direction away from reform philosophy. Further item analysis of the 21 changed items found that only 3 items changed more than 1 degree on a 4 point Likert scale, for example items changing only one degree may have changed from TRUE to MORE TRUE THAN FALSE. The three responses that had more than one degree of change came from three different teachers.

Comments

This project followed a group of 8 teachers who participated in a two-year alternative preparation program to become elementary teachers in urban classrooms. During the first year (Phase I) they participated in integrated mathematics content/

mathematics methods coursework and were supervised in their mathematics teaching by the same instructor. At the end of that first year they demonstrated beliefs and practice that were more consistent with the philosophy of reform. During the second year (Phase II) of the program they were coached in their classrooms and participated in coursework again by the same instructor. At the end of that experience the teachers demonstrated a strong resilience in their newfound beliefs and practice.

What did we learn from these results? As preservice teachers the participants had experienced all their placements in urban classrooms where they observed, almost without exception, traditional practice; however, they were clearly not prepared for the difficulty in working within such a culture. This is important information as we develop and reform our professional program. It appears that even in the face of this frustration they held on to beliefs that are consistent with reform practices. They appear to understand and value student construction of knowledge and the importance of exploring the content they are teaching. They have attempted to integrate behaviors that support these beliefs within the traditional culture within which they work. They appear to believe reform practices can and will work in urban setting. We also learned that while the integration of content and methods appeared to provide a solid foundation for establishing new beliefs about teaching and learning mathematics and about pedagogical methods that supported a reform philosophy, the mathematics content was still not sufficiently developed to produce teachers who felt confident in their understanding of much of that content. This is clearly an area that deserves attention as we plan future iterations of the program. Teaching within a reform philosophy requires a deep understanding of elementary mathematics content. These teachers seem to understand their own limitations in this area.

The study is clearly limited by the small number of teachers I was able to follow through the two-year program. The results, however, are useful to us as we plan for the future. It is important that we prepare teachers for working in a culture that may not be supportive of their philosophical position about learning and teaching mathematics. They must understand that there will be many different perspectives about what it means to know and do mathematics. We must also attend to the need for developing deeper content knowledge of elementary mathematics. The integrated content and methods presented within an environment that supports a constructivist theory of learning appeared to facilitate the development of beliefs that were consistent with a reform position. However, there was insufficient time to attend to the complex and diverse content issues they needed. As mathematics educators we need to think deeply about how we can impact preservice elementary teachers more deeply within the limited amount of time we have to work with them. Ma & Kessel (2001) state "... content and pedagogy may be two sides of the same coin." (p. 16). I would amend that slightly and say that content and pedagogy should be two sides of the same coin.

696

References

Ball, D. L. (1994, November). *Developing mathematics reform: What don't we know about teacher learning – but would make good hypotheses?* Paper presented for the Conference on Teacher Enhancement in Mathematics, K-6, National Science Foundation, Washington, DC.

Brown, C. A. (1985). *A study of the socialization to teaching of a beginning mathematics teacher.* Unpublished doctoral dissertation, University of Georgia, Athens.

Confrey, J. (1987). *The current state of constructivist thought in mathematics education*. Unpublished manuscript.

Cooney, T. J. (1985). A beginning teacher's view of problem solving. *Journal of Research in Mathematics Education, 16*, 324-336.

Fennema, E., & Nelson, B. S. (Eds.). (1997). *Mathematics teachers in transition*. Mahwah, NJ: Erlbaum.

Gergen, K. (1992, February). *From construction in contest to reconstruction in education*. Paper presented at the Conference on Constructivism in Education, University of Georgia, Athens, GA.

Hart, L. (in press). Changes in preservice teacher's beliefs and practice after participating in an integrated content/methods course. *School Science and Mathematics*.

Hart, L. (1999). Teachers' perceptions and beliefs about factors that influence change in their pedagogy. In E. Pehkonen & G. Torner (Eds.), *Mathematical beliefs and their impact on teaching and learning mathematics: Proceedings of the Conference on Mathematical Beliefs* (pp. 43-50). Oberwalfach, Germany: Universitat Duisburg.

Hart, L. (1994). Shared authority: Roadblock to teacher change? In. J. R. Becker & B. Pence (Eds.), *Proceedings of the Fifteenth Annual Meeting of the Psychology of Mathematics Education – North American Chapter* (Vol. 2, pp. 189-197). Asilomar: University of California.

Hart, L. (1993). Issues of potential conflict in purposeful sampling, access and the development of trust in teacher change research. In K. Shaw (chair), symposium on *Issues in Teacher Change Research.* Symposium presented at the annual meeting of the AERA Special Interest Group: Research in Mathematics Education, Seattle.

Hoffman, K. (1989). *The science of patterns: A practical philosophy of mathematics education.* Paper presented for the Special Interest Group on Research in Mathematics Education at the Annual Meeting of the American Educational Research Association, San Francisco, CA.

Lerman, S. (1997). The psychology of mathematics teacher learning: In search of theory. In E. Pehkonnen (Ed.), *Proceedings of the Twenty-First International*

Meeting of the Psychology of Mathematics Education (Vol 3, pp. 200-207). Lahti, Finland.

Lortie, D. (1975). *Schoolteacher: A sociological study.* Chicago: University of Chicago Press.

Ma, L., & Kessel, C. (2001). Knowledge of fundamental mathematics for teaching. In *Knowing and learning mathematics for teaching.* (Available online at http://books.nap.edu/0309069955/html/)

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research*, 62 (1), 307-332.

Phillips, D. C. (1995). The good, the bad, and the ugly: The many faces of constructivism. *Educational Researcher*, *24* (7), 5-12.

Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula (Ed.), *Handbook of research on teacher education* (pp. 102-119). New York: Simon & Schuster.

Schifter, D. (1998). Learning mathematics for teaching: From a teachers' seminar to the classroom. *Journal of Mathematics Teacher Education*, *1* (1), 55-87.

Schifter, D. (Ed.). (1996). *What's happening in math class? Reconstructing professional identities* (Vols. 1 & 2). New York: Teachers College Press.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal of Research in Mathematics Education*, *26* (2), 114-145.

Steffe, L., & D'Ambrosio, B. S. (1995). Toward a working model of constructivist teaching: A reaction to Simon. *Journal of Research in Mathematics Education*, *26* (2), 146-159.

Stein, M. K., Smith, M. S., Henningsen, M. A, & Silver, E.A. (2000). *Implementing standards-based mathematics instruction*. New York: Teachers College Press.

USING DISCOURSE ANALYSIS TO "SEE" TEACHER-POSITIONING: A TALE OF TWO 8TH GRADE MATHEMATICS CLASSROOMS

Beth Herbel-Eisenmann
University of Wyoming
herbelei@uwyo.edu

Introduction[1]

Mathematics Education researchers have only recently begun to bring discourse analysis perspectives to data from mathematics classrooms; the majority of the mathematics education literature on this topic has been published in the last decade or so. Increased attention to discourse in mathematics classrooms may be due in part to the *Professional Standards for Teaching Mathematics* (NCTM, 1991) which discussed the teacher's and student's role in fostering discourse in the mathematics classroom. For example, it states:

> The discourse *embeds* fundamental values about knowledge and authority. Its nature is reflected in what makes and answer right and what counts as legitimate mathematical activity, argument, and thinking. Teachers, through the ways in which they orchestrate discourse, convey messages about whose knowledge and ways of thinking and knowing are valued, who is considered able to contribute, and who has status in the group. (p. 20, emphasis added).

"Embeds"—what does that mean? What does it look like? The following discourse analysis helps us see more clearly how issues of teacher positioning with respect to authority and knowledge might be "embedded" in teacher-talk.

Learning to teach in the ways advocated by the *Standards* means learning many new types of knowledge, including understanding content and pedagogical content in new ways. It also needs to include having teachers examine the beliefs and conceptions they bring with them from their own experience as students of mathematics. As Thompson (1992) pointed out:

> The tendency of teachers to *interpret new ideas and techniques* through *old mindsets*—even when the ideas have been enthusiastically embraced—should alert us against measuring the fruitfulness of our work in superficial ways (p. 143, emphasis added).

Even when teachers' beliefs and conceptions have changed, the mindsets that have been inherited as the result of being students and teachers in more traditional mathematics classrooms may not have. These mindsets can be embedded in and carried by the language teachers use, which can directly influence the norms that teachers establish in their classrooms.

In this paper, I show how two "reform-oriented" middle school mathematics teachers who are very much alike invite students to engage in mathematics in different ways. To do so, I focus on the norms[2] that were *embedded in and carried by* the discourse[3] in the two classrooms, paying particular attention to how the teachers drew on other sources of knowledge (i.e., the textbook and students) in their classrooms.

Ethnography of Communication

How might these norms be "embedded in and carried by" the discourse? Drawing on a perspective of discourse analysis called ethnography of communication, I focused on "the social functions of particular forms of language" (Mishler, 1972, p.268) that were used in the context of two middle school mathematics classrooms. This focus on form and function was emphasized throughout my analysis and is highlighted in the following excerpts about each teacher, as well as in the cross-analysis. I located reoccurring forms (which consisted of particular words and how they came together) and looked at how those forms seemed to function (i.e. what purpose they served) in the classroom.

In addition to this emphasis on the social functions of particular forms, this approach to discourse analysis puts forth that the language used helps understand the cultural norms that motivate the way we act toward one another:

> …culture is continually created, negotiated, and redefined in concrete acts between persons who are participating in some kind of interactive situation […] the way we communicate with each other is constrained by the culture, but it also reveals and sustains culture (Schiffrin, 1994, p. 139).

When I located the three most prominent forms in each teacher's classroom and identified how those forms seemed to function, I argue that particular norms were apparent in the teacher talk because the major forms/functions occurred not only in all of the classrooms sessions but also often occurred repeatedly throughout each class session. In this paper, I focus on illustrating and elaborating how the two teachers drew from the students and textbooks as sources of knowledge in the classroom in different ways.

Background on Curriculum, Teachers and Students

I began my work in two "reform-oriented" classrooms as a graduate research assistant, focusing on student understandings of algebra. The "reform-oriented" teachers/classrooms I am referring to are ones that not only agree with the reform proposed by the NCTM *Standards* (1989, 1991), but that also piloted and use a curriculum that was developed in the spirit of this reform—the Connected Mathematics Project (CMP) (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998a). CMP is a middle school mathematics program that uses problem solving to introduce students to mathematical ideas. In the two 8th grade classrooms in which I worked, a major emphasis was placed on algebraic ideas. The curriculum offers linear, exponential and quadratic functions, emphasizing four representations: contexts, graphs, tables, and symbols.

As I became acquainted with the teachers, Karla and Josh, I came to find out that they had many attributes in common, including similar academic backgrounds, same teaching certification, same school, same curriculum and a similar enthusiasm for it, same heterogeneous group of 8th grade students, similar involvement in professional development activities, etc. One might expect that two such similar teachers with so many similar experiences teaching such similar populations of students would teach very much alike. However, as a participant-observer, I felt these two classrooms were different places to be.

Data Collection and Analysis

As a participant observer for over two years in these two classrooms, I took detailed field notes on my laptop both years and during the second year of my work audio- and video-taped 2-5 class periods per week while students were using two of the CMP algebra units—one at the beginning of the school year (*Thinking with Mathematical Models*) and one toward the end (*Say it with Symbols*). I also conducted 4-5 interviews with both the teachers and the students in the two classrooms. The teacher-interviews focused on having the teachers talk about: their backgrounds, their beliefs about teaching and learning mathematics, how they establish norms in their classrooms, etc. In addition, the teachers were presented with transcripts illustrating some preliminary patterns from their talk and were asked for their interpretation about those patterns[4].

The process of discourse analysis I employed included first listening to the data from *Thinking with Mathematical Models*[5] on three separate occasions and recording re-occurring forms in each teacher's language. Through examining these individual recordings, I located the three most common forms of talk in each classroom. I examined all instances of these forms of talk as separate excerpts and then chose two focal class periods to examine the forms more closely. The focal class periods were chosen based on common activities taking place across the classrooms. For example, each observation included the teacher handing back a quiz and going over it, going over a homework assignment, introducing students to a new Investigation, and exploring[6] a problem. When examining the forms within the context of the focal class periods, I identified what I believed the function of the forms to be, drawing on a range of discourse analysis perspectives in my interpretations, including speech act theory, pragmatics, ethnography of communication, etc. I then returned to my audio-tapes to: search for discrepant events, check to be sure that the patterns were consistent and prominent across all of the class sessions, and follow up on any questions that had arisen in my analysis[7]. Below I illustrate one set of the forms and functions of the teacher-talk and make inferences about the norms that were carried by and embedded in these.

Illustration of the Form/Function Analysis

To exemplify this form/function analysis with the purpose of establishing norms, I offer an example from each classroom, focusing on possible sources of knowledge from which a teacher may draw, including the textbook and the students. One result of my analysis showed that the teachers drew on the textbook and students in ways that were different from one another.

Drawing From the Textbook

Karla often referred to the textbook through stating her own narrative interpretation of the activity in which they were going to engage, using the pronoun "they" to refer to the authors of the textbook. This form seemed to function as a particular type of hedge, which are words or phrases whose job it is to make things fuzzier (Lakoff, 1972). The particular type of hedge that this seemed to function as was as an Attribution Shield (Rowland, 1995, following Prince, Frader, & Bosk, 1982). An Attribution Shield implicates a degree or quality of knowledge to a third party.

> An example of this appeared when students had a difficult time with the following problem on a quiz:
>
> To plant potatoes, a farmer cuts each potato into about 4 pieces, making sure each piece has an "eye." The eye contains buds that *will become* new plants. Each new plant *will produce* 5 potatoes. Thus, a single potato *will yield* 20 potatoes.
>
> A. Make a table and a *graph model* that show how the number of potatoes grown depends on the number of potatoes cut and planted.
>
> B. Write an equation that describes this situation (*Thinking with Mathematical Models*, Teacher's Edition, p. 64, emphasis added).
>
> Subsequently, this discussion took place when Karla handed back the quiz:
>
> Example 1: Karla's Form/Function
>
> K[8]: The potato problem, *they* gave more information than we needed to know. I've heard this--to try to get you to sort through the information that was necessary. All *they* wanted to know was that if I plant an old potato, about how many new potatoes are going to grow from it? All that stuff about cutting it up and dividing some from each piece and all that, that was just some background information about how if you wanted to go home and do this, you could do this.
>
> Ms: It was easy.

Sammie: I was confused.

K: I know [directed at Sammie].

[Ss comment—overlapping speech]

K: [inaudible] 20 potatoes. So then two potatoes and we could get forty and three, hopefully sixty. So, your table is going to increase that way. Why did *they* use the word "graph model" when the data seemed perfectly linear?

[…] [Comment regarding student behavior]

K: *They* used the word graph model when it seemed to be a perfect fit. […]

In comparing the language Karla used to the language the text used, an interesting contrast was taking place. The language that Karla used was hypothetical ("if," "could") and vague ("about," "seemed," "hopefully"). The language of the text was absolute and precise ("will produce," "will yield") and may have been confusing to students because when students collected data of their own or when they were presented with data that had been (supposedly) collected by someone else, the increases were not exact increments to account for errors collecting data. In addition, students seemed to have come to understand "graph models" as not being completely regular or precise because they referred to mathematical models as "kinda linear" or "almost linear" or "sorta linear." In fact, both Karla and the students used the word "perfect" to distinguish data that was exactly linear from a modeling situation. To deal with this conflict of language use, Karla may have used "they" as an Attribution Shield, identifying the authors of the textbook as a third party and positioning herself away from its external authority. This seemed to allow her to act as interpreter for her students, distancing herself from the authority of the textbook and the authors.

The norm that I've associated with this form is: "One of the teacher's roles is to offer or interpret the author's intentions." It was tacitly the student's role to accept the interpretations as she stated them. This expectation appeared in student talk on a few occasions when a student would request that Karla "say that in English" when something was not clear to a student.

In contrast, Josh explicitly drew from the textbook in many ways. One of the forms Josh repeatedly used to do this was to state that students should "Look back [in the book…]." This form seemed to function as a way to help students see that the text could be used as a resource when students were uncertain or unable to answer a question, especially the questions that he himself asked. A quick but important point is that "question-forms" in general were the most prominent discourse pattern I identified in Josh's talk[9].

An example of this form and function occurred during talk about the purpose of mathematical models.

Example 2: Josh's Form/Function

J:	[…] Remember what a graph model is—it's a line or a curve that shows the trend in the data, it fits the data-so that all the points are pretty close [to the line of best fit]. Um, I don't know. Why do we do this? What's the purpose of a graph model?
	[Abram raises his hand right away]
J:	Abram, what's the purpose?
Abram:	To show the linear relationship.
J:	Yeah. I could maybe see that it's linear just from looking at the table or just looking at the way the points are plotted. Why did I draw the line in? Just to show the pattern? Christy?
Christy:	To get a better look at what the data is trying to tell you.
J:	Well, maybe that's part of it. Look back at your definition for a graph model. Look at your definition of a graph model. What does it [the book] say? Read that last paragraph to yourself on page seven. Lance, what's the purpose here? Why do we even bother doing this?

Josh began this portion of the class by preformulating (French & MacLure, 1983) his question about the purpose of graph models as he reminded students of the textbook's definition of a graph model. He then asked a series of questions, reformulating his first question "Why do we do this? What's the purpose?" to "Why did I draw the line in?" to "What does it [the book] say?" When there was not an immediate response, Josh told them the exact page and paragraph to look at to answer the question in the "correct" manner. In doing so, he may have tacitly let students know that they should refer to the textbook when they were uncertain of something, especially when they were unsure about how to answer his questions.

The norm that I've associated with this form/function is: "The textbook is part of the mathematical authority and is a place in which to find correct answers." This norm seemed to be accepted by students because there were some instances in my data of students looking ahead in the textbook, answering Josh's questions by reading from the book and justifying their answer by saying that it "was in the book."

A closer look at forms and functions revealed that the teachers used different surface forms that appeared to serve the same purpose in each classroom. These differences had implications for teacher-positioning with respect to authority and knowledge. In the next section I will first demonstrate how different surface forms can serve the same broad purpose by focusing on how the teachers drew on other sources of knowledge (i.e. the textbooks and students) in the classroom. I will then describe how these may have affected how the teacher positioned him/herself with respect to the external mathematical authority and his/her epistemological stance.

Different Surface Forms: Same Broad Purpose

Both Josh and Karla drew from other knowledge sources in the classroom. However, the forms that each used for doing this were different. So, their broad purpose was the same, but the words they used were not. I've already described how Josh explicitly drew from the textbook, locating it as a source of authority in the classroom and how Karla used the word "they" when referring to the textbook, acting as an interpreter of the textbook. In addition, they each had other ways that they drew from knowledge sources in the classroom, which are summarized in Table 1 below.

I'll first summarize how the teachers drew from the textbook as a source of knowledge in the classroom and then illustrate the differences in how they drew from students.

Drawing From the Textbook

By referring to the authors of the textbook as "they", Karla positioned herself as separate from the external mathematical authority. Even though she may have separated herself from the locus of authority, she sometimes seemed to be playing the role of interpreter who had access to their intentions. When I asked Karla about her relationship with "they," she talked about how much respect and reverence she had for the authors and how accountable she felt to them. In a sense, it appeared that Karla viewed "they" as an authority to which she must render—"I feel that much respect for them that I worry about, am I really trying to hold true to what their intent was?" (Interview

Table 1.

Common Function	Karla's Forms	Josh's Forms
Drawing from other sources of knowledge	**Textbook:** • "They"; **Students:** • "If you go too fast, I get lost"; "Help us out here" • "Slantiness" (classroom-generated language)	**Textbook:** • "Read on page…"; "Look back in the book…"; "What do you know?" **Students:** • Multiple Questions; Open → Closed questions • "Slope" (more official mathematical language)

K-4, 1/07/00). In fact, she seemed to use "they" as an Attribution Shield, attributing the "knowing" to the authors-a nameless group of people that she calls "they." By doing so, Karla seemed to be letting students know that there was a much bigger authority to which they all must answer, downplaying her own role as authority in the classroom.

In contrast, Josh often drew explicitly from the textbook. As I showed earlier in the example from Josh's talk, he sometimes pointed to the fact that answers to his questions could be found in the book. He typically defined the classroom activity or introduced and defined mathematical terminology in his direct references to the textbook. In addition, Josh attributed "knowing" to the textbook through his use of the question form "What do you know?" That is, when students were given mathematical information in the textbook, Josh would refer to that information as things students "knew." In Josh's classroom, the textbook was positioned as a source of knowing and as part of the external mathematical authority.

This latter claim was further evidenced by the fact that Josh and the textbook had a similar "voice;" they had common ways of addressing students. The most prominent discourse pattern I identified in Josh's classroom was that he often asked strings of questions (sometimes up to five in a single turn). After I realized that each teacher drew from the textbook in different ways, I also did a detailed analysis of *Thinking with Mathematical Models* (Herbel-Eisenmann, 2000). In that analysis I found that the book also often used strings of questions. In his interviews, Josh seemed to equate asking the "right kinds of questions" with "developing a true understanding." However, because both Josh and the textbook used this similar form, students may have viewed them as having a similar voice-the voice of authority.

Drawing from Students

In drawing from students as a source of ideas/knowledge in the classroom, Josh and Karla also used different surface forms. There were at least two ways in which Josh and Karla differed in the manner in which they drew from students: how they engaged students with their own and each other's ideas, and how they attended to student's ways of talking about mathematics.

Engaging Students with Ideas

When engaging students, Karla drew them into articulating their thinking by saying that she was "lost." She also asked students to interpret each other's ideas when they offered non-conventional ways of solving problems by telling students she was "confused". In Goffman's (1974; 1981) terms, she had students "animate" one another by calling on other students to explicate, putting them in the position of interpreting each other's thoughts. This is the "converse of externalizing one's own reasoning" (p. 91), which O'Connor and Michaels (1996) claim is "necessary activity within collaborative intellectual work".

In contrast, Josh asked multiple questions and many that appeared "open" were reformulated and became more "closed" (Barnes, 1969). This may have happened because Josh had a particular answer in mind, as I illustrated earlier. Many of his questions may have appeared test-like in nature because there really was only one acceptable answer.

Talking about Mathematical Content

The second way Josh and Karla differed in their use of student contributions was associated with the ways they advocated talking about mathematical ideas. Through my observations and analysis, I have attended to the multiple ways that the teachers and students talked about important mathematical ideas, including slope and y-intercept (Herbel-Eisenmann, 2000; Herbel-Eisenmann, Smith, & Star, 1999). By doing so, I have examined both the "mathematical register" (Pimm, 1987, following Halliday, 1978) and other ways of talking about mathematical content, examining the range of "more" and "less" mathematical talk (Chapman, forthcoming). While both teachers used a mix of language that seemed closely linked to the tables (e.g., "what it [the *y*-value] goes up by [as the *x*-value increases by 1]"), the graphs (e.g., the "steepness") and the equations (e.g., "it's the number attached to the *x*"), their use of language differed on the more extreme ends of the continuum.

In Karla's classroom, appropriation of student language for mathematical terms was quite common. For example, they often referred to the slope of a line as the "slantiness" of the line or to decreasing exponential functions as "swoopy down curves." In addition, Karla did not explicitly correct students when they chose to use this language over more mathematically appropriate language. In Josh's classroom, the focus was more on using official mathematical language like "slope" and "y-intercept". In fact, I have no instances of the type of classroom-generated language in Josh's classroom that I have in Karla's classroom.

Discussion

How might these different ways of talking influence the classroom? The way that Karla seemed to position herself was more deferential to the external mathematical authority; the epistemology that she seemed to be advocating was that students were also a source of knowledge who could create and discuss each other's mathematical ideas. She acknowledged and appropriated student's ways of talking about mathematical ideas, allowing students to engage in "exploratory talk," which Barnes (1976) claims is a different way of rehearsing knowledge than "final draft" talk.

The way that Josh seemed to position himself was as part of the external mathematical authority; the epistemology that he seemed to be advocating was that the mathematics already existed and it was something that he and the textbook "know." The role of students was to find answers in the book, and remember things Josh said. In addition, his focus on official mathematical language indicated that he valued talk associated with the external mathematical community, rather than the community of the classroom.

In the teacher-interviews, when I offered transcripts of particular patterns I had noticed in each classroom, the response was one of surprise. Both Josh and Karla did not seem to notice they were repeating these patterns of talk[10] and found it difficult to try to tell me what they thought the talk patterns were accomplishing in the classroom. However, some consistency did appear between what the teachers said their beliefs and expectations were with the practice that I observed. Karla said that she did not want students to think she was the "all-knowing" and that she wanted to take learners from where they were at and help guide their thinking, keeping in mind the big mathematical ideas. She often defined the mathematical topic to be discussed, but made the processes students were offering the center of the class discussion. Listening to student ideas guided many of the decisions she made in her classroom, although she did not feel like she did that well enough and was working with a colleague to improve on this.

In contrast, Josh often relayed that he was concerned with "control" in the classroom and said that he'd seen some very powerful mathematical discussions that were student-centered. He also acknowledged the fact that he just couldn't see himself doing that—it was not comfortable for him to let the conversation go like that. He wanted to keep student's ideas on line with the big mathematical idea that was the focus of the lesson. One way that he did this was to ask multiple questions that began open and became closed through his reformulations—directing students toward the "right" answer.

Implications

I believe bringing a discourse perspective to the mathematics classroom can help teachers think about many things. The discourse analysis that I have offered helps us to see more clearly how issues of positioning with respect to the external authority and one's epistemological stance might be "embedded" in teacher-talk. It can help teachers see more clearly how using language may be undermining or promoting the kind of discourse teachers may want to establish in their classroom. This type of analysis gives us a lens to think about *how* these things are inherent in the discourse—it makes it more concrete and tangible.

This work extends much of the work on teacher beliefs/knowledge by beginning to define a concrete way that these beliefs/knowledge play out in a teacher's practice. As Thompson (1992) pointed out,

> …as teacher educators we must investigate ways of helping teachers reflect on their [practice]. We must find ways of helping teachers become aware of the implied rules and beliefs that operate in their classrooms and help them examine their consequences (p. 142).

I propose that one way to do that is to engage experienced teachers in thinking about their practice from a discourse analysis perspective, allowing them to examine their beliefs/knowledge/conceptions in a more concrete way. One reason for engaging teachers in this perspective would be to bring these issues to a level of consciousness. The use of language in these two classrooms did not seem to be functioning at a conscious level of awareness. That is, the teachers were doing these things, but as I pointed out earlier they were not really aware of what they were doing or what purpose their patterns seemed to serve.

Next Steps

In 1987, Shulman began to outline the knowledge base for teaching and distinguished between content knowledge and pedagogical content knowledge. In addition, there seems to be *something* else interacting in the classroom with these—content knowledge, pedagogical content knowledge and this type of *pedagogical discourse knowledge/beliefs* all seem to be interacting as a triad. Currently, many preservice and inservice programs include experiences related to content knowledge and pedagogical content knowledge. However, there is no real place for studying the interaction between discourse and beliefs in classrooms and the effect that aligning these more closely might have on the learning environment. I am currently developing such an experience for teachers and intend to look at how studying one's discourse may affect the mathematical learning environment in the classroom.

References

Barnes, D. (1969*). Language in the secondary classroom*: *Language, the learner and the school* (pp. 11-77). Baltimore: Penguin Books.

Barnes, D. (1976). *The teacher's control of knowledge*: *From communication to curriculum* (pp. 108-138). London: Penguin Books.

Cazden, C. (1988). *Classroom discourse: The language of teaching and learning*. Portsmouth, NH: Heinemann.

Chapman, A. (forthcoming). A social semiotic of language and learning in school mathematics. In V. Cifarelli (Ed.), *Semiotic perspectives on mathematics education*. Purdue University Press.

French, P., & MacLure, M. (1983). Teachers' questions, pupils' answers: An investigation of questions and answers in the infant classroom. In M. Stubbs & H. Hillier (Eds.), *Readings on language, schools and classrooms* (pp. 193-211). New York: Methuen.

Goffman, E. (1974). *Frame analysis*. Cambridge, MA: Harvard University Press.

Goffman, E. (1981). *Forms of talk*. Philadelphia: University of Pennsylvania Press.

Halliday, M. A. K. (1978). *Sociolinguistic aspects of mathematics education, Language as social semiotic: The social interpretation of language and meaning*. Baltimore: University Park Press.

Herbel-Eisenmann, B. A. (2000). *How discourse structures norms: A tale of two middle school mathematics classrooms.* Unpublished dissertation, Michigan State University, East Lansing, MI.

Herbel-Eisenmann, B. A., Smith, J. P., & Star, J. (1999). *Middle school students' algebra learning: Understanding linear relationships in context.* Paper presented at the American Educational Research Association, Montreal, Quebec.

Lakoff, G. (1972). *Hedges: A study in meaning criteria and the logic of fuzzy concepts.* Paper presented at the Eighth Regional Meeting of the Chicago Linguistics Society, Chicago.

Lappan, G., Fey, J., Fitzgerald, W., Friel, S., & Phillips, E. (1998a). *The connected mathematics project.* Palo Alto, CA: Dale Seymour.

Mishler, E. G. (1972). Implications of teacher strategies for language and cognition: Observations in first-grade classrooms. In C. B. Cazden, V. P. John, & D. Hymes (Eds.), *Functions of language in the classroom* (pp. 267-298). Prospect Heights, IL: Waveland Press.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards.* Reston, VA: The Council.

National Council of Teachers of Mathematics. (1991). *Professional standards.* Reston, VA: The Council.

O'Connor, M. C., & Michaels, S. (1996). Shifting participant frameworks: Orchestrating thinking practices in group discussion. In D. Hicks (Ed.), *Discourse, learning and schooling* (pp. 63-103). New York: Cambridge University Press.

Pimm, D. (1987). *Speaking mathematically.* London & New York: Routledge & Kegan Paul.

Prince, E. F., Frader, J., & Bosk, C. (1982). On hedging in physician-physician discourse. In R. J. di Pietro (Ed.), *Linguistics and the professions* (pp. 82-96). Norwood: Ablex.

Rowland, T. (1995). Hedges in mathematics talk: Linguistic pointers to uncertainty. *Educational Studies in Mathematics, 29*, 327-353.

Schiffrin, D. (1994). *Approaches to discourse.* Cambridge, MA: Blackwell.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*, 1-22.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. New York: Macmillan.

Notes

[1] This work was funded, in part, by a Spencer Research Training Grant at Michigan State University.

[2] The "norms" of these classrooms are defined by the expectations, roles, responsibilities, and rights that are established in the classroom.

[3] For the purposes of this paper, discourse includes mainly spoken language, but also the written language of the textbook. As the 1991 NCTM *Standards* point out, inherent in the discourse are issues pertaining to what it means to know and do mathematics—where the locus of authority lies, who gets to speak, what students get to speak about, etc. Intertwined in focusing on written and spoken language are the interactions between people that help establish the environment of the classroom and that send subtle messages about the norms in the classroom.

[4] More specifically, the teachers were asked: Did you notice that you often say this? If you were to name this talk-pattern, what would you call it? What do you think this talk-pattern is doing in your classroom?

[5] The data from this unit consisted of 8 observations in Karla's classroom and 7 in Josh's classroom over about 3 weeks in October, 1998. Josh was absent during one of the scheduled observations. Also, the two teachers typically paced their lessons differently, so during most observations Josh was 1-2 lessons ahead of Karla. This resulted in my observing only a few overlaps in what each teacher was teaching.

[6] The pedagogical format introduced by CMP is that teachers "launch" a problem, then typically students "explore" it in small groups, and finally the teacher finishes the problem by "summarizing" it with the class.

[7] An example of sometning I had to follow up on included tracing Josh's questioning patterns to see how they played out in the classroom. One of his question-forms appeared to be a form of scaffolding (Cazden, 1988). I investigated the data further to see if this line of questioning appeared only after Josh handed back quizzes (as it had in the focus class periods) or if they appeared more regularly throughout other class periods.

[8] I use "K" to indicate when Karla is talking. "Ms" is used when a male student is talking; "Fs" for a contribution from a female student. Whenever multiple students were talking, I use "Ss" to indicate this.

[9] I will address this further when I return to issues related to the textbook later in the paper.

[10] The exception here was Josh's recognition of his use of questions in general. He was not aware of the particular forms of questions that showed up repeatedly in his talk. For example, he didn't realize that he often used forms that ended with a tag like, "That's the x-intercept, isn't it?"

A TEACHER IN CONFLICT: ROBIN AND OPEN-ENDED ASSESSMENT

Wendy B. Sanchez
Kennesaw State University
wsanch@bellsouth.net

Abstract: The purpose of this study was to understand factors that influenced a teacher's use of open-ended assessment items. Findings indicated that the teacher's beliefs, the constraints of her system, and her level of reflectivity affected her use of open-ended assessment. Characteristics that enabled the teacher to use open-ended assessment powerfully are discussed and implications for teacher education are provided.

Studies concerning teachers' assessment practices (Cooney, 1992; Hancock, 1994; Senk, Beckman, & Thompson, 1997) have painted a grim portrait of the movement toward the vision for assessment outlined in the National Council of Teachers of Mathematics' [NCTM] *Standards* documents. Along with the call for understanding as the focus of mathematics teaching, the NCTM has called for assessments that reflect an emphasis on understanding. Whereas prior research on teachers' assessment practices has focused primarily on the kinds of questions that teachers use, this study focused on a particular kind of question (open-ended items) and how one teacher used those kinds of questions.

Open-ended assessment items, broadly defined, are questions that have either more than one solution or more than one way to arrive at a solution. Moon and Schulman (1995) explained,

> Open-ended problems often require students to explain their thinking and thus allow teachers to gain insights into their learning styles, the 'holes' in their understanding, the language they use to describe mathematical ideas, and their interpretations of mathematical situations. When no specific techniques are identified in the problem statement…teachers learn which techniques the students choose as useful and get a better picture of their students' mathematical power (p. 30).

Responses to open-ended items can provide teachers with information about their students' understanding that responses to traditional items fail to provide. Teachers can use the information gained from responses to open-ended items in a variety of ways to inform their teaching.

This paper presents the results of a study of a secondary mathematics teacher (Robin) who had participated in three consecutive projects designed to enable teachers to create and use open-ended assessment in their teaching. The third of these projects was concurrent with data collection. This paper describes both Robin's use and the factors that influenced her use of open-ended assessment items. The purpose of the paper is to provide an example of a teacher who used open-ended items in a meaningful way and to provide insights into the characteristics that enabled her to do so. Struggles that Robin faced in using open-ended assessment are also highlighted.

Theoretical Perspectives

Ernest (1988) identified beliefs, constraints, and reflectivity as three elements essential for understanding the practice of teaching mathematics. These three elements were explored as possible factors that could affect Robin's use of open-ended assessment items. Dewey (1933) defined reflective thinking as the "active persistent, and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it and the further conclusions to which it tends" (p. 9). He further explained that reflective thinking "includes a conscious and voluntary effort to establish belief upon a firm basis of evidence and rationality" (p. 9). Dewey described reflective thinking as constructing a metaphorical chain with each link built from the previous one. Ideas, when tied together in a consecutive way to form a consequence, make up reflective thinking. Random ideas that pass through our heads do not constitute reflective thinking. Not all thinking is reflective thinking, but not all thinking that is not reflective should be considered low-level. King and Kitchener (1994) explained that thinking may be high level and require a good deal of knowledge or intelligence, but not be reflective thinking.

Dewey (1933) identified two phases of reflective thinking. In the first phase, an individual recognizes something as problematic, similar to the notion of a perturbation. It is during this phase that thinking originates. Subsequent to the recognition of a situation as problematic, an individual initiates the second phase, an act of search in order to solve the problem. During this search, ideas come to mind (and are linked together as a chain, as described above) in order to generate hypotheses. These hypotheses are tested in an effort to create evidence. Beliefs can be established as a result of this reflective process.

People come to believe through a variety of means. The reflective thinking described above can be viewed as a way to establish, confirm, or refute a belief, or more generally, to come to believe. In their reflective judgment model, King and Kitchener (1994) posited three categories of judgment: pre-reflective, quasi-reflective, and reflective. It is in the reflective category that King and Kitchener (1994) claim that true reflective thinking (in the Dewey sense) occurs. Pre-reflective thinkers view knowledge as certain. Viewing knowledge as certain does not leave room for recognizing situations as truly problematic. Thus, reflective thinking is cut off at the first phase. Quasi-reflective thinkers recognize that some situations are problematic, and they recognize the need for evidence, but they use evidence in an idiosyncratic way. These thinkers get cut off in the second phase of reflective thinking. Although ideas might come to mind, they might not be linked together in a logical way. These quasi-reflective thinkers may grab onto any evidence they come across without testing it or judging its viability. In contrast, true reflective thinking guides the formation of beliefs in the reflective category.

Thinking can be classified as reflective or not, but reflective thinking can be further classified according to the depth of reflection. Van Manen (1977) described three levels of reflectivity that demonstrate this depth. The first level of reflection is concerned with the means needed to obtain given ends. A person reflecting at the first level takes for granted that the ends are desirable and does not question them. The criteria for reflection at the first level are limited to economy, efficiency, and effectiveness. Individuals reflecting at the first level are mostly concerned with utility, with what works. They are looking for the most efficient path to a goal. At the second level of reflection, a debate occurs over principles and goals. At this level, one assesses the implications and consequences of actions and beliefs. The ends are as yet not questioned at this second level. Individuals reflecting at this level follow their actions to their logical conclusions and assess whether or not the results are aligned with their principles and goals. In contrast to the first level, the concern is greater than merely what works. The concern is more directed at why things work or if the way they work is appropriate in a given context. At the third level of reflection, the ends themselves are questioned. At this level, one questions the "worth of knowledge and…the nature of the social conditions necessary for raising the question of worthwhileness in the first place" (Van Manen, 1977, p. 227). Here, individuals question the morality or value of propositions given the conditions of society at large.

Methodology

Three secondary mathematics teachers were observed for seven weeks. This paper provides the results of the study from one of the three teachers. I observed Robin's teaching approximately four 50-minute class periods for each of the seven weeks of data collection. I interviewed her once each week, and each interview lasted about an hour. The data for this study consisted of 7 individual interviews, field notes from 26 class-period observations, and artifacts such as student work, handouts, assignments, tests, and quizzes. Each interview was audio-taped and transcribed. Data analysis was done through categorical aggregation, in which "the researcher seeks a collection of instances from the data, hoping that issue-relevant meanings will emerge" (Creswell, 1998, p. 154). This process was aided by the creation of cards containing relevant pieces of data that were sorted into groups. Themes emerged from the sorting of the cards.

Getting to Know Robin

Robin was an experienced teacher of 26 years. She was the mathematics department chair at her school. Robin was a leader in her county, serving on several committees and project teams. She had taught many staff development courses in her county.

She had taught every level of secondary school mathematics from pre-algebra through calculus, and she had also taught middle school mathematics. As a magna cum laude undergraduate, Robin took 60 quarter hours of mathematics courses, including courses in number theory and topology. She completed a master of education degree in mathematics education, during which she took courses in modern algebra, geometry, and analysis. The year following data collection, Robin was voted teacher of the year at her school and also teacher of the year for her county.

Robin believed that teaching should promote understanding. She did not want her students to merely be able to compute correct answers to mathematical problems. She wanted her students to understand why procedures work, how they are connected to other procedures and how the procedures could be applied in the world. She believed that it was important for students to see how certain skills were developed from other skills and how many mathematical skills are the reverse processes of previously learned skills. During instruction, Robin asked questions such as, "Why is 40 the common denominator?" (Algebra I, 2/11), "Tell me why $x^2 + 4 \neq (x + 2)(x + 2)$?" (Algebra I, 1/27), "What does it mean that I have two answers?" (Algebra I, 1/19), and "Do you notice a pattern?" (Calculus, 1/27). Her focus was clearly on more than computing answers.

Robin believed that mathematics fits together, that the pieces of mathematics can be linked together. This view of mathematics played out strongly in Robin's teaching, as she sought to help students link the pieces of mathematics together in a meaningful way. The concept of connections was central to Robin's beliefs about teaching. She explained, "I like to always give my students a reasoning or a kind of a developmental sequence of how this came about or how you can fit this in with this" (Interview 1, 1/26). Later she said,

> If there is anything that I think I do better than other teachers, it's showing the connection from one topic to the next....My whole emphasis in mathematics is, don't learn math as a bunch of rules, but develop the why and the understanding of it or the connections between. (Interview 5, 3/10)

Perhaps more important than Robin's emphasis on connections was that she carried it out in her teaching. "Connections" was not merely an NCTM buzz-word to Robin. More than rhetoric, it was an essential aspect of mathematics and mathematics teaching. For example, when she taught simplifying rational expressions, she began by asking students to recall how to simplify rational numbers. She connected the way she taught rational expressions with the her students' previous knowledge of rational numbers. This was common practice for Robin.

Robin was a reflective person who thought deeply about teaching. Several times during data collection, Robin told me when we sat down for an interview that she had been thinking more about what we talked about in the last interview. She wanted to revisit topics from previous interviews once she had time to further think about things.

Robin and I discussed the fact that students have a hard time understanding factoring. I asked Robin what it was that the students did not understand. She said,

> I guess…they don't know what factors are. I mean, they don't realize that, maybe it boils down to the symbolism, the variables. Maybe we haven't built up the whole concept of the expression with x's in it. (Interview 3, 2/11)

During a subsequent observation, she explained to me that she had been thinking about what we talked about in interview 3. She said she had been thinking about how teachers emphasize the procedural, especially with factoring. She felt that students do not understand that $x^2+8x+15$ is a number, and that its factors are two numbers that multiply to yield that number. Perhaps, she explained, if we would emphasize plugging in some values for x, it would help, e.g., $x^2+8x+15 = (x+3)(x+5)$. Let $x=2$. The factors would be 2+3 and 2+5 (5 and 7) so the product $x^2+8x+15$ should be 35, and it is (because $2^2 +8(2)+15$ is 35). After our interview, Robin kept thinking about what it was that students did not understand about factoring and she devised an alternative way of presenting that material that she thought might enhance understanding.

Robin's reflective nature kept her thinking about her responsibilities and her beliefs and she often found herself in conflict between the two. Robin believed that it was her responsibility to cover the entire curriculum, but this belief sometimes conflicted with her belief that teaching should promote understanding. Teaching for understanding was difficult given the curriculum and time constraints under which she worked. Robin was strictly tied to her county's curriculum (see below). When asked if she was able to be the best teacher she could be, she responded by saying,

> You are just so pushed for covering this curriculum objective, and this curriculum objective, that you can't really slow down and do it as thoroughly as you want to do it. And so, in that sense, no, I think I could be a much better teacher if I had more time to develop topics. (Interview 2, 1/31)

At times Robin chose to teach a topic differently due to lack of time. For example, I observed her the day she taught the slope-intercept form of a line. The last fifteen minutes of class, she explained that in the equation $y = mx + b$, the "m" was the slope and the "b" was the y-intercept, meaning where the line crossed the y-axis. She explained how to put a point on the y-intercept and then count using the slope to determine a second point and then connect the points to graph the line (Observation, 3/10). Later that day, during an interview, Robin discussed a different method of teaching the slope-intercept form of a line. She described a discovery, inductive approach to teaching the topic where students graph lines using a table of values (a skill they had already mastered) and then notice patterns, eventually discovering that the y-intercept is the "b" in the slope-intercept form of a line and that the slope of the line is the "m" in the equation. Robin described how she could have taught the lesson in a way that would better promote student understanding, but in action, she made a decision to just tell. She made a conscious decision to not use an inductive approach and to tell students "the rule." She was not comfortable with this decision. Reflecting on it, she said the lesson was unsuccessful. But finishing the curriculum was important enough to Robin to sacrifice for it.

Robin felt obligated to finish the curriculum for primarily two reasons. First, she had been involved in writing curricula for her county. She felt that if the writers could not finish it, how could they expect others to? Secondly, as department head at her school, she felt that she should be a model for the other teachers in her school. Robin planned to serve on a curriculum committee the year following data collection, and she said, "I'm hoping that on this curriculum committee that we're on, that we do start narrowing down the topics" (Interview 5, 3/10).

When Robin used open-ended items, she used them in a meaningful way. She felt that student responses to assessment items could and should give her feedback about her teaching. When she assigned an open-ended item regarding factoring that few students were able to answer, she commented that

> Next year when I start talking about factoring, I am going to at least include that idea because I felt like I overlooked it or I would have had at least one answer from everybody. So that made it a good item, too, that fact that not only did I get information about what they could do, I also got information about what I could do. (Interview 2, 1/31)

There were instances where Robin saw a student's method for working a problem and decided that the student's method made more conceptual sense than the method she taught. After teaching percent problems, Robin noticed that one of her students had a method for answering percent questions in which she related everything back to one percent. Robin said,

> To me, that girl knew what she was doing. I mean, she related it to and she could do all three of those types of problems by relating it to, okay, if I have got five percent and it is this, what is one percent? And to me, that's so much more meaningful than to set up this proportion, cross multiply, and divide, you know? Number one it shows a little bit of sense with numbers. And I think that other method is just pushing numbers around on the page…they don't understand the relationship between numbers. And so, I am going to use that [method the next time I teach this topic]. (Interview 7, 5/1).

Robin did not merely grade her students' responses to assessment items, she thought about how the students were thinking and how her instruction impacted or should impact the way they were thinking. She was not willing to place blame immediately on the students when they missed an item. She did not assume that students were not prepared or did not practice enough. She was willing to consider that her teaching may not have facilitated the student's understanding the way she had anticipated it would.

Robin used the information in student responses not only to ascertain that a student did not master a skill or concept, but also to understand exactly what misconception that student had. After reading a particular student's response to an open-ended item concerning rational expressions, Robin determined that this student believed that one could square the numerator and denominator of a fraction and have an equivalent fraction. This was powerful information that Robin could use in further instructing this student.

It was a daily struggle for Robin to mediate between her beliefs about teaching for understanding and her commitment to finishing the curriculum. During data collection, I asked Robin questions about her beliefs and we discussed her teaching. We discussed that her beliefs about teaching were not always aligned with her actual teaching practice. This really bothered Robin, and in fact, one day she got quite emotional, believing herself to be a hypocrite. Robin professed a belief that using open-ended items was an important way of gathering information about student understanding. Yet she rarely used open-ended items during class. Open-ended items did appear on tests, but were seldom used as part of instruction or as homework questions. Robin began to question why she was not doing what she believed she should do. Eventually, she concluded that she used much more open-ended assessment than she used to, and that everything could not change overnight. She recognized changing her teaching practice to include more open-ended assessment as a process—one with which she was not finished.

Discussion and Implications

Robin's beliefs, the constraints of her system, and her level of reflectivity each affected the way she used open-ended assessment. Because she believed that mathematics teaching should promote understanding, she believed it was important to assess understanding. Too, her belief that teaching should promote understanding led her to value student approaches to mathematics that made more conceptual sense than her own approach.

Though Robin believed in using open-ended assessment, there were constraints that limited her use of open-ended items during class. Because of her commitment to finishing the curriculum, Robin was not always willing to use valuable class time to explore open-ended questions. Robin's commitment to her curriculum was not a result of her dependence on authority for her decision making. She was willing and capable of making her own decisions about teaching. Her commitment to the curriculum stemmed from her involvement in the curriculum-writing process. She recognized that the curriculum contained too many objectives, but she also understood that the state guidelines prevented individual school systems from removing objectives. She recognized a need to change the state guidelines.

The most powerful factor that influenced the way Robin used open-ended assessment was the extent of her reflectivity. Robin's thinking about her teaching was reflective thinking, in the Dewey sense. Recall her thoughts about teaching factoring. She recognized the situation as problematic (students did not understand factoring). She then initiated a search in order to solve the problem (she came up with a new approach to teaching factoring). She intended to test her method the next time she taught factoring. Robin approached her teaching reflectively, and her approach to assessment was no different. She used the results of assessment for more than just assigning grades. When she looked at student responses to open-ended assessment, she gained information about student understanding and about her teaching. She used this information to adjust her teaching in general, and for particular students.

Robin was concerned with more than whether students were getting correct answers. She wanted her methods of teaching to be aligned with her purposes and goals for teaching. If her teaching did not promote understanding, then she did not feel successful, even if students got correct answers. She used results of assessment to check for understanding, and learned new ways of presenting material for understanding from her students' responses to open-ended items. Robin consistently reflected at the second of Van Manen's levels, and sometimes on the third level, and this deeper level of reflection empowered her use of open-ended items.

From conducting this study, I have learned important lessons about teacher education, both preservice and inservice. First, I have learned that my job as a teacher educator is more than merely *convincing* teachers to use open-ended items. Teachers may use open-ended items but not in a way that moves them toward the NCTM's vision for assessment that reflects understanding. Rather, we must help teachers learn *how* to use open-ended items in powerful ways. Robin believed that teaching should promote understanding and approached assessment (and teaching) in a reflective way. These characteristics enabled her to use open-ended items powerfully. However, even though Robin was *able* to use open-ended items in a way that meaningfully affected her teaching, she did not use them often because of the constraint of her curriculum. This is a teacher who is able and willing to teach for understanding, but sometimes chooses not to because of time. If reform is truly to take hold, the community must address the constraints under which teachers work. Finally, I have learned that reflective thinking facilitates a meaningful use of open-ended items. The next question (and certainly not a new one) is how do we promote reflection in our teachers?

References

Cooney, T. J. (1992). *A survey of secondary teachers' evaluation practices in Georgia* (A study supported by the Eisenhower Program for the Improvement of Mathematics and Science Education). Athens: University of Georgia, Mathematics Education.

Creswell, J. W. (1998). *Qualitative inquiry and research design: Choosing among five traditions*. Thousand Oaks, CA: Sage.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. Boston: Heath.

Ernest, P. (1988, August). *The impact of beliefs on the teaching of mathematics*. Paper presented at the Sixth International Conference of Mathematical Education, Budapest, Hungary.

Hancock, C. L. (1994). Coping with a mandate: Effects of a revised end-of-course test for first year algebra. *Dissertation Abstracts International, 58* (02A), 0483. (University Microfilms No. AAI9520826)

King, P. M., & Kitchener, K. S. (1994). *Developing reflective judgment.* San Francisco: Jossey-Bass.

Moon, J., & Schulman, L. (1995). *Finding the connections: Linking assessment, instruction, and curriculum in elementary mathematics*. Portsmouth, NH: Heinemann.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Senk, S. L., Beckman, C. E., & Thompson, D.R. (1997). Assessment and grading in high school mathematics classrooms. *Journal of Research in Mathematics Education, 28*, 187-215.

Van Manen, M. (1977). Linking ways of knowing with ways of being practical. *Curriculum Inquiry, 6* (3), 205-228.

ANALYZING PROFESSIONAL DEVELOPMENT BY THINKING OF TEACHERS AS LEARNERS

Jeffrey M. Choppin
University of Wisconsin-Madison
jmchoppin@students.wisc.edu

Professional development is one of the most important components of mathematics education reform, yet there is still considerable debate about what constitutes effective professional development practices. I will use Simon's (1997) model of teaching and Webb, Heck, and Tate's (1996) model of teacher growth to analyze professional development practices described by several reviews of large programs. I chose these two models because: (1) they describe teacher change in the context of reform in mathematics education; (2) they represent both theoretical and empirical emphases; and (3) they connect teaching and learning (that is, they treat teachers as learners). The purpose of this paper is to inform the discussion of how best to facilitate teacher practices that reflect reform in mathematics education.

Webb, Heck, and Tate's (1996) model, based on analysis of teachers in the Urban Mathematics Collaborative, features seven elements of professional growth. These seven elements were rarely found in isolation and the extent and combination of their presence varied with each teacher. Together, they portray teacher growth as a complex, difficult, and highly variable process.

Simon's 1997's model describes the Mathematics Teaching Cycle. In this cyclical model, interaction with students influences both teachers' mathematical and pedagogical knowledge, which influences teachers' goals, plans, and assumptions about student learning, which influences teachers' interactions with students. The teaching cycle is guided by a teacher's knowledge and goals and by the mathematics of students. This model implies a dynamic relationship between students' performance, classroom norms, and teacher knowledge.

I will assess the alignment of these models with prevalent professional development practices. The purpose will be to determine if the presence of aspects of these models align with professional development practices that are considered effective.

The professional development literature that will be discussed are a review of the Eisenhower program (Garet et al., 1999), a review of state systemic initiatives in mathematics (Corcoran, Shields, & Zucker, 1998), a review of California mathematics reforms (Acquarelli & Mumme, 1996; Cohen & Hill, 1998), and work done at the National Institute for Science Education (Kennedy, 1998; Loucks-Horsley, Hewson, Love, & Stiles, 1998).

This literature reflects large-scale empirical observations of professional development practices, searches for best-case examples, in-depth observations of particular programs, and reviews of the professional development literature.

References

Acquarelli, K., & Mumme, J. (1996). A renaissance in mathematics education reform. *Phi Delta Kappan, 77* (7), 478-484.

Cohen, D. K., & Hill, H. C. (1998). *State policy and classroom performance: Mathematics reform in California* (Report RB-23). Philadelphia: Consortium for Policy Research in Education.

Corcoran, T. B., Shields, P. M., & Zucker, A. A. (1998). *Evaluation of the National Science Foundation's Statewide Systemic Initiatives (SSI) Program: The SSI's and professional development for teachers* (Project 3612). Menlo Park: SRI International.

Garet, M. S., Birman, B. F., Porter, A. C., Desimone, L., Herman, R., & Yoon, K. S. (1999). *Designing effective professional development: Lessons from the Eisenhower Program* (Report). Washington, DC: Department of Education.

Kennedy, M. (1998). *Form and substance in inservice teacher education* (Research monograph 13). Madison, WI: University of Wisconsin-Madison.

Loucks-Horsley, S., Hewson, P. W., Love, N., & Stiles, K. E. (1998). *Designing professional development for teachers of science and mathematics*. Thousand Oaks, CA: Corwin Press.

Simon, M. A. (1997). Developing new models of mathematics teaching: An imperative for research on mathematics teacher development. In E. Fennema & B. S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 55-86). Mahwah, New Jersey: Erlbaum.

Webb, N. L., Heck, D. J., & Tate, W. F. (1996). The urban mathematics collaborative project: A study of teacher, community, and reform. In S. A. Raizen & E. D. Britton (Eds.), *Bold ventures: Case studies of U.S. innovations in mathematics education* (Vol. 3, pp. 246-360). Boston: Kluwer.

TOWARD A COHERENT NETWORK OF LEARNING GOALS FOR PRE-K-GRADE 5

Karen C. Fuson
Northwestern University
fuson@northwestern.edu

This theoretical paper seeks to initiate a conversation within the field about 1 or 2 or 3 paths through the Pre-K to Grade 5 curriculum that will permit widespread attainment of the NCTM Standards 2000. Presently, the fragmentation of grade level goals across states and districts greatly impedes coherent progress toward reform. The U.S. "mile wide and inch deep" curriculum (McKnight & Schmidt, 1998) with its large amount of review in each year is one product of such fragmentation. Textbooks that wish to reach nationwide markets must include all topics for a grade level that are in the goals of the most populous states. The inclusion of so many topics does not permit sufficient time on core grade-level goals for all students to reach understanding. Most other nations have national grade-level goals that are focused on core concepts at each grade. Subsequent grades build on rather than repeat earlier work. Mathematics educators need to begin a conversation about what such a sequence of core grade-level goals, or standards, might be appropriate for this country. The presentation will outline some elements of such a proposal in order to initiate such a conversation.

I, of course, am keenly aware of the risks of national standards. However, equity issues and a deep experience with the costs of the present fragmentation convince me that any increased coherence will be beneficial. Diverse learners need more time on coherent curricular chunks if they are to master the concepts and skills of those chunks, but they can master them (e.g., Fuson, Smith, & Lo Cicero, 1997; Knapp, 1995). Learning mathematical ideas with understanding requires more time than does skill-only teaching, and it requires building connections. Present fragmented curricula do not support such connections within or across grades. For example, in many textbooks fractions are not related to division. Many teachers do not have sufficient understandings to make these connections for students. Core grade-level topics that are more coherent across grades can support learning and teaching with understanding. These goals would not limit what children would or could learn. They would simply be the baseline mastery goals for all that would ensure that teachers and schools were more ambitious than at present for children living in poverty and for less-advanced children in all classrooms.

My efforts to begin to draft a proposal for ambitious but attainable grade-level standards build on many years of carrying out research on how children learn and then designing and testing ways to help all children learn in effective ways. All of this work also draws on the extensive research of others from around the world and of curricular designs and textbooks from around the world, especially East Asian, European, and Russian materials. More recently I have been engaged in a sustained 8-year project to design and test with urban and suburban students curricular materials focused on core grade-level goals. These materials were first developed in English-speaking and Spanish-speaking urban classrooms and were then continually adapted to work well in upper-middle-class suburban schools and in schools ranging between these two extremes. These experiences revealed much about the ecologies of different schools and school systems and the affordances and constraints of these teaching-learning settings. Involvement in the national and state policy and political dialogues concerning reform in this country and abroad has added another layer of design issues to this already complex task. Finally, domain analyses of relationships among mathematical constructs has been a continual thread in these efforts.

A new more coherent view of goals, standards, and testing for Pre-K through Grade 5 is necessary to achieve the new more complex goals of mathematics learning and teaching necessary for the 21st century. These goals, standards, and testing require focused concentrated ambitious grade-level topics that build coherently across grades. Some children may need additional learning time and support to achieve mastery goals. Having ambitious goals but providing extra support is more equitable than the present practice of allowing children to move on with quite different amounts of learning and falling increasingly (and eventually hopelessly) behind. To initiate a conversation in the field about what such mastery standards might look like, an overview of 12 conceptual chunks within the PreK-Grade 5 curriculum will be advanced in the presentation

References

Fuson, K. C., Smith, S. T., & Lo Cicero, A. (1997). Supporting Latino first graders' ten-structured thinking in urban classrooms. *Journal for Research in Mathematics Education, 28*, 738-760.

Knapp, M. S. (1995). *Teaching for meaning in high-poverty classrooms*. New York: Teachers College Press.

McKnight, C. C., & Schmidt, W. H. (1998). Facing facts in U.S. science and mathematics education: Where we stand, where we want to go. *Journal of Science Education and Technology*, *7* (1), 57-76.

CRITERION REFERENCED TESTING: AN AGENT OF CHANGE IN MATHEMATICS EDUCATION

Jean J. McGehee
University of Central Arkansas
jeanm@mail.uca.edu

Linda K. Griffith
University of Central Arkansas
lindag@mail.uca.edu

The national scene for school evaluation and assessment has evolved from standardized norm-referenced tests and tests of basic skills to more comprehensive assessment plans that also include criterion-referenced tests. The large-scale assessments communicate what is important for students to know and strongly influence what they are taught. Alignment of assessment with curriculum standards has potential as a positive agent of change and an effective tool in professional development. A broad question for the mathematics educator is how this opportunity to use assessment as a change agent will produce meaningful changes in teacher practice and student learning as opposed to "teach to the test" strategies and superficial improvements in scores.

This study investigates the effectiveness of a professional development model based on coaching (Joyce & Showers, 1988; Weathersby, 2000) and alignment of the intended (written), the taught, and the tested curricula (Glatthorn, 1999; Hayes Jacobs, 1997). Project participants are 38 teachers in grades 3-8 from two of the five school districts in a large population center. This report summarizes the first year of the three-year project and makes specific references to one sixth grade teacher as an example.

In June 2000, the teachers began the curriculum alignment process. The tested curriculum is based on the state criterion-referenced tests given at the Benchmarks of grades 4, 6 and 8. Each Benchmark exam consists of 40 multiple-choice items in which computation is contextually based and 5 open-response rubric scored items. Open-response items account for 50% of the raw score. All items are categorized in the strands of the state framework: Number, Property, and Operations; Geometry and Spatial Sense; Measurement; Patterns, Functions, and Algebra; Data, Probability, and Statistics. A powerful teacher-made wall display of released test items arranged by strand and ordered from worst to best scores sparked an insightful discussion about the taught curriculum and student learning. It motivated teachers to develop a vertically articulated curriculum. In August, we facilitated grade level groups as they made detailed plans for the general curriculum outlines, found Standards-based material to supplement their texts, and planned alternative assessments.

During the school year we collected data from teacher-reported reflections, observations, sample student work and teacher/researcher interactions in school visits consisting of planning, model teaching, and reflection sessions. The data shows the ways in which teachers have made initial changes to move from listing generalized topics to articulating conceptually-focused major teaching objectives and from using a static, traditional model based on covering a textbook to a dynamic model based on student learning. Our sixth grade teacher originally tried to cover all objectives tested up to eighth grade. She now focuses on rational number concepts and concrete pre-algebra concepts. At the beginning of the year, she *told* her students answers to critical thinking questions; at the end of the year, she *listened* more to students and allowed more than one answer. She is still very technical in presentations; the model teaches show her where stronger content knowledge opens the lesson.

The project also tracks progress in student achievement. The research team designed pre/post tests for each grade level by correlating items to released Benchmark items and modifying them as appropriate for each grade level. With pretests given in September and posttests in April, early analysis shows significant gains in paired scores (t-value of 8.04 at $p < .01$). Using the Benchmark's four categories: Below Basic, Basic, Proficient, and Advanced, we also have noted the movement from below Proficient categories since the beginning of the year. A typical change in 6th grade shows 83% to 22% BB; 11% to 44% B; 6% to 17% P; 0% to 11% A.

This summer will produce a more complete analysis of the pre/post test data, and the actual Benchmark scores will become available for these schools. Also we will have a new set of teacher curriculum documents to compare with last year's beginning efforts to change.

References

Glatthorn, A. (1999). Curriculum alignment revisited. *Journal of Curriculum and Supervision, 15* (1), 26-34.

Hayes Jacobs, H. (1997). *Mapping the big picture: Integrating curriculum and assessment K-12.* Alexandria, VA: Association for Curriculum Supervision and Development.

Joyce, B. & Showers, B. (1988). *Student achievement through staff development.* White Plains, NY: Longman.

Weathersby, J., & Harkreader, S. (2000). Staff development and the link to student achievement. *Catalyst for Change, 20* (3), 9-12.

TEACHERS' CONCEPTIONS OF "DOING MATHEMATICS"

Neil Portnoy
California State University, Chico
nportnoy@csuchico.edu

Karen Graham
University of New Hampshire
kjgraham@cisunix.unh.edu

Teachers' beliefs about mathematics affect their instructional strategies (Thompson, 1992). Many secondary school mathematics teachers have limited experience in mathematical inquiry. In schooling through their undergraduate years, teachers have had little, if any, opportunity to move beyond memorizing facts and procedures into the realm of creating mathematics as is typical in the profession (Lampert, 1990). In addition, there is a relationship between "content conceptions and teaching practice" (Lloyd and Wilson, 1998, p. 249). For example, when teachers believe that mathematics is a set of rules and procedures, instruction in their classrooms can reflect this belief.

This study attempts to describe how a group of practicing secondary school teachers interprets the meaning of "doing mathematics" in the context of a master's level course in geometry. The seven-week summer course involved 16 secondary school teachers in the study of Euclidean and spherical geometry. Pedagogical strategies in the course were purposefully designed to encourage *doing mathematics* through guided inquiry and writing assignments. Three of the sixteen teachers volunteered to participate in the study. Dialogue journals completed by the participants were the main source of data for the study.

The meaning of *doing mathematics* for the teachers in this study was different from the individual meanings they brought into the course. The teachers reported that *doing mathematics* involved much more freedom as well as frustration. Issues arose for the teachers as learners involved in the variety of pedagogical strategies employed in the course. Frustration emerged with the lack of lecture, the cooperative investigations, and the need to write mathematics. The teachers' mathematical assumptions were challenged. They found that a change of axiom or postulate could drastically change subsequent mathematical statements or consequences. Also, teachers noted a change in their perspective about mathematics, from correctness being absolute to correctness being relative.

References

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27* (1), 29-63.

Lloyd, G. M., & Wilson, M. S. (1998). Supporting innovation: The impact of a teacher's conceptions of functions on his implementation of a reform curriculum. *Journal for Research in Mathematics Education*, *29* (3), 248-274.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

WHAT TO TEACH ABOUT VOLUME? SOME MEXICAN PRIMARY TEACHERS' BELIEFS

Mariana Saiz
Universidad Pedagógica Nacional,
México
msaiz@upn.mx

Olimpia Figueras
Cinvestav del IPN, Mexico
dfiguera@conacyt.mx

This document deals about the partial results obtained from a more general research project which is to point out teachers' beliefs in regard to the concept of volume and the way in which it is being taught in primary schools. For the procedure of this research, a starting theoretical Local Model was designed (see Saiz & Figueras, 1999). The teaching models component include among other things, the results of an analysis of the plans and programs, cost free textbooks and other bibliographic material that integrate the national curriculum and the bibliographic sets that children and teachers receive from the Ministry of Education (SEP) and which use is compulsory in Mexican elementary schools. In this national curriculum a didactic treatment of volume has been proposed which gives special importance to the aspects of qualitative type.

As it is well known, to perceive ideas and beliefs of teachers is not an easy task. If they are directly asked, you run the risk of obtaining the repetition of a discourse, in which, their activity in the classrooms is not always reflected. Observing teachers in front of their groups could reflect information about how they teach volume. Nevertheless, since they know they are being observed, that also affects their behavior in the classroom. Also, by these means it is impossible to obtain any additional information concerning their mental object volume, which characterization is a central part of the research that is being carried out. Due to the above mentioned, a workshop course was designed in which teachers could work and freely discuss with their peers problems and situations related with volume and teaching techniques. By these means, the observer researcher could gather information to comply episodes for a qualitative analysis.

Using this methodology, information was complied, concerning: a) Identification of lessons that deal with the study of volume in the text books. b) Items in the didactic materials that could be useful when teaching this concept. c) Difficulties teachers think children might be facing when they learn this concept and have to carry out the concerning activities. d) The way teachers solve problems included in the materials of the SEP.

Among the most important outcomes derived from the information described in the above paragraphs the following are accounted: 1) In spite of the fact that the curricula reform dates from 1993, teachers, in general, are still not acquainted with the bibliographic materials that have been worked up to back their classroom activities. 2) Verbal arithmetic problems concerning the concept of volume are identified, jus because expressions as liters and cubic centimeters appear in statements. 3) In contrast with the previous point, qualitative type tasks, like filling of boxes or pouring of liquids are associated to capacity and are not considered as part of the teaching of volume. 4) Even though qualitative type activities are thought as being valuable in teaching volume, a tendency is appreciated to over-estimate the use of formulas, and consequently, calculus and volume problems. 5) There exists a tendency to write numeric quantities to solve or think about situations of the qualitative type and one single calculation leads them to generalize a result.

Reference

Saiz, M., & Figueras, O. (1999). A conceptual network for the teaching-learning proceses of the concept of volume. *Proceedings of the Twenty-First Annual Meeting of the Psychology of Mathematics-North America* (pp. 436-442). Cuernavaca, Morelos, Mexico. Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

THE PROFESSIONAL ROAD LESS TRAVELED

Bridget Arvold
University of Illinois at Urbana-Champaign
arvold@uiuc.edu

As teachers attempt to meet the needs of underachieving students, the alignment of their embedded traditions and popular curricular materials does not always produce the expected results. The findings of a case study of the intervention efforts of two teachers exposed a professional development road less traveled. The teachers took it upon themselves to initiate and facilitate an intervention designed to help students who as a result of failing the first semester of an algebra course had been removed from their algebra classes with no place to go. The teachers' decision to select curricular materials that reinforced basic skill practice reflected their belief that proficiency in basic skills is a prerequisite to success in algebra.

The teachers' primary goal was to provide students with ample opportunity to practice basic skills in order to equip them for success when they repeated algebra the following year. Another goal was to create a relaxed atmosphere in which students could enjoy mathematics, yet as the teachers partnered to assist 15-18 students in a semester long intervention using computer generated worksheets, they were forced to make decisions that undermined their efforts. In addition to the teachers' beliefs about mathematics, student needs, and student learning, the goals and expectations that were part of their embedded traditions served to maintain an environment in which most students continued to experience frustration doing mathematics and a rather negative view of mathematics classes.

In alignment with the aims of PME, the research group analysis of data that were collected through classroom observations, teacher interviews, and joint discussions furthered the understanding of the psychological aspects of teaching and learning mathematics. The findings imply that the isolated efforts and unidirectional focus of well-intentioned teachers do little to promote the success of underachieving students but can serve as a springboard for professional development.

BELIEVE IT OR NOT

Jason K. Belnap
University of Arizona
belnap@math.arizona.edu

Abstract: This preliminary study focused on teaching assistants in the department of mathematics at the University of Arizona. The study had the following three purposes: first, to determine how aligned their teaching beliefs were with the NCTM standards; second, to see how their practices as teachers compared with those used in the classrooms in which they were students; and third, to determine what things they felt influenced them to change their teaching practices (if anything).

Over the past couple of decades, it has become evident that the views and beliefs that teachers hold impacts their teaching preparations and practices. Researchers have looked into ways of encouraging teachers to reevaluate their teaching practices in an effort to reform the teaching of mathematics—most of this research has focused on preservice and inservice teachers. Recently, research has begun to look more into teaching at the collegiate level. A recent study, by Victoria Boller LaBerge, Linda R. Sons, and Alan Zollman, focused its attention on collegiate faculty (1999). Their study revealed that although awareness of reform documents, like NCTM standards, was low among college and university faculty, the beliefs and views were fairly consistent with the principles of these documents (LaBerge et al., 1999). Because such faculty usually do not receive the pedagogical training that preservice teachers receive, the question remains: what influences shape the pedagogical beliefs of collegiate and university faculty?

This study begins to address this question by focussing on teaching assistants (TA's), since that is where most of collegiate faculty begin their teaching experience. An anonymous survey was a modification of the survey tasks used by LaBerge et al., (1999). Surveys were distributed to all TA's in the mathematics department. The survey was comprised of three main parts: the first dealt with the mathematics and teaching beliefs of the TA, the second determined what teaching practices the TA had encountered as a mathematics student (and how often), and the third dealt with what teaching methods the TA used in the classroom (and how often). The survey also included an opportunity for TA's to report what factors influenced them to change their teaching practices.

Responses to the survey were grouped for comparison in a variety of ways, including comparison by teaching experience and comparison by area of study—math, applied math, and math education. Similar to the results of the study by LaBerge et al., (1999), this study showed that the views of TA's overall were fairly consistent with the NCTM views. Further, analysis of teaching practice showed significant difference between the teaching practices of TA's and the pedagogy they experienced as mathematics students. In addition to this, comments by TA's indicated the existence of influences creating inconsistencies between their teaching beliefs and practices, similar to those given by faculty (LaBerge et al., 1999) and similar to those encountered in a beliefs study by Anne M. Raymond (1997).

This poster reports the results of this preliminary study, the details of its findings, and its implications for upcoming research. It also discusses implications for TA training and reports on the factors that TA's listed as factors that influenced them to change their teaching practices.

References

LaBerge, V. B., Sons, L. R., & Zollman, A. (1999). Awareness of NCTM standards, mathematics beliefs, and classroom practices of mathematics faculty at the collegiate level. *Focus on Learning Problems in Mathematics*, *21* (3), 30-47.

Raymond, A. M. (1997). Inconsistency between a beginning elementary school teacher's mathematics beliefs and teaching practice. *Journal for Research in Mathematics Education*, *28* (5), 550-575.

AFFECTIVE CHANGES TIED TO A SECONDARY TEACHER'S REFLECTIONS ON DISCOURSE

M. Lynn Breyfogle
Bucknell University
mbreyfog@bucknell.edu

The purpose of this study was to investigate the ways in which participation in a teacher-researcher collaborative intervention focused on mathematical discourse affected the nature of the teacher's reflection about and vision of mathematical discourse. This collaboration, based on a new paradigm for professional development (Loucks-Horsely, Hewson, Love, & Stiles, 1998; Stein, Smith, Henningsen, & Silver, 2000), focused on the mathematical discourse in a teacher's classroom. An experienced secondary school mathematics teacher, Donald (a pseudonym), was chosen for his past experience in NSF-funded curriculum workshops and current teaching utilizing NSF-funded curricular materials, was a participant in a teacher-researcher collaboration. Video-taped episodes of Donald's classroom, along with a researcher developed set of questions, were the primary instruments used to encourage this focused reflection.

A series of 12 focused reflection sessions and three interviews took place over the Fall 2000 semester. Each of these sessions occurred the day following a video-taped observation, and was audio-taped, transcribed, and coded. The initial coding scheme was based upon the cognitive stages as described by Shaw and Jakubowski (1991), and research on aspects of promoting meaningful mathematical discourse (e.g. Pimm, 1987; Yackel & Cobb, 1996).

There were a variety of changes that occurred in Donald's reflections, the most salient of these was a shift toward becoming more reflective, which took place in conjunction with a series of affective phases. At first, he was unsettled or perturbed by what he witnessed on the video tapes. For example, in an early reflection session he remarked, "Maybe I was a little harsh or I was a little [pause] they did have a good response and it wasn't the one I was looking for. And I locked myself into that one response I was looking for so much that I didn't give enough credit to what was said." (9/22/00) This was one such example of references he made to not keeping an open mind and listening to students' mathematical thinking.

As a result of the many perturbations he experienced, he entered a phase of confusion and frustration. For example, in the mid-interview he said, "Just learn how to keep my mind open to many possibilities while still being aware of what I want, what needs to get discussed or needs to get spoken or needs to be understood. Have that in the back but keep, learn how to keep, an open mind so that we can come to that in many different ways where I'm not looking for it or to hear it in a particular fashion." Later in the same interview he goes on to say, "So it's hard, it's dangerous to keep an open mind. It's scary and it's unpredictable. You're just dangling, sometimes you feel like you're just dangling there." (11/02/00) It became evident that he wanted to change his ways, but was struggling with various tensions, including his inability to keep an open mind in the face of obstacles such as time and his own agenda. Toward the end of the collaboration, Donald seemed to be entering a third phase, which might be termed *contented reflection*. Donald was less agitated and more thoughtful about what was happening in the classroom. He was no longer defensive and viewed the classroom episodes almost as an outside observer.

The identification of the affective phases of change has practical implications for professional development. Knowing this might enable teacher educators to empathize with and encourage teachers in different ways as they travel this emotion-filled path on their way to becoming reflective practitioners. For example, when working with teachers who are not familiar with certain teaching philosophies and practices, it would behoove them to use activities like video-tape comparisons to perturb teacher thinking. Whereas, when teachers are already perturbed and aware of their incongruous teaching practices but frustrated or confused, activities that provide guidance without evaluation would be more appropriate. This also raises questions for the research community to investigate the relationship between the cognitive readiness and affective stages in order to understand better how to assist teachers in changing their practices.

References

Loucks-Horsley, S., Hewson, P. W., Love, N., & Stiles, K. E. (1998). *Designing professional development for teachers of science and mathematics.* Thousand Oaks, CA: The National Institute for Science Education.

Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. New York: Routledge.

Shaw, K. L., & Jakubowski, E. H. (1991). Teachers changing for changing times. *Focus on Learning Problems in Mathematics, 13* (4), 13-20.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development.* New York: Teachers College Press.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27 (4), 458-477.

TEACHERS' BELIEFS AND EXPECTATIONS: INFLUENCES ON MATHEMATICAL LEARNING

Lecretia A. Buckley
University of Illinois at Urbana-Champaign
lawilson@uiuc.edu

This research study examines two high school mathematics teachers' efforts to address student failure in mathematics classes. The teachers embrace a low-level skills-based curriculum in an attempt to fortify students' proficiency in pre-algebra and algebra. The teachers' beliefs about the nature of mathematics were integral to the selection of this curriculum and to the goals and standards the teachers set for the course and student performance.

Data were collected in the form of field notes from weekly classroom observations, audio-taped semi-structured interviews with the teachers, and journaling. Data were also collected from other department activities such as department meetings. I transcribed the audio-taped interviews and class segments, and I analyzed them line-by-line. I developed a coding system to analyze data, triangulated across sources, and participated in a weekly research meeting in which data were presented and analyzed. The research team provided alternative analytic perspectives on initial inferences and subsequent areas of focus. Prevalent themes that I identified while implementing constant comparison analysis (Strauss, 1987) guided the focus of subsequent observations. Emergent themes included a procedural view of mathematics (Hiebert & Wearne, 1986), low student expectations, and skill-acquisition, focused goals for students' mathematical learning.

This study demonstrates the connection between teachers' beliefs about the nature of mathematics and how mathematics is learned and the curriculum they implemented in their classes. Selection and implementation of curriculum were further influenced by the teachers' low expectations for the students. Students' poor performance in the class, however, appeared to have opened a door. The teachers reconsidered the efficacy of a low-level skills-based curriculum in meeting the needs of students who have failed high school mathematics courses. The teachers experienced frustration with the curriculum and the students' poor performance. These experiences have called into question the teachers beliefs about how mathematics is learned. Findings from this study suggest that efforts to enhance mathematics instruction must take into account teachers' beliefs about the nature of mathematics and how mathematics is learned as well as teachers' expectations of students.

References

Hiebert, J., & Wearne, D. (1986). Procedures over concepts: The acquisition of decimal number knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 199-223). Hillsdale, NJ: Erlbaum.

Strauss, A. L. (1987). *Qualitative analysis for social scientists.* Cambridge, UK: Cambridge University Press.

Teacher Education

CLASSROOM COACHING: AN EMERGING METHOD OF PROFESSIONAL DEVELOPMENT*

Joanne Rossi Becker
San Jose State University
Becker@mathcs.sjsu.edu

Purposes

The main purpose of this ongoing project is to investigate the efficacy of classroom coaching in improving instruction in elementary mathematics classrooms. The coaches involved in this study have been participants in a state-funded professional development program for a number of years. That program includes three major aspects:

- an intensive 3-week summer institute focusing on mathematics content, pedagogical content knowledge, and leadership skills;
- summer lab schools for children organized and run by participants, who themselves, with staff support, provide professional development for team teachers who teach the classes;
- comprehensive follow-up activities including workshops with leading national and international mathematics educators.

Part of the leadership development strand has included training in classroom coaching, using a peer coaching model. With private foundation funding, the coaches in this study have been released from classroom duties to be full-time coaches in mathematics in their districts. This ongoing study has been designed to ascertain the impact these coaches are having in the classrooms in which they work, and indirectly, the impact of the professional development in which they have participated. In particular, the study was designed to document how coaches worked, how they interpreted their roles, and how they affected the teachers with whom they worked.

Background

In the last edition of the *Handbook of Research on Teaching*, the chapter on mathematics education (Romberg & Carpenter, 1986) hardly mentions research on in-service teacher education. As Grouws pointed out (1988), and as is still the case, there is little information available about the overall design features of in-service education programs which maximize changes in teacher beliefs and ultimately classroom practices. Grouws called for studies that focus on the impact of various features of in-service education on classroom practice. And the *Handbook of Research on Teacher Education* (Sikula, 1996) does not even include coaching in the index of the volume.

The meager research reported in mathematics about classroom coaching as a means of professional development predicts considerable promise for this technique. Becker and Pence (1999a, b) identified classroom coaching as the most important component of a professional development program for secondary teachers. In these studies, the coaching was done by the authors, who also designed and implemented the whole professional development program. Coaching that was intended as a non-evaluative mechanism for identifying the impact of the professional development itself became the most important aspect of the in-service for participant teachers. Those studies concluded that coaching might itself be a worthy, though time-consuming and expensive, planned component of professional development.

There are a number of models of coaching extant within the educational community. For example, Evered and Selman (1989) define coaching as conveying a person from where he or she is to where he or she wants to be. The metaphor of an old stagecoach communicates this perspective. In this model the teacher is considered a thoughtful decision-maker who, through support and collaboration, can further develop her/his ability to reflect on and improve instruction. A second model is content-focused coaching (Institute for Learning, 1999), which focuses on the content of the lesson in relationship to issues at the core of the teaching-learning process. From my reading and viewing of videotapes in which content-focused coaching is used, it appears to be a bit more directive, in that the coach may use the pre-conference to "teach" content to a teacher who seems to lack content knowledge related to the lesson, may interrupt the lesson and even take it over, and may provide her/his own solutions during pre- or post-conferences. However, both models have the following characteristics in the ideal: a pre-conference to discuss the lesson and its goals and the teacher's focus for the observation; an observation of the lesson in which the coach records as much data as possible; and a post-conference to debrief. Coaching might also include demonstration lessons, co-teaching, or joint lesson planning. In this study I searched for aspects of both models during observations of coaching sessions. In addition, I focused on interactional moves of the coach, such as listening skills, strategic questions, and use of feedback, as well as content specific moves, such as clarifying the goals of the lesson, anticipating and diagnosing difficulties, or reflecting on students' attainment of lesson goals.

Mode of Inquiry

This was a qualitative study using participant observation techniques (Glaser & Strauss, 1967). Observation sessions varied depending upon what the individual coach had planned and how s/he worked with teachers. For example, one coach, Lewis, is working with two fourth grade teachers at the same school. They plan lessons together in a meeting a day or two before the lesson. Then the coach views half of the lesson with one teacher and half with the other, and holds a joint post-conference with both teachers during lunch. Because of scheduling and prohibitive distances involved, in this case I meet with the coach before the lessons to determine what was discussed in the pre-conference. Then we jointly observe the classes, interacting with the children as they work on activities. I observe the post-conference, providing my input when asked or when it adds to discussion of, for example, student work. In this case I am more on the observer end of the participant-observer continuum. In another case, I spend the whole morning at a school with the coach and the two fifth grade teachers with whom she is working. We have a brief pre-conference with each separately, one before school, the other during a break, identifying areas of focus for the observation. We observe the whole mathematics lesson of each teacher, with the coach making notes that she hands to them during the post-conference. The post-conference usually takes place with both teachers during lunch. There are other variations as other coaches fit in time to meet their coachees in teachers' busy schedules, but space limitations preclude discussing these.

All notes from observations and interviews with teachers and coaches were typed and expanded, with patterns and questions to investigate further identified as work progressed (Glaser & Strauss, 1967). The aim is to identify patterns of coaching work and its impact on teachers, and subsequently, to ascertain how participation in the professional development program has affected the coaches and their work. Data include field notes, interview transcripts, and artifacts from the classrooms such as assessments.

Data Sources

The study is ongoing during the 2000-2001 academic year. Six coaches are being observed, each with at least one teacher, for a total of 12 teachers and 6 coaches. Due to space limitations this paper will discuss the cases of three of the coaches, one male and two female.

Lewis is a former middle school teacher who has been with the professional development project for three years. This is his third year as a coach. Lewis is a European American male who has been teaching 11 years. He works in a small district of eight K-8 schools in northern California. Lewis is working with two fourth grade teachers, Sally and Susan who were using TERC units (Clements, Battista, Akers, Woolley, Sarama, & McMillen, 1998).

Anita is a European American primary school teacher who has extensive professional development, has been teaching 13 years, and has been coaching for three years. However, for the first two years Anita coached only half time, and she described that work as demonstration teaching, not really coaching. She now coaches primary teachers in her K-12 district full time.

Dana is a European American female who taught upper elementary school. She works as a coach in a K-12 district in northern California. Dana has been coaching 3 years, and also has over 20 years of teaching experience. The teachers with whom she works use a variety of materials, with TERC (Clements et al., 1998) forming the core of the content I observed Dana coaching a fourth grade male teacher, Neil, and a fifth grade female teacher, Melanie, both with less than four years of experience.

Results

The coaches were classified into three types related to patterns of how they interacted with their coachees: coach as collaborator; coach as model; coach as director. These three models range from least to most directive in coaching style. However, I would classify all as falling into the category of the stewardship, internal commitment, and learning model (Hargrove, 1995) which emphasized helping people set goals for self-directed learning and providing feedback as a way to create a shared vision of improvement, in this case, in mathematics instruction.

The Case of Lewis

Lewis is an example of what I am calling a "coach as collaborator." He endeavors to be one of the group of three who are working on a lesson together. Thus the post-conferences tend to be about the structure of the lesson rather than specific as to how each teacher implemented the planned lesson. In fact, by viewing half of each lesson for Sally and Susan, Lewis cannot really ascertain how the second teacher developed the core of the lesson [he does switch order each visit]. Lewis does not keep written notes from the lessons, and does not give the teachers written feedback. However, he works closely with children, frequently asking questions, and seems to have a good sense of what they are understanding. For example, in one lesson the teachers were developing multiplication facts greater than 10; they wanted children to work them out without use of the standard algorithm. In Sally's class, as students shared their methods orally, it was clear that this was difficult for those who knew the algorithm. One girl even verbalized the whole standard algorithm by visualizing it in her head (the problem was 12x6). Both Sally and Susan noted in the post-conference that students seemed wedded to an algorithm. Lewis suggested that they ask children to find more than one way to do 12x6 to get them beyond an algorithm. Sally and Susan liked this suggestion, and in later observations, both were observed asking for more than one way in other contexts.

Although much of his work is collaborative, it is clear that Lewis has a slightly different role from that of the teachers. He provides performance assessment practice items for teachers' use, scores them for the teachers, and does the class presentations of the problems and the rubric scoring to help children get familiar with that type of testing. Although Lewis does not provide feedback specific to how a teacher organized the lesson, he does concentrate on what students seemed to understand. By being active in the classroom, watching and questioning students, he gleans considerable information about student understanding to share with teachers. From Lewis' perspective, perhaps the most important part of his role is encouraging and facilitating the team planning and reflection that are occurring. Without his presence as coach, this level of collaboration would not be taking place. The planning time forces each teacher to think through the lesson, its goals, and how they plan to implement them beforehand. Because they are working as a team in this way, they have a mutual responsibility for the lesson and its pros and cons. The teaming that Lewis has encouraged has extended to consistent planning throughout the week, even when he is not visiting. Thus Lewis' model encourages the elimination of the isolation many teachers feel by working alone in their own classrooms.

On the other hand, lack of specific feedback to each teacher precludes Lewis from the possibility of influencing each teachers' teaching strategies. The lesson may be the same but may be implemented in quite different ways. Thus Sally seems to have a need for full control at all times in her classroom, so that she leads students to do problems exactly how she wants them done. This style mitigates against students developing multiple methods of solution, such as sought for 12x6. Susan's more open style generates more ways of solving problems. Peer visits or feedback on pedagogy might provide both with more ideas on instructional strategies that would lead to further mutual professional growth.

The Case of Anita

Anita exemplifies "coach as model." Anita has developed a special way of working with teachers new to her as coachees. First she presents several model lessons, leaving the teachers materials and ideas on how to continue that work until her next visit. Then she moves into modeling peer coaching, in which she is the teacher and the classroom teachers act as coach for her. Then she facilitates, by covering their mathematics classes, teachers serving as peer coaches for each other.

For example, I observed two second grade lessons that Anita did several weeks apart. In the first lesson, Anita had students investigate growing patterns. She first modeled finding the first five steps in a geometric pattern on the overhead projector, engaging the children in finding the pattern and describing how it was growing. Next children were given pre-made patterns, first three steps, to copy with cubes, then extend to the fourth and fifth steps. Patterns ranged in difficulty and were exchanged as children completed them. Anita did this lesson in both classes for each teacher, discussing with the teachers the goals for the lesson and what individual children seemed to be understanding. Both teachers were impressed with what their students were able to do in this lesson, although it seemed that they had not done any work with patterning before this. Then Anita left the materials behind asking teachers to give children more practice in finding growing patterns and extending them. At the next lesson, a two-day one several weeks later, children had to complete five steps of a pattern with cubes, then color in the first five steps on inch grid paper, then make a poster of their pattern and a description of how it grew. This lesson ramped up the concept as children also had to fill in a table showing how many cubes were used at each step; this was also modeled with the whole group. Interestingly, Anita adjusted her instruction of the second lesson in the second class, which immediately followed the first. One aspect of the lesson, looking for patterns in the 100s chart, confused the children. This difficulty and her adjustment provided interesting topics for discussion with the teachers after the lesson. Contrasting the children's interactions in the first set of lessons with those in the last, it was obvious that the classroom teachers had worked with patterns in the interim, as the children seemed much more comfortable copying a pattern concretely and semi-concretely, expressing the patterns they saw in words, and creating the next steps in the patterns.

In another lesson, Anita was acting as the coachee and the two teachers observed her teaching their classes, taking notes on areas of the lesson of interest to Anita: questioning; and checking for understanding via questions, asking for clarification, and asking students to repeat back directions. During the debriefing after both lessons, teachers were able to share concrete data relative to these focus areas of Anita's, which helped generate further discussion about students' understanding. This experience also served to lessen any anxiety the teachers might have had about getting feedback from Anita, as she volunteered to be first to receive feedback.

Thus Anita is acting as a model on several dimensions. She presents exemplary lessons and is always prepared with materials and manipulatives needed for the lesson. Her lessons always begin with a whole-group activity in which she models what she would like the children to do. She clearly does long-range planning as teachers can infer from the work she leaves for them to continue. She wants teachers to do peer visits, so she first models that to help them understand the process and feel comfortable with it. That is, she takes the first risks. Anita has become a model of coaching among peers.

The Case of Dana

Dana is an example of what I am calling "coach as director." She always begins the debriefing session by asking the teacher how s/he thought the lesson went. These two coachees usually give a rather general response to this opening. Then Dana proceeds to guide the teacher to reflect very specifically on the way the content was presented in the lesson and to generate alternate strategies and follow-up activities to enhance student understanding. She takes very detailed notes on the lesson, and, considering she does not have time between the lesson and the debriefing to review her notes, recalls very specific aspects of the lesson on which to focus. She will challenge the teacher mathematically by asking probing questions about the content and is not timid about pointing out errors or omissions related to content.

For example, in one lesson with Neil, the class was discussing a performance assessment item on a practice test:

> "Judy and her two friends buy a bar of candy. The candy bar can be divided into sixteen square pieces. Alice eats four pieces and Kerry eats eight pieces.

> Judy eats the other pieces herself. What fraction of the candy bar does each of the three girls eat? Use the diagram to show how you figured it out."

Included was a figure of a square divided into 16 equal squares.

In the discussion of the problem that ensued, some students expressed the answer in sixteenths, while others used fourths. In trying to explain the equivalence, the teacher wrote: 4/4 x 4 = 16/16 In the debriefing, Dana began with the start of the class and discussed various aspects of the lesson before this incident. Then she showed him what he had written and asked for his reaction. Neil began a discussion of why 4/4 and 16/16 were the same because each gave 100% or 1 whole. He commented that he had not explained it well and asked if he were making sense. It seemed that he was trying to clarify that 4/4 of a whole small unit, e.g. a pizza, was not the same as 16/16 of a large pizza because the unit was different. Dana asked if he were trying to build equivalence at that point in the lesson, pointing out that the problem involved the same size model of one candy bar. Neil commented that he had four (pointing) columns to get 16/16. It seemed that Neil was confused between the small squares (1/16 of the whole candy bar) and a column of 4 small squares (4/16 of the candy bar) and thinking that 4 of the 1/4 pieces (or 4/16) should equal the whole candy bar. But he wrote 4/4 x 4 instead of 1/4 x 4 because he wanted the answer in sixteenths. Dana proceeded to help Neil, through a series of questions, to sort out the difficulty. Neil was very open to discussing this and clarifying his thinking.

Dana was quite astute in pinpointing the critical areas of possible misunderstanding in a lesson. For example, in another lesson observed, students were dividing various figures in half. Nearly all the students were striving to find two congruent pieces (not always possible given the irregular shapes). In the debriefing, Dana pointed out that the salient feature to ensure one half was area and that the two halves did not have to be congruent. In the final interview, Neil picked this incident out as an example of how the coaching had helped him. He felt he would not have analyzed that activity in that way without Dana's help in focusing on what was important mathematically. Thus in addition to teaching techniques that helped expand his repertoire of strategies, Neil felt he learned a great deal mathematically as well.

These mathematical issues, I think, would not have arisen in the debriefing if Dana had not raised them. That is, the teacher was not experienced enough to understand the nuances that Dana gleaned from the lesson. Thus her directness in guiding the teacher to new and deeper understandings was a critical component of her success as a coach. However, Dana was not confident that this level of intervention was ideal; she asked me several times for feedback on her coaching and whether she should intervene as she did. Note that, during the year of this study, the coaches in the sample attended a full-day workshop with Lucy West on content-based coaching. Dana and I discussed this model; while she felt that what we had seen on video had been too direct, with the coach interrupting the class at one point, she did feel the obligation to initiate discussion of content issues even if not broached by the teacher.

Discussion

This research identified three unique styles of coaching. These could be considered to range on a continuum from less to more directive. In this paper, three coaches were discussed who each modeled a style of coaching identified from the data: coach as collaborator; coach as model; and coach as director. Lewis, while quite non-directive, does raise questions regarding instruction that he tries to work through with the teachers through collaborative dialogue. However, the model Lewis uses precludes his focusing on individual teachers' instruction specifically. Lewis seems to have had a positive effect on the teachers with whom he works, however, by encouraging their own decision-making and by encouraging the collaboration that they have extended to all mathematics lessons on their own initiative. Anita uses a long-range plan of working with teachers by modeling instruction that actively involves the children in high level tasks as well as modeling the coaching process itself with her as teacher. Dana is much more of a guide to the teachers with whom she works. However, this direct guidance is accepted, at least on content issues, because of the way it is approached. By grounding comments in what the teacher did and what the students did and seemed to understand, Dana and her coachees became collaborative problem solvers in designing next steps in instruction. Of the three coaches discussed in this paper, Dana perhaps best represents the interweaving of content and pedagogy necessary for the improvement of instruction (Ball & Bass, 2000).

Although these three coaches had differing styles of working with teachers, there was one common thread through all of their work. All coaches aimed to develop a shared vision with client teachers of what a mathematics classroom should look like (Hargrove, 1995). Whether it was through cooperative lesson planning, modeling of instruction, or guiding the teacher to deeper reflection, the goal was to improve mathematics instruction by meeting individual needs. Coaches had a coherent, well-articulated conception of mathematics instruction themselves; this vision guided their work.

From the perspective of teachers who were coached, there were some salient characteristics that make coaches effective. These included: openness; fairness; non-judgmental demeanor; helpful; dependable; approachable; and, experience. Although the three coaches discussed here had different styles of coaching, all three exemplified these characteristics albeit in different ways. Coaches all perceived the coaching experience as improving their mathematics instruction. Most felt more comfortable and confident teaching mathematics now, and thought they had a more coherent view of the whole mathematics curriculum at their grade level. Teachers seemed to focus more now on the big ideas of mathematics rather than just following the textbook from page to page. They were more concerned with improving students' understanding of mathematics, and discussed more processes, such as problem solving, than skills when asked about their goals for mathematics instruction. The extra set of eyes and ears in the classroom helped teachers really focus on what their students were understanding; together with the coaches, they were able to find ways to enhance that understanding.

The cases discussed here are thought-provoking because of their contrasts, and stimulate questions that will be investigated with the rest of the sample of coaches:

> Is there a style of coaching that is most efficacious in promoting growth in teachers?
>
> Is there a range of skills, dispositions, and domains of knowledge that are needed by a coach?
>
> What is effective coaching?
>
> How does a coach develop a practice of effective coaching?
>
> Can coaching scaffold a teacher's lack of content knowledge?

Essentially I am looking for evidence that coaching is an effective form of staff development, one that supports effective implementation of reform principles (Saxe, Gearhart, & Nasir, 2001), as well as evidence that helps inform professional development of coaches themselves.

Finally, I would like to comment on my role as the observer. This varied considerably depending on the coach and the teachers with whom they were working. In all classes in which the coach was observing, we both actively participated in the lesson while children were working in groups or individually. During debriefing sessions, I mainly took notes, but occasionally I was asked my opinion of children's understanding by either the classroom teacher or the coach, or to check my notes to compare with the coach's. At their invitation, I would add my opinion. But I did not perceive my role as one of expert and neither did the coaches or teachers. Perhaps the most interesting discussions to me as a teacher educator were those with the coaches after their debriefing with the coachees. These discussions gave me further insight into the coach's intentions and future directions with specific teachers, a valuable window into the coaching process that could not be obtained by observation alone.

References

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 83-104). Westport, CT: Ablex.

Becker, J. R., & Pence, B. J. (1999a). Classroom coaching: A critical component of professional development. In F. Hitt & M. Santos (Eds.), *Proceedings of the Twenty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 771-777). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Becker, J. R., & Pence, B. J. (1999b). Classroom coaching: Creating a community of reflective practioners. In O. Zaslavsky (Ed.), *Proceedings of the Twenty-Third Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 89-96). Haifa, Israel: Technion-Israel Institute of Technology.

Clements, D. H., Battista, M. T., Akers, J., Woolley, V., Sarama, J., & McMillen, S. (1998). *Investigations in number, data, and space.* White Plains, NY: Dale Seymour.

Evered, R. D., & Selman, J. C. (1989). Coaching and the art of management. *Organizational Dynamics, 18* (9), 16-32.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research.* Chicago: Aldine.

Grouws, D. (May 1988). *Overview of the role of research on mathematics teaching and learning in improving research in classroom instruction and teacher education.* Paper presented at the First Wisconsin Symposium on Teaching and Learning Mathematics, Madison, WI.

Hargrove, R. (1995). *Masterful coaching: Extraordinary results by impacting people and the way they think and work together.* San Francisco: Josey-Bass Pfeiffer.

Institute for Learning (1999). *Content-focused coaching.* Pittsburgh: University of Pittsburgh.

Romberg, T., & Carpenter, T. (1986). Research on teaching and learning mathematics: Two disciplines of scientific inquiry. In M. Wittrock (Ed.), *Handbook of research on teaching.* New York: Macmillan.

Saxe, G. B., Gearhart, M., & Nasir, N. S. (2001). Enhancing students' understanding of mathematics: A study of three contrasting approaches to professional support. *Journal of Mathematics Teacher Education, 4,* 55-79.

Sikula, J. (1996). (Ed.). *Handbook of research on teacher education.* New York: Macmillan.

Note

*This research has been partially supported by a grant from The Robert Noyce Foundation. Views expressed are those of the author alone, and do not represent those of the Foundation.

CURRICULUM REFORMS THAT INCREASE THE MATHEMATICAL UNDERSTANDING OF PROSPECTIVE ELEMENTARY TEACHERS

Irene Bloom
Arizona State University
bloom@math.la.asu.edu

As the mathematics curriculum in the United States is being transformed from a fact and algorithm based curriculum to a concept based curriculum (NCTM, 1989, 2000), universities and colleges have been taking a closer look at their teacher preparation programs (Arizona Board of Regents, 1995). It has been recommended that the experiences that preservice teachers have should conform to the standards developed by professional organizations, that the quality of the mathematics content courses should be strengthened, and that preservice teachers should experience the type of classroom environment they will be expected to develop.

Recently, a group of post-secondary institutions in a southwestern state were awarded a grant from the National Science Foundation to reform its preparation of teachers in Mathematics and Science. One of the courses targeted for improvement by the grant was a mathematics course that taken by all prospective elementary teachers. A team of instructors was assembled and current educational research was consulted in an effort to develop a rich curriculum designed to deepen preservice teachers' understanding of the fundamental concepts in mathematics. As a result, this required course has been dramatically modified to reflect the recommendations of NCTM (1991) and other national and local agencies. The course is now taught in what is considered a highly reformed manner (as opposed to a traditional lecture format). That is, the course is currently conceptually based, integrating tools and technology into the instructional delivery of the curriculum. These preservice teachers are now being asked to solve problems, write mathematically, think critically and construct concepts in the manner in which we expect them to provide for their students. For instance, these students used a variety of concrete models to study fractions. Activities with varied representations allowed them to see equivalent fractions, and to understand the motivation for the use of common denominators for fraction addition and subtraction. Array models used for whole numbers were extended to encompass fraction multiplication, and repeated subtraction was used to model and explain fraction division.

This study proposes to answer the following research question: What impact does the reformed curricula and teaching methods instituted in this course have on prospective teachers' understanding of rational numbers and integers. In order to obtain more in-depth insights regarding prospective teachers' concept development, these two fields were investigated rather than the entire curriculum. Research shows that students, as well as teachers, traditionally have difficulty both understanding and conveying these concepts. Insights into students' understandings and misconceptions should shed light on the efficacy of the described reforms.

Framework

Understanding the concepts of signed numbers and rational numbers are fundamental for the acquisition of higher mathematical concepts. Understanding these sets of numbers involves more than knowing the operational algorithms; it also includes the ability to:

- Add, subtract, multiply and divide integers and rational numbers.
- Order integers and rational numbers.
- Predict how the basic operations affect size and order of integers and rational numbers.
- Explain how the basic operations affect size and order of integers and rational numbers.
- Solve problems using integers and rational numbers.
- Solve ratio and proportion problems.

These aspects of understanding provided the groundwork from which the assessment instruments were developed, and the methods in this study were designed to capture the presence or absence of these abilities.

Background

Most of the research conducted in this area either highlight areas where preservice teachers are mathematically deficient, or make recommendations for the mathematical development of preservice and in-service teachers. Studies show that both in-service and preservice teachers have been inadequately prepared in the past. Practicing elementary and middle school teachers in Arizona have self-reported that their preparation was fair at best and many had reservations about their ability to teach mathematics (Arizona Board of Regents, 1995, 1997). Post (1991) found that 25 percent of the middle school teachers studied could not successfully find the answers to problems that required basic computations with rational numbers. When asked to solve problems requiring conceptual knowledge, only half of them were successful. Ball (1990, 1990) studied the understandings that preservice elementary and secondary teachers have of division. She found that both groups studied had a shallow and fragmented view of division. While the prospective high school teachers had a better grasp of the rules, they were no better at explaining how the rules were derived or what they meant. Tirosh and Graeber (1990) found similar results when studying preservice teachers' understanding of division of rational numbers. They found that certain ways of thinking—for example the idea that "division makes numbers smaller"—while were true for whole numbers became obstacles when reasoning about rational numbers. In a 1997 study, Behr, Khoury, Harel, Post and Lesh investigated the various ways preservice teachers approached a single problem involving rational numbers. In that study, the authors concluded that teachers must be aware of the many problem-solving strategies that a single problem can generate, and must provide activities that can develop a variety of cognitive structures. In a study that compared the conceptual understanding of a few Chinese elementary school teachers and American elementary school teachers, Ma (1999) found that even though those Chinese teachers had less formal education than their American counterparts, they demonstrated what she calls a profound understanding of fundamental mathematics, while teachers in the United States did not.

This collection of research studies suggests that teacher education programs in the United States need to better prepare prospective teachers by developing the conceptual underpinnings necessary to implement the new school curriculums driven by the NCTM Standards (1989). Many colleges and universities have responded to these finding by reforming their mathematics classes for prospective teachers. Yet there is little data available to suggest that these efforts are producing the desired results.

The Study

The subjects for this study were approximately 225 students enrolled in *Theory of Mathematics for Elementary School Teachers;* a one-semester course designed to promote deep understanding of the mathematics taught in elementary school. Also included in the study were the seven instructors assigned to teach the class. The research involved two groups -- a treatment group which involved four of the instructors and their students and a control group which involved the other three instructors and the balance of the students. Both groups were administered identical pretest and posttests. Participants from the treatment group were interviewed after both the pretest and the posttest. Interviewees were asked to solve selected problems, explain their reasoning, and express their general views about the class and their attitudes about mathematics. The interviews were tape-recorded and tapes were analyzed to look for insights into students understanding as well as misconceptions.

During the treatment phase of the study, the participants in the treatment group explored the topics in the context of activities that utilized concrete models and technology. Regular homework was assigned from a textbook and graded weekly, and an exam was administered at the end of the unit. In addition to tests, homework and class activities, students were asked to write about these topics on a regular basis in a math journal. While the control group did experience some "student-centered" curricula, they did not use the materials developed for the treatment group.

However, curriculum is only one facet of instruction. Implementation of the curriculum is critical to its success. For this reason, the instructional strategies used in the classrooms studied were also investigated. Since the treatment claimed to be "highly reformed", it was important to not only substantiate that claim, but to also observe and compare the instructional strategies of both groups. The level of "reformed teaching" observed in these classrooms was measured using the Reformed Teacher Observation Protocol (RTOP). RTOP (Piburn et. al., 2000) was designed to quantify the level of inquiry or student-centered teaching that takes place in a classroom. Two researchers used the RTOP to independently evaluate the classrooms involved at least twice. The scores were averaged, and then correlated to the scores on the students' tests to determine if a relationship existed.

The pretest and the posttest consisted of three subtests: Computational Skills, Number Sense, and Conceptual Understanding. Many of the questions in this instrument have been used by other researchers studying the knowledge and misconceptions of prospective teachers. Detailed rubrics were developed for each of the conceptual questions during the pilot phase of the study. These questions were scored on a five-point scale, where 0 indicated that no attempt was made, and 5 indicated no flaws in work or reasoning. Questions from the other subtests were either correct or incorrect. Scores were established for each subtest as well as a total score for each participant.

Results

The pretest results show that a startling number of students enter the program under-prepared in mathematics. Even though the prerequisite for this course is College Algebra, 58% of those students tested were unable to execute simple computations involving rational numbers and integers without the use of a calculator. The poorest performance was reflected in those problems involving multiplication with mixed fractions, and division of two decimals. Interviews provided evidence for their strictly mechanical approach to these types of questions. Frequently students could not articulate their reasoning or their solution pathway, nor did they express confidence in their abilities.

Interviewer: *How did you arrive at this answer?*

Student: *I don't know – I just cross-multiplied.*

Interviewer: *Why did you do that?*

Student: *Well, I just remembered something about that with fractions you just cross over.*

Even when students arrived at the correct answer, their thinking and understanding of the problem was incorrect. When asked how she determined where to place the decimal point in the answer to $085 \div 0.2$, Carole replied, " *The decimal point was over 2 [places] so I figured it should be over two or three, so I put it over two.*"

When asked to estimate $72 \div 0.025$, most students responded that the answer would be less than 72. As Terry explained, "*When you divide something it gets smaller.*"

In response to the question, write a word problem that uses $1\frac{3}{4} \div \frac{1}{3}$, most students wrote a problem that required division by 3.

Dana: *Joe and his 2 friends ordered two pizzas. They ate one pizza and three quarters of the other pizza altogether eating* $1\frac{3}{4}$ *of a pizza. How much did Joe eat, considering they each ate equal amounts?*

Interviewer: *You got as an answer* $3\frac{1}{2}$*. Does your answer make sense?*

Dana: *No—that doesn't make sense at all.*

Paired t-tests confirmed that while both groups made significant gains from the pretest to the posttest, the treatment group scored higher on the posttest than the control group. Not only did they perform better on the test as a whole, but also their sketches, drawings and explanations on the test papers revealed a deeper understanding of the content. The most dramatic and exciting increases were from the Number Sense and Conceptual Understanding subtests. Posttest interviews confirmed better understanding as well as more confidence in the subject matter.

Interviewer *How did you do this one, where you were to estimate* $\frac{12}{13}+\frac{7}{8}$*?*

Alice: *I figured* $\frac{12}{13}$ *is close to* $\frac{13}{13}$ *which is 1, and* $\frac{7}{8}$ *is close to* $\frac{8}{8}$ *which is 1, and so I figured 1+1 is 2*

Many more of them were able to come up with appropriate word problems using division with fractions.

Dana: *I have* $1\frac{3}{4}$ *cantaloupes, and I want to give everyone* $\frac{1}{3}$ *of one. How many pieces can I get?*

Some of the interviewees were still confused division by $\frac{1}{3}$ with division by 3 when trying to create a problem. Yet, when those students were offered examples of both types of problems during interviews, they were able to select the correct one.

The interviewees also expressed enthusiasm for the format of the class. When asked if the activities they experienced helped them to understand fractions, decimals, and integers better, all but one stated that the treatment definitely helped their understanding of these concepts. They cited fraction manipulatives and base 10 blocks as well as paper folding as tools that enhanced their knowledge.

Alice: *I think a lot of the stuff helped me...it more. Like the reasoning behind it.*

Dana: *I think drawing it out, like say you have a certain fraction and you want to divide by another fraction...I wouldn't have thought before that I could divide each one into one third and then count them. I [also] like working in groups, I learn a lot more.*

Karen: *I'm like totally hands on. I've got to get into it.*

Ben was more reserved.

Ben: *A little bit. I'm still kind of confused. Like I said, I got very confused on how much exactly* $\frac{4}{9}$ *is. So a little bit. It seems like I do the Base 10 stuff and all in class, and then I try and apply it again and it gets more confusing. And then I go back and look at my activity book and it's not as clear for some reason as when I first did it.*

Comparing the Groups

There was a statistically significant gain from pretest to posttest for all subjects. In addition, the RTOP scores for the Treatment group were significantly higher then the RTOP scores for the Control Group. When RTOP scores were compared to the gains achieved from the pretest to the posttest, there was no correlation. (See Table 1) It would appear that the level of reform teaching had no impact on student achievement. However, an examination of the sub-tests revealed a different story. When RTOP scores were compared with the normalized gain (that is the gain divided by the potential gain) for the Computation Sub-test, there existed only a very weak correlation, indicating that a higher level of reform does not necessarily lead to better computational skills.

Table 1. RTOP Correlated with Selected Results

Group	RTOP	Overall Percent Increase	Overall Normalized Gain (Hake)(%)	Normalized Gain on Computational Subtest (%)
Treatment	83	11	26	59
Treatment	71	22	34	35
Treatment	73	32	50	72
Treatment	60	25	35	63
Control	61	25	31	55
Control	45	20	23	46
Control	61	18	24	64
Correlation		-0.21	0.37	0.22

Yet, when the RTOP scores are compared with the just post-test scores, the correlation jumps to 0.88. When considering only the Number Sense Subtest, RTOP correlates with both the posttest scores (r =0.76) as well as with the normalized gains attained (r =0.73). (See Figure 1.) A closer look at the Conceptual Understanding Subtest reveals a similar relationship. For a comparison of RTOP and the posttest scores of that subtest, r =0.89, and when compared with the normalized gains, r =0.73. (See Figure 2.)

Conclusions and Discussion

These results suggest a relationship between a "reformed" mathematical environment and enhanced student achievement. When prospective teachers experience a student-centered, conceptually based, Standards driven mathematics course, they benefit in at least two ways. They develop a deeper, richer understanding of mathematical concepts than they would in a traditionally taught, or even moderately reformed classroom setting. The treatment group displayed stronger "number sense" was better able to estimate reasonable solutions, to solve problems, and their computational skills were equal to those of the control group. They were also more articulate about their mathematical thinking. This is especially heartening; since number since and estimation are areas at which our students traditionally do not excel (TIMSS, 1996).

They also experience first hand the classroom setting that is advocated by various professional organizations. These students were able to feel the power of discovery, and the satisfaction of understanding. As a result, their attitude toward mathematics

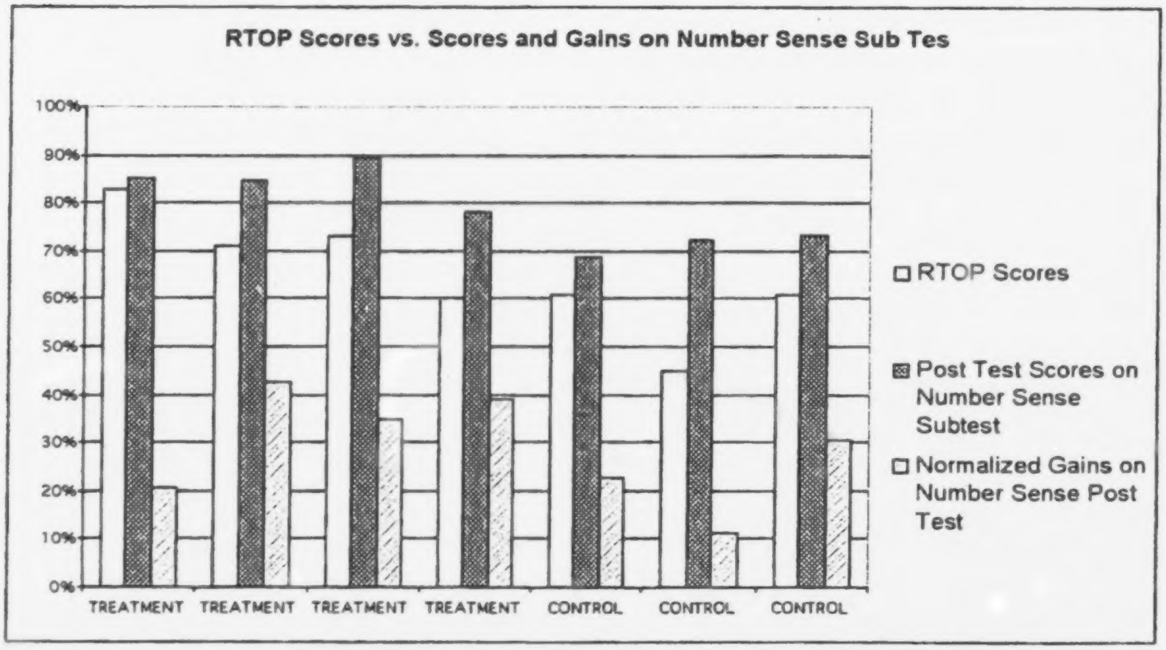


Figure 1.

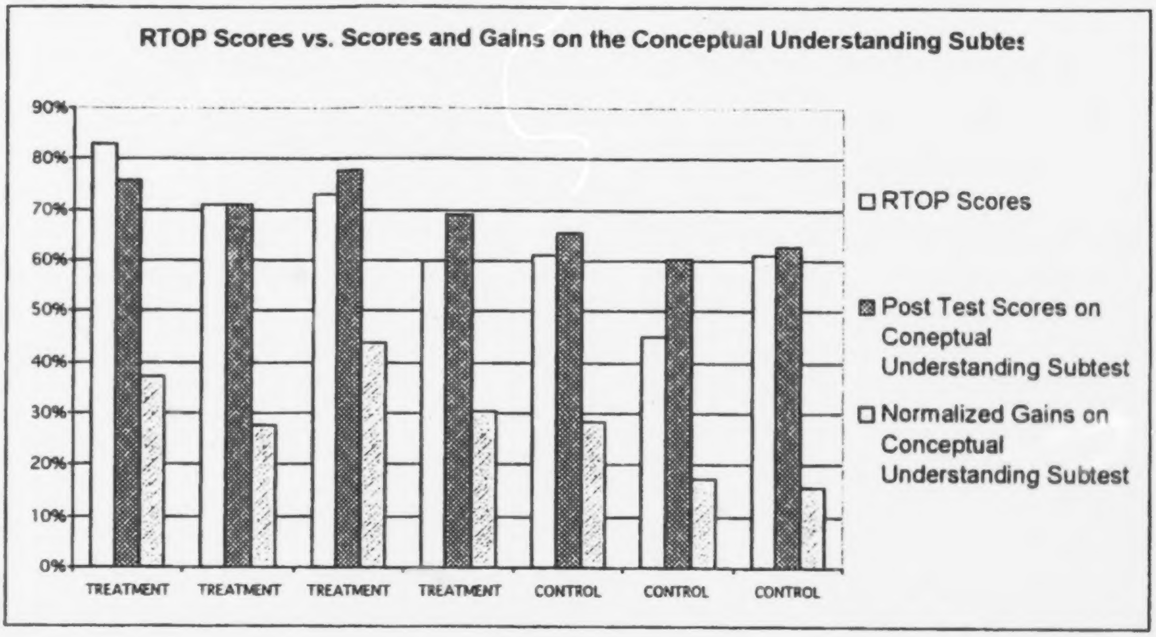


Figure 2.

learning and mathematics teaching changed. Most of the students interviewed liked the approach and reported that it enhanced their understanding of rational numbers and integers. They enjoyed working with the manipulatives and saw the value of using these methods with children. They benefited from the working in a group and having the opportunity to brainstorm and share with their peers. They also reported greater enjoyment of mathematics in general, more persistence, as well as more confidence in their own abilities.

These results also confirm the recommendations of many of the experts in this field, yet it also raises some issues that teacher preparation programs need to be aware of. Although all these students have a high school diploma and at least 3 years of high school math and the prerequisite of college algebra, they came to the program under-prepared. Not only did they not have the conceptual knowledge that one would expect of a teacher, they couldn't perform basic computations, and were unable to see that an answer did not make sense.

A short treatment like the one these students experienced may not be sufficient for all the students. One semester may not be enough to address the knowledge gaps that many prospective teachers bring with them. For instance, this treatment appeared to be deficient in proportional reasoning and decimal operations, as those are the areas of least improvement. However, even a short treatment can be valuable as demonstrated by the strong improvement noted in the estimation problems.

This is a positive start to developing a generation of teachers who understand the concepts, and who do not avoid mathematics. But this is only a start. One semester is clearly not enough. Many students still think in terms whole numbers only, and when unsure, tend to revert to mechanical methods. Students like Ben, need more exposure and reinforcement, or they will fall back into old habits. Methods curricula and professional development activities must be developed to continue this process of building conceptual knowledge, and to support them as they try to impart this conceptual knowledge in their classrooms.

Resources

Arizona Board of Regents. (1995). *K-12 mathematics and science education in Arizona: A status report* (pp. 55-78). Eisenhower Mathematics and Science Program.

Arizona Board of Regents. (1997). NAU Symposium on Systemic Reform in Science and Mathematics Education. Eisenhower Mathematics and Science Program.

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal, 90* (4), 449-467.

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21* (2), 132-144.

Behr, M., Khoury, H., Harel, G., Post, T., & Lesh, R. (1997). Conceptual units analysis of preservice elementary school teachers' strategies on rational number-as-operator task. *Journal for Research in Mathematics Education*, *28* (1), 48-69.

Bruno, A., Candelaria, E., & Martinon, A. (1997) Prospective teachers solve additive problems with negative numbers. *Focus on Learning Problems in Mathematics, 19* (4), 36-55.

Ma, L. (1999). *Knowing and teaching mathematics.* Mahwah, NJ: Erlbaum.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for the teaching of school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (1991). *Professional standards for the teaching of school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Piburn, M., Sawada, D., Turley, J., Falconer, K., Benford, R., Bloom, I., & Judson, E. (2000). Reformed teaching observation protocol (RTOP): Reference manual. *Arizona collaborative for excellence in the preparation of teachers, Technical report IN00-3.*

Post, T., Harel, G., Behr, M., & Lesh, R. (1991). Intermediate teachers knowledge of rational number concepts. *Integrating research on teaching and learning mathematics* (pp. 177-198). State University of New York, Albany, NY.

Rey, Robert E. (1998). Computation vs. number sense. *Mathematics Teaching in the Middle School, 4* (2), 110-112.

Third International Mathematics and Science Study. (1996). *U. S. National Research Center* (7).

Tirosh, D., & Graeber, A. (1990). Evoking cognitive conflict to explore preservice teachers' thinking about division. *Journal for Research in Mathematics Education, 21* (2), 98-108.

THE RELATIONSHIP BETWEEN PRESERVICE ELEMENTARY TEACHERS' CONCEPTIONS OF MATHEMATICS AND MATHEMATICAL REASONING

Linda A. Bolte
Eastern Washington University
lbolte@mail.ewu.edu

Abstract: This study explored preservice elementary mathematics teachers' conceptions of mathematics and the relationship between these conceptions and demonstrated levels of mathematical reasoning. As a group, students' conceptions of mathematics tended toward a relational/active viewpoint, as opposed to an instrumental/passive perspective. Preliminary analysis of the data indicates a correlation between a relational/active viewpoint and depth of mathematical reasoning.

The primary objectives of this study were to 1) determine and categorize preservice elementary mathematics teachers' conceptions of mathematics, 2) analyze the relationship between scores on a Conception of Mathematics Inventory and demonstrated levels of mathematical reasoning, and 3) analyze the relationship between a change in conception of mathematics score and demonstrated level of mathematical reasoning.

Conceptual Framework

One current area of interest within the mathematics education community explicitly focuses on the need to prepare preservice mathematics teachers to implement curricular and pedagogical recommendations made at both the national and state levels. The National Council of Teachers of Mathematics (NCTM) and the Washington State Commission on Student Learning describe a K-12 curriculum that stresses a connected body of mathematical understandings and competencies. The ten Standards presented in *Principles and Standards for School Mathematics* "specify the understanding, knowledge, and skills that students should acquire from kindergarten through grade 12" (NCTM, 2000, p. 29). A similar list of eight Essential Academic Learning Requirements has been compiled by the Washington State Commission of Student Learning (1988).

The Algebra and Representation standards are of particular interest to this study. Specifically with respect to the concept of function, the Algebra standard recommends all students demonstrate the ability to select appropriate representations, (i.e., numerical, graphical, verbal, and symbolic), for a given function; flexibly convert among different representations; interpret representations; and be proficient in the use of multiple forms of representation to interpret mathematical relationships and real-world situations. The notion that the development of a deep understanding of the function concept is closely tied to facility with representing functions in multiple ways is reiterated in the Representation standard.

> The ways in which mathematical ideas are represented has a fundamentally important role in shaping the ways people can understand and use those ideas. . . . Representations should be treated as critical elements in supporting student understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understanding to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations via modeling. (NCTM, 2000, p. 67)

Despite past efforts to prepare preservice teachers to implement these recommendations as originally proposed over a decade ago in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), "the portrayal of mathematics teaching and learning in *Principles and Standards for School Mathematics* is not the reality in the vast majority of classrooms, schools, and districts" (NCTM, 2000, p. 5). In addition to continued exposure to rich mathematical tasks that focus on developing both conceptual and procedural understanding, one factor that has emerged as playing a critical role in development of mathematical knowledge is the learner's conception of what mathematics is, what it means to do mathematics, and how one learns mathematics (Thompson, 1992; Treagust, Duit, & Fraser, 1996).

This study focuses on the interaction between preservice elementary teachers' conceptions of mathematics and the depth of their mathematical reasoning in a course titled, Functions and Relations. This is the first content course beyond our general mathematics requirement that is required for preservice elementary education students with a major or minor in mathematics. Historically, many of these students have experienced difficulty with this course. Their previous exposure to functions has largely been procedural, and an emphasis on writing and utilizing multiple representations, which included the use of a graphing calculator, was a new experience. Students' conceptions of mathematics were addressed by a written inventory (Grouws & Howald, 1994); depth of mathematical reasoning was indicated by students' ability to integrate numerical, graphical, and symbolic representation of functions and their proficiency with central aspects of the function concept. This dual emphasis is consistent with research findings related to how beliefs and conceptions of mathematics influence the teaching and learning of mathematics (Cifarelli & Goodson-Epsy, 1998; Edwards, 1999; Emenaker, 1995; Otto, Lubinski, & Benson, 1997; Williams, Pack, & Khisty, 1997), as well as the integral role played by multiple representations in developing an overall understanding of functions (Brenner, Mayer, Moseley, Brar, Duran, Reed, & Webb, 1997).

Methodology

Survey data, based on a Conception of Mathematics Inventory administered at the beginning and end of the quarter, were collected in the Functions and Relations course during fall quarter, 2000, and winter and spring quarters, 2001 [n=70]. Mathematical reasoning data, which included select homework assignments, regular tests, and the final exam, were collected during fall quarter, 2000 [n=16].

The Conception of Mathematics Inventory is composed of four themes and seven related dimensions, as indicated in Figure 1. Students responded to eight statements related to each of the seven dimensions, indicating they strongly disagreed (1), disagreed (2), slightly disagreed (3), slightly agreed (4), agreed (5), or strongly agreed (6)

I. The Nature Of Mathematics	
1. *Composition of Mathematical Knowledge*	
Knowledge as concepts, principles, and generalizations	Knowledge as facts, formulas, and algorithms
2. *Structure of Mathematical Knowledge*	
Mathematics as a coherent system	Mathematics as a collection of isolated objects
3. *Status of Mathematical Knowledge*	
Mathematics as a dynamic field	Mathematics as a static entity
II. Nature Of Mathematical Activity	
4. *Doing Mathematics*	
Mathematics as sense-making	Mathematics as results
5. *Validating Ideas in Mathematics*	
Logical thought	Outside authority
III. Learning Mathematics	
6. Learning as constructing understanding	Learning as memorizing intact knowledge
IV. Usefulness Of Mathematics	
7. Mathematics as a useful endeavor	Mathematics as a school subject with little value in everyday life or future work

Figure 1. Themes and related dimensions for Conceptions of Mathematics Inventory

with the statement. Of the eight items related to a particular dimension, a relational/active perspective is reflected by responding "strongly disagree" to four questions and "strongly agree" to four statements. An instrumental/passive perspective is indicated by rating the respective statements in the opposite manner (i.e, strongly agree and strongly disagree). The following are representative items for each dimension: 1) Composition of Mathematical Knowledge – "There is always a rule to follow when solving a mathematical problem" and "Essential mathematical knowledge is primarily composed of ideas and concepts"; 2) Structure of Mathematical Knowledge – "Diagrams and graphs have little to do with other things in mathematics like operations and equations" and "Concepts learned in one mathematics class can help you understand material in the next mathematics class"; 3) Status of Mathematical Knowledge – "When you learn something in mathematics, you know the mathematics learned will always stay the same" and "New mathematics is always being invented"; 4) Doing Mathematics – "One can be successful at doing mathematics without understanding it" and "When working mathematics problems, it is important that what you are doing makes sense to you"; 5) Validating Ideas in Mathematics – "When two students don't agree on an answer in mathematics, they need to ask the teacher or check the book to see who is correct" and "When one's method of solving a mathematics problem is different from the instructor's method, both methods can be correct"; 6) Learning as constructing understanding – "Learning to do mathematics problems is mostly a matter of memorizing the steps to follow" and "When learning mathematics, it is helpful to analyze your mistakes"; and 7) Mathematics as a useful endeavor – "Students should expect to have little use for mathematics when they get out of school" and "Students need mathematics for their future work". Responses were linearly transformed to express a relational/active perspective as a score of 1 and an instrumental/passive perspective as a score of 6. An average score for each theme and related dimensions was calculated for each student.

The level of mathematical reasoning score was based on written responses to questions that focus on topics identified in previous research as critical indicators of depth of knowledge of functions: definition of function, functional notation, composition of functions, domain and range, multiple representations of functions, zeros of a function, solving equations and inequalities, the relationship between a function and a corresponding equation, characteristics of families of functions, and modeling real-world situations. Relevant data from select homework assignments, four regular tests, and the final exam were tabulated and analyzed to determine an overall level of mathematical reasoning.

Results

Data from the 70 preservice elementary preservice teachers enrolled in the Functions and Relations course indicates the majority of the students are somewhat closer to relational/active perspective of mathematics than the instrumental/passive viewpoint. The range of scores was 1.0 to 3.8 for the four themes and 1.0 to 4.1 for the related dimensions. With respect to Nature of Mathematics, 60 students scored between 2.2 and 3.1.; for Nature of Mathematical Activity, 59 students scored between 2.2 and 3.1; and for Learning Mathematics, 46 students scored between 2.2 and 3.1. Only with respect to Usefulness of Mathematics is there a strongly held relational/active viewpoint, with 67 students scoring between 1.0 and 2.5.

Analysis of scores for each dimension shows the relative strength of a perspective varies within each theme. Within Nature of Mathematics, Composition of Mathematical Knowledge tends more toward instrumental/passive (58 students between 2.2 and 3.5) than Structure of Mathematical Knowledge (60 students between 1.6 and 2.8). Of the seven dimensions, Status of Mathematical Knowledge had the largest range, 1.2 to 4.1, with 56 students between 1.9 and 3.5. Within Nature of Mathematical Activity, Doing Mathematics tended more toward the relational/active perspective (66 students between 1.2 and 3.1) and Validating Mathematics toward the instrumental/passive (67 students between 2.2 and 3.8).

Preliminary analysis of data from fall quarter, 2000, indicates students' conception of mathematics is related to their level of reasoning with respect to functions. No correlation existed between students' score for Usefulness of Mathematics and level of reasoning. However, scores for Nature of Mathematical Activity and Learning Mathematics were significantly correlated with level of reasoning. Students with a relational/active viewpoint tended to obtain higher mathematical reasoning scores. Although Nature of Mathematics as a theme was not highly correlated to mathematical reasoning, one dimension within this theme, Composition of Mathematical Knowledge, was significantly correlated, and a second dimension, Structure of Mathematical Knowledge, was somewhat correlated. Within Nature of Mathematical Activity, Doing Mathematics was more highly correlated to reasoning score than Validating Mathematics. The difference between the beginning and end of the quarter survey scores was compared to the demonstrated level of mathematical reasoning for students completing the course during fall quarter [n=16]. Although the Conception of Mathematics scores did not change significantly during the 10-week term, there was some correlation between these scores and demonstrated levels of mathematical reasoning [$r = -0.44$, $r^2 = 0.20$, $p < 0.08$]. Analysis of data collected will continue, and, if warranted, further results will be reported.

Summary

Although the conceptions of preservice elementary teachers varied within the four themes addressed by the Conceptions of Mathematics Inventory, there was a tendency toward the relational/active perspective. This perspective was significantly correlated to the demonstrated level of mathematical reasoning, as indicated by performance on a selection of questions focusing on critical topics related to functions or the final course grade. These preliminary results indicate investigating the relationship between pre-

Table 1. Characterization of Conceptions of Mathematics on Pretest, Fall - Spring

Theme	1.0 - 1.1	1.2 - 1.5	1.6 - 1.8	1.9 - 2.1	2.2 - 2.5	2.6 - 2.8	2.9 - 3.1	3.2 - 3.5	3.6 - 3.8	3.9 - 4.1
I. Nature of Mathematics	1	2	23	18	19	6	1			
II. Nature of Mathematical Activity	2	5	21	21	17	3	1			
III. Learning Mathematics	2	1	9	12	17	17	11	1		
IV. Usefulness of Mathemataics	10	21	10	21	5	1	2			
Dimension										
1. Composition of Mathematical Knowledge				4	10	12	19	17	4	4
2. Structure of Mathematical Knowledge		1	8	17	23	12	7	2		
3. Status of Mathematical Knowledge		2	2	12	11	12	12	9	7	3
4. Doing Mathematics		4	6	16	24	9	7	3	1	
5. Validating Mathematics			1	2	8	17	25	14	3	
6. Constructing Understanding		2	1	9	12	17	17	11	1	
7. Useful Endeavor	10	21	10	21	5	1	2			

Table 2. Relationship Between Conception of Mathematics and Mathematical Reasoning

Conception of Mathematics Themes to Mathematical Reasoning Score [n=16]	**Correlations** r	 r2	 p-value
Nature of Mathematics	- 0.31	0.10	0.23
Nature of Mathematical Activity	- 0.56	0.32	0.02
Learning Mathematics	- 0.52	0.27	0.04
Usefulness of Mathematics	.00	0.0	
Conception of Mathematics Dimensions to Mathematical Reasoning Score [n=16]			
Composition of Mathematical Knowledge	- 0.53	0.28	0.03
Structure of Mathematical Knowledge	- 0.43	0.29	0.09
Status of Mathematical Knowledge	- 0.23	0.02	0.37
Doing Mathematics	- 0.56	0.31	0.02
Validating Mathematics	- 0.36	0.13	0.16
Constructing Understanding	- 0.52	0.27	0.04
Useful Endeavor	.00		

service elementary teachers' conceptions of mathematics and their level of mathematical reasoning and how to change these conceptions toward a more relational/active perspective is worthy of further study.

References

Brenner, M., Mayer, R., Moseley, B., Brar, T., Duran, R., Reed, B., & Webb, D. (1997). Learning by understanding: The role of multiple representations in learning algebra. *American Educational Research Journal*, (Vol. 34, pp. 663-689).

Cifarelli, V., & Goodson-Epsy, T. (1998). Mathematical beliefs and conceptions of college algebra students. In S. Berenson, K. Dawkins, M. Blanton, W. Coulombe, J. Kolb, K. Norwood, & L. Stiff, (Eds.), *Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 429-434).

Edwards, T. (1999). Altering elementary education students' conceptions of mathematics. In F. Hitt & M. Santos (Eds.), *Proceedings of the Twenty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 717-722).

Emenaker, C. (1999). The influence of a problem-solving approach to teaching mathematics on preservice teachers' mathematical beliefs. In F. Hitt & M. Santos (Eds.), *Proceedings of the Twenty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 135-139).

Grouws, D, & Howald, C. (copyright, 1994). *Conceptions of mathematics inventory.* Unpublished document resulting from project supported by the Iowa Measurement Research Foundation. Obtained through personal correspondence with the authors.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Otto, A., Lubinski, C., & Benson, C. (1997). The relationship of undergraduates' beliefs about learning algebra and their choice of reasoning strategies for solving algebra problems. In J. Dossey, J. Swafford, M. Parmantie, & A. Dossey, (Eds.), *Proceedings of the Nineteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 81-89).

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

Treagust, D., Duit, R., & Fraser, B. (1996). *Improving teaching and learning in science and mathematics*. New York: Teachers College Press.

Washington State Commission on Student Learning. (1998). *The Washington state essential academic learning requirements in mathematics.* WA: The Commission.

Williams, S., Pack, M., & Khisty, L. (1997). Teachers' beliefs about mathematics as assessed with repertory grid methodology. In J. Dossey, J. Swafford, M. Parmantie, & A. Dossey, (Eds.), *Proceedings of the Nineteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 452-458).

THE EFFECTS OF STAFF DEVELOPMENT USING A STUDY GROUP MODEL

Daniel J. Brahier
Bowling Green State University
brahier@bgnet.bgsu.edu

Abstract: A study group of 16 elementary school mathematics teachers was established for biweekly meetings through the Spring and Fall and seven full days in the Summer as a year-long staff development program. The group, as a whole, showed a significant increase in their scores on a survey instrument which measured their attitudes, beliefs, and classroom practices. Inner-city teachers from both the public and parochial school settings showed significantly more gains in their scores than their suburban counterparts. Classroom observations, however, indicated that many of the teachers possessed the knowledge of what constitutes effective teaching but experienced difficulty in translating reform into practice. A second year of study group meetings has been funded, and research will continue to monitor the long-term effects of professional development using this format.

Introduction

Funded by a grant from the Dwight D. Eisenhower Mathematics and Science program in a Midwestern state, 16 elementary mathematics teachers participated in a staff development study group in 2000, from January through late-November. The teachers were selected from five different elementary schools – three in an inner-city setting (one public and two parochial) and two from a suburban setting (one public and one parochial). The teachers volunteered to be part of the project and varied in teaching experience from 2 to 30 years. Their grade level assignments represented a mix of Kindergarten through Grade 6, with two Special Education teachers also participating. The study group met for two hours every two weeks from January through May, for seven full days in the Summer (five days in June in a retreat format and two days in August in a conference format), and for two hours every two weeks from September through November. The program was entitled SUCCESS – Suburban-Urban Collaborative for Classroom Educators in a Study-group Setting. Sessions focused on implementation of *Principles and Standards for School Mathematics* by the National Council of Teachers of Mathematics (2000) and how to use an inquiry approach to teaching and learning in the classroom. Teachers were responsible for implementing their own inquiry-based units and providing outreach to others in their building and districts (through staff development programs, family math nights, etc.). A Web site was developed as a forum to share experiences with a broader audience.

The purpose for this study was to measure and document the effects of using the study group model for professional development. Specifically, research was conducted to determine the following:

1. What effect, if any, did the study group have on the attitudes and beliefs of teachers involved in the program?
2. Were there any measurable differences in the changes of attitudes and beliefs within subgroups (e.g., did teaching experience, grade level, or school setting contribute significantly to changes)?
3. What effect, if any, did the study group have on classroom behaviors of the teachers involved in the program?

Perspectives/Theoretical Framework

The document *Principles and Standards for School Mathematics* (NCTM) was released in Chicago in April of 2000. This report is clear about the direction of mathematics education in schools – that mathematics needs to be taught in an inquiry-based manner with the students at the focal point of the classroom if educators are to teach significant mathematical concepts to children. Battista (1994) pointed out that "teachers are key to the success of the current reform movement" and that "many teachers have beliefs about mathematics that are incompatible with those underlying the reform effort" (p. 462). Therefore, to reform the process of teaching and learning mathematics, it is important that the mathematics community addresses the underlying attitudes and beliefs held by teachers of mathematics. Teachers need first-hand experiences with inquiry-based lessons and the opportunity to field tests these types of lessons and reflect on them with peers. The authors of an article in *Phi Delta Kappan* journal (Bay, Reys, & Reys, 1999) noted that the implementation of inquiry-based curricula depends, in part, on "interaction with experts" and "collaboration with colleagues," both of which were cornerstones of the SUCCESS project.

Furthermore, the *Principles and Standard for School Mathematics* document (NCTM, 2000) emphasizes that there are six Guiding Principles that underlie the teaching of mathematics and that all of these need to be addressed in the classroom on a regular basis. These Principles include Equity, Mathematics Curriculum, Teaching, Learning, Assessment, and Technology. In the chapter dealing with the underlying Principles, the authors of the document emphasize that the likelihood of students meeting stated objectives is greatly increased when "these efforts include [teachers] learning about mathematics and pedagogy, benefiting from interactions with students and colleagues, and engaging in ongoing professional development and self-reflection" (p. 19). Indeed, ongoing and significant staff development is crucial to helping teachers make the connection between research underlying reform and classroom practice.

While designing the SUCCESS project, a local curriculum coordinator commented that teachers are too isolated and often believe that their classroom, whether in the inner-city or the suburb, public or parochial, is "unique" and that no one could be experiencing the same difficulties. In reality, teachers share many of the same concerns, regardless of their classroom settings. Therefore, it seemed fitting and appropriate to design a program that would place teachers in a study-group environment to meet with others over an extended period of time, reviewing exemplary teaching practices, designing inquiry-based teaching units, and developing leadership skills to assist other teachers in their buildings, districts, and State.

The study-group concept was described at the 1999 NCREL Advisory Board meeting in Chicago as the fastest-growing method of staff development in terms of popularity and impact. A well-known publication by Murphy and Lick (1998) entitled *Whole-Faculty Study Groups: A Powerful Way to Change Schools and Enhance Learning* describes, in detail, the power of establishing long-term study groups for teachers with common concerns. They defined a study-group as "a small number of individuals joining together to increase their capacities through new learning for the benefit of students" (p. 4). Furthermore, the authors stated that "as teachers work together in these study-group approaches, they alter their practices to provide new and innovative opportunities for their students to learn in challenging and productive new ways" (p. 2). It is important to keep a study-group to a manageable size so that each member has a regular opportunity to contribute at meetings and to allow the Project Director an opportunity to interact with each and to visit their classrooms as part of the group process. The May 1999 issue of *Educational Leadership* journal also emphasized the importance of reflection on practice and the use of study-groups as effective means of improving instruction. Rogers and Babinski stated that study-groups "offer a safe place where...teachers can voice their concerns, share their joys and frustrations, and help one another deal with problems. Establishing regularly scheduled times when...teachers can talk and listen to one another...gives them the chance to learn and grow professionally" (1999, p. 38).

Other publications, such as *Learning Circles: Creating Conditions for Professional Development* (Collay, Dunlap, Enlose, & Gagnon, 1998) and *Promoting Reflective Thinking in Teachers* (Taggart & Wilson, 1998) have also endorsed the use of small study groups to develop teachers. Most recently, the Glenn Commission Report (2000) stated that "building- and district-level Inquiry Groups can provide venues for teachers to engage in common study to enrich their subject knowledge and teaching skills" (p. 8). Also, the authors of the TIMSS-R report (2000) stated, "Some research suggests that the experience of teachers observing other teachers can contribute to the sharing of good practices...(however), on average, U.S. eighth grade students were taught by mathematics teachers who spent 1 class period during the 1998-99 school year observing other teachers and were observed by other teachers during 2 class periods" (p. 50). Clearly, the message of the power of a study-group and reflection on practices of self and others are universally recognized but not widely practiced. More research on the topic is needed to document the effects of this staff development tool.

Methods/Modes of Inquiry & Data Sources/Evidence

The following served as sources of raw data for this study:

1. Surveys (pre- and post-project)
2. Interviews
3. Journal entries
4. Inquiry-based teaching units
5. Field visits (2) to classrooms and related videotapes
6. Written "learning episodes" by participants

The data were collected and used as follows: Participants in the SUCCESS project were administered a survey at the beginning of the program. The instrument measured their attitudes, beliefs, and (participant-reported) classroom practices as mathematics teachers. The instrument was a modified version of a questionnaire developed to measure a teacher's perception of what it means to teach mathematics and their pedagogical approaches to instruction. Throughout the Spring sessions, the teachers wrote entries into their journals and were visited by the Project Director who observed their teaching and recorded field notes. The journal entries were analyzed by the Project Director and the Project Evaluator, as were the field notes on what was observed in the classrooms (including physical setting, design of the observed lesson, and interactions with children). In the Summer sessions, participants were interviewed by the Project Evaluator and designed inquiry teaching units. The interviews were transcribed and organized by themes by the Project Evaluator. In the Fall, the Project Director returned to the participants' classrooms and videotaped lessons for discussion and reflection. Participants were given copies of their classroom videotapes and asked to write in-depth analyses of the lessons and to share segments of the tapes and their reflections with SUCCESS participants later in the Fall. In addition, participants were each required to formally "write up" two learning episodes from their teaching in the Fall. The episodes served as 2-3 page descriptions of how students reacted to specific inquiry-based lessons. These episodes were shared at SUCCESS meetings, collected, and organized onto the project Web site for others to access. Finally, participants were administered a post-project survey at the end of the SUCCESS program. The survey data was quantitative in nature, and a series of statistical tests were conducted to identify effects of the project as a whole and for subgroups. The researcher triangulated data from the pre- and post-project surveys, interviews, journals, teaching units, field notes, videotapes, and learning episodes to address the research questions.

Results

A t-test was conducted for the total group of 16 teachers, based on results from the pre- and post-project surveys. A significant, positive shift in attitudes, beliefs, and practices was identified from these surveys. Teachers' opinions about the nature of mathematics and how it is most effectively taught clearly showed a marked change over the 11-month period. Further statistical analyses using the teaching experience, grade level, and school settings as independent variables and pre-test and post-test totals as dependent variables were conducted, but no interactions or overall significance were identified. However, a univariate one-way ANOVA showed a significant F-ratio for the school setting ($F_{3,11} = 5.14, p < .05$). Individual dependent t-tests for the school settings showed that the teachers in the inner-city parochial schools underwent the most significant change ($t_4 = 5.30, p < .01$) and that the inner-city public school teachers also performed significantly higher on the post-test ($t_2 = 6.08, p < .05$). The suburban teachers, however, did not show a significant change in performance from their pre-tests to post-tests in the program. In fact, the mean pre-test score for the suburban public school (154) was higher than the post-test mean score (152.6) for the inner-city parochial school teachers. Clearly, the project had the most profound effect on teachers in the inner-city setting.

The field interviews and participant journal entries indicated that participants valued the study group sessions and were heavily-influenced by the instructional videotapes, case studies, and other presentations at their sessions. However, field visits and videotapes of participant lessons showed that some of the teachers illustrated a halo effect on the surveys and interviews, reporting that they had significantly changed their practices, while demonstrating only superficial changes (such as using manipulatives but not effectively or posing problems that were neither rich nor developmentally appropriate) in the classroom. For example, one participant expressed having become "a cheerleader in the building for implementation of the Standards," yet an observation in the Fall revealed a fairly traditional classroom setting and a lesson that was neither developmentally appropriate nor in accord with the recommendations of the NCTM in the Standards documents. Another teacher wrote an excellent inquiry unit on patterning, but the lesson videotape from the unit illustrated an inability to effectively use student responses to guide the discourse. The result was a lesson that appeared much better "on paper" than it did in its actual classroom application.

Overall, the participants did appear to undergo attitudinal changes, as evidenced through their survey results, journal writing, and interviews. In particular, teachers frequently commented on the power of sharing their ideas with others and recognizing that teachers face similar problems in all of the different classroom settings represented. They commented on how infrequently teachers are generally given the opportunity to share ideas with others in the field and how the study group unified and supported one another through the process. The interviews showed that participants saw the week-long retreat in June as the most significant contributor to group cohesion – even more so than a shared trip to the NCTM Annual Conference in Chicago in the Spring of the year. However, the infancy of implementing inquiry lessons led to few observable changes in classroom practices for several of the participants. In short, it appears that some of the teachers have adopted reform principles, have undergone some change, but need more experience with leading inquiry-based lessons to show long-term observable shifts in their teaching practices.

Discussion

The SUCCESS project has certainly demonstrated the potential for using study groups as a staff development tool. While more traditional models of teacher in-service feature guest speakers and workshops, the study group offers teachers a long-term opportunity to obtain new information, field test ideas, and share the results and their reflections with others. The diverse settings of the teachers in the project allowed participants to compare teaching situations and reflect on whether their challenges were a result of the school environment or something more realistically attributed to the nature of teaching and learning with any children, in a general sense. The survey instrument clearly showed a marked change in the attitudes, beliefs, and practices of SUCCESS participants.

The additional statistical analyses also showed that while the "entire group" underwent significant shifts in their thinking and practice, the changes were primarily attributed to subgroup changes of inner-city teachers. The eight teachers in the inner-city entered the project with much more traditional views about teaching and the nature of mathematics than their suburban counterparts. Of course, the research that continues to be needed is the study of whether the nature of children in the urban setting *causes* the teachers to develop these attitudes or if urban teachers who choose urban settings, by nature, tend to possess a more simplistic view of mathematics and the role of the teacher. If we accept the premise that all children can learn mathematics, as has been the overriding message of the NCTM, then the latter explanation appears plausible but is not proven here. The results of this study also did not show whether those inner-city teachers were more significantly impacted by presentations from instructors or through interactions with teachers from the suburban settings who entered the program with considerably more optimism and knowledge about recent reform efforts.

Particularly troubling in this study were the teachers who identified themselves on surveys and interviews as having "changed" due to SUCCESS, but their classrooms appeared very traditional, and their teaching illustrated little more than an early attempt to run a discussion out of the ordinary. Fortunately, when those teachers later viewed their videotapes and reflected on them, they acknowledged the shortcomings of the lesson and their inability to translate the vision into practice. One teacher stated that she waited for over a week to view her videotape because she "knew it was a bomb and just couldn't bring myself to look at it." If these teachers had felt that their lessons were reasonably in accord with contemporary teaching philosophies, this could have created another stumbling block to reform. But they were able identify their errors and developed ideas for how the lessons could have been improved. Other SUCCESS teachers also provided many suggestions for change as they viewed one another's videotapes.

Since it was determined that several of the teachers were in their "infancy" toward implementing reform and needed additional time and support to develop their ideas, a proposal was written and funded for a continuation of the project -- SUCCESS II. In the new project, four new schools and 16 additional teachers began to meet on a regular basis, while the original SUCCESS teachers continued to meet monthly during the school year and for two full days in the Summer. A proposal for SUCCESS III has already been submitted. Over time, the members of the study group continue to develop and support one another, and this growth is being measured over the second year. Further research needs to be conducted to determine if the results of this study will be mirrored in SUCCESS II and to consider the degree to which a second (or even a third) year of meeting in the group will enhance the participants' teaching practices.

References

Battista, M. T. (1994, February). Teacher beliefs and the reform movement in mathematics education. *Phi Delta Kappan*, 462-470.

Bay, J. M., Reys, B. J., & Reys, R. E. (1999). The top 10 elements that must be in place to implement standards-based mathematics curricula. *Phi Delta Kappan, 80* (7), 503-506.

Collay, M., Dunlap, D., Enloe, W., & Gagnon, G.W. (1998). *Learning circles: Creating conditions for professional development*. Thousand Oaks, CA: Corwin Press.

Gonzales, P., Calsyn, C., Jocelyn, L., Mak, K., Kastberg, D., Arafeh, S., Williams, T., & Tsen, W. (2000). *Pursuing excellence: Comparisons of international eighth-grade mathematics and science achievement from a U.S. perspective, 1995 and 1999.*

Hole, S., & McEntee, G. H. (1999). Reflection is at the heart of practice. *Educational Leadership, 58* (8), 34-37.

Murphy, C. U., & Lick, D. W. (1998). *Whole-faculty study groups: A powerful way to change schools and enhance learning*. Thousand Oaks, CA: Corwin Press.

National Commission on Mathematics and Science Teaching for the 21st Century. (2000). *Before it's too late*. Washington, DC: U.S. Department of Education.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Rogers, D. L., & Babinski, L. (1999). Breaking through isolation with new teacher groups. *Educational Leadership, 56* (8), 63-66.

Taggart, G. L., & Wilson, A. P. (1998). *Promoting reflective thinking in teachers: 44 action strategies*. Thousand Oaks, CA: Corwin Press.

ENHANCING PROSPECTIVE TEACHERS' UNDERSTANDING OF WHAT AND HOW TO TEACH THROUGH SMALL-GROUP COLLABORATION

Laurie O. Cavey
North Carolina State University
locavey@unity.ncsu.edu

Sarah B. Berenson
North Carolina State University
berenson@unity.ncsu.edu

Matthew R. Clark
North Carolina State University
mrclark@unity.ncsu.edu

Katrina Staley
North Carolina State University
kstaley@unity.ncsu.edu

Abstract: This study, embedded within the context of a large longitudinal study, examined how the collaboration of a group of four prospective teachers enhanced their understanding of how to teach within the context of a geometry lesson-planning task. Guided by a conceptual framework that we've developed (Berenson, Cavey, Clark, & Staley, 2001), we assumed that engaging prospective teachers in meaningful teaching tasks provided opportunities for them to make connections, via reflection and communication, between their primitive knowledge of mathematics and what is taught in school and to construct an array of effective teaching strategies. Preliminary results indicate that several factors contributed to the effectiveness of this group. In particular, the nature of the varying focuses of each prospective teacher when initially planning the lesson, the opportunity for each to create individual images of what and how to teach prior to group planning, and their willingness and ability to negotiate added to the effectiveness of this group.

Introduction

Much has been learned about how children construct an understanding of mathematics (for example, Davis, 1997; Maher, 1998; Pirie & Kieren, 1994) and what classroom characteristics facilitate this construction. Student communication and reflection are heralded as vital mechanisms for developing mathematical understanding (Hiebert et al., 1997; Sfard, 2000), whereas teachers' selection of tasks and appropriate uses of tools are critical for creating mathematics classrooms that facilitate understanding (Hiebert et al.). Mathematics teachers are asked to create opportunities for social interaction that involve students working in cooperative groups on appropriate tasks and with the right tools (National Council of Teachers of Mathematics [NCTM], 2000).

There is also a call for collaboration among mathematics teachers to adequately inform their practice (Ma, 1999; NCTM, 2000), and there is evidence that collaborative efforts used in both inservice and preservice education programs are proving worthwhile (Oakley, 2000; Steele, 1994). Oakley's work with elementary teachers demonstrates how collaborative inquiry with other teachers can help increase a teacher's focus on teaching for understanding. Cooperative groups can also serve to challenge prospective teachers' conceptions of mathematics, helping them develop mathematical understanding and change their image of mathematics instruction (Steele). Nevertheless, substantial research supporting the educational practice of teacher educators is necessary to adequately inform the practice of teacher educators (Ma, 1999; Schoenfeld, 1999; Simon, Tzur, Heinz, Kinzel, & Smith, 2000).

This study is embedded within the context of a longitudinal study that tracks prospective teachers' understanding of what (school mathematics) and how (teaching strategies) to teach (Berenson & Cavey, 2000) and is aimed at furthering a deeper and broader understanding of the psychological aspects of learning to teach school mathematics. The question of this part of the study asks: how can small-group collaboration enhance prospective teachers' understanding of how to teach within the context of a geometry lesson-planning task?

Theoretical Perspective and Methodology

Eisenhart (1991) emphasized the importance of using conceptual frameworks to "facilitate more comprehensive ways of investigating a research problem" (p. 211). This study relies heavily on a conceptual framework we've developed (Berenson et al., 2001) within the context of the longitudinal study to understand how prospective teachers develop an understanding of what and how to teach. The foundation of the framework is the model of mathematical understanding proposed by Pirie and Kieren (1994), which depicts understanding mathematics as a dynamical process and illustrates the mental activities necessary for mathematical understanding to occur. The conceptual framework extends the ideas of Pirie and Kieren into the realm of teacher knowledge by considering "the understanding of school mathematics to be a function of mathematical understanding in which prospective teachers access their primitive knowledge of mathematics to make connections with what is taught in high school" (Berenson et al., 2001). In addition, preliminary results from the longitudinal study have led to several conjectures about prospective teachers' understanding of what and how to teach.

> *Conjecture 1.* Folding back is critical to the development of prospective teachers' understanding of what and how to teach and can be prompted by conversations with others.
>
> *Conjecture 2.* Deep stacks of primitive mathematical knowledge may hinder the process of folding back to revise images of what to teach in school mathematics. The difficulty may stem from one's memory search functions or the ability to unpack one's knowledge base of college mathematics to find the connections to school mathematics.

Conjecture 3. Shallow stacks of teaching strategies are to be expected among prospective teachers. The lack of primitive knowledge allows the prospective teachers to search through their strategy stacks quickly. Additionally, they appear to be more receptive to interviewer's probes so as to revise their images of how to teach and add to or pack these stacks with new ideas.

Hiebert et al., (1997) define understanding in terms of identifying relationships and connections. Specifically, they stated, "that we understand something if we see how it is related or connected to other things we know" (p. 4). They pinpointed reflection and communication as two processes that play a critical role in the making of connections and described core features of a classroom that facilitate understanding. In particular, classroom tasks that facilitated mathematical understanding were described as being problematic, promoting insights into the structure of mathematics, and making use of available tools as resources. Problematic tasks were defined as challenging and mathematically intriguing, and therefore required student communication. Insights into the structure of mathematics were thought to occur when mathematical relationships and connections were revealed. Tools were defined broadly to include students' prior conceptions, physical materials, written symbols, and verbal language.

In brief, it was assumed that engaging prospective teachers in meaningful teaching tasks provided opportunities for them to make connections, via reflection and communication, between their primitive knowledge of mathematics and what is taught in school and to construct an array of effective teaching strategies.

Participants and Procedures

We report here on four prospective teachers' thinking about introducing right triangle trigonometry to a high school geometry class. All four were white males between the ages of 20-23 and preparing to teach high school mathematics. They were participants in a larger teaching experiment conducted in the first methods class for undergraduates. Individually they were interviewed before and after planning a lesson to introduce the concept of right triangle trigonometry to a high school geometry class. Additionally they were asked to relate their lessons to similarity and ratio. A week later they were assigned to a group to develop a "group" lesson plan on the same topic and were given their notes from their individual interviews. The group had one hour to plan, and then they were interviewed about their ideas.

Data and Analysis

The individual interviews, the group lesson planning process, and the group interview were all videotaped. Additional sources of data were notes made by the participants during the individual interviews. Data were sorted first to identify the lesson tasks planned by individuals and then the group. Additional sorts considered the extent to which the prospective teachers' tasks were problematic, promoted mathematical insights, and used available tools. These sorts informed the process of tracing ideas from individual efforts to collaborative lesson planning.

Results

Individual Images

When sharing his lesson ideas, David seemed to have an image of the sequence of the activities. He knew that he was going to follow a prescribed formula for his plan that involved beginning the class with an opening problem, stating the objective of the plan, and then giving the students the right triangle definitions of the sine, cosine and tangent ratios. He also seemed to have an image of the instructional sequence of the related school mathematics, which is indicated when he says, "The students are supposed to already know the Pythagorean theorem…I plan to go into the unit circle the day after this." David described his role as teacher as giving examples and explaining how to use the trigonometric ratios. His one defined task for students was a means to practice using the trigonometric ratios and hence would not be considered problematic.

In contrast, Craig had no image of the sequence of his activities when he introduced his lesson plan to the group: "I just threw a lot together…not in any particular order." His first goal was to "use these fun things"—the manipulatives that had been provided for the prospective teachers to work with—to enable his students to visualize the prerequisite concepts and relationships. Craig's second goal, finding trigonometric applications by searching through the textbooks that were provided and by folding back to his primitive knowledge of teaching strategies, suggests that he would facilitate understanding through activities that would catch his students' attention and that would be problematic in the sense that the students could not find the answer by using methods that they had already learned. "Let [the students] know that there's stuff you can't measure that you can find, and there's stuff you can measure that you can find to prove that it's right," he said, trying to convince the group to construct their lesson plan around various applications, such as measuring the length of a beam from a lighthouse and how far a ship drifts from its original place of anchor.

Patrick had a clear image of how he planned to start that involved each student constructing an equilateral triangle of any size and bisecting one of its angles to produce two 30-60-90 triangles: "I didn't do much of a lesson plan but I came up with a pretty good problem." He planned for students to calculate various ratios of the sides of one of their 30-60-90 triangles so students would notice that they each obtained basically the same ratios even though the triangles were different sizes. We thought Patrick's task for the students could be insightful into the structure of mathematics by revealing connections to similarity, but he seemed to overlook the physical materials it would take to create the geometric figures.

Mark, like David, fit his lesson plan into a generic outline that he had learned in his methods class. He explained that he would begin with problems to review the Pythagorean theorem and similarity of triangles and then introduce the definitions of trigonometric functions, stressing mnemonic devices. His emphasis seemed to be on quickly getting his students to a formalized understanding of the concepts. He approved of Patrick's suggestion for an opening investigation but wanted to make sure that it didn't take too much time: "Yeah, I mean that's fine with me. I'm just trying to figure out when to tell them the definitions. I like the idea, but let's just do one [ratio of side lengths]." After giving the students tips on how to remember the formulas for the trig functions, Mark's plan was to divide the students into groups to find the sine, cosine, and tangent of various angles. He did not mention the use of any physical materials in the description of his lesson plan.

Group Images

During an hour of planning, the group decided to use a modified version of Patrick's opening triangle task as the "class opener", restricting the number of ratios computed to one at the request of Mark. In addition, they worked through many of the details needed for implementation by discussing the physical materials and instructions that students needed to complete the triangle task. They also planned for the "teacher" to lead a brief discussion about the students' results of this task. In fact, after much discussion, Patrick suggested that they plan to "ask students why they think the ratios are the same," instead of just telling them why, which was advocated by David.

After the triangle task, they planned to follow up with a statement of the lesson's objective. They didn't discuss the details of this objective statement; they only discussed where it should be included in the lesson. Mark suggested that the definitions of three trigonometric ratios—sine, cosine, and tangent—be introduced next, with an opportunity for students to practice using the definitions with their own triangles created earlier and share their results at the board. After trying to get students to see the connection to proportions, as proposed by Patrick, the group planned to get students involved with better defined versions of some of the application problems, like the lighthouse problem, suggested by Craig.

Summary

Individually, these four prospective teachers seemed to be focused on very different aspects of the lesson-planning task prior to the group task. David was focused on the sequence of the mathematics, whereas Craig was focused on finding real-world applications. Patrick was primarily concerned with how the lesson should be started and Mark was concerned about quickly getting his students to a formalized understanding of the concepts. Although all four prospective teachers showed an interest in motivating their students, Patrick and Craig were the only two that came up with tasks that were potentially problematic. In addition, Craig was the only prospective teacher to generate some possible applications and the only one that seemed to attempt to use any of the available physical materials.

As a group, David, Craig, Patrick, and Mark planned a lesson that incorporated the images of each individual and some images made by the group. There were still some characteristics of tasks that facilitate mathematical understanding that were not addressed by the group plan, such as, the appropriate use of tools. However, the group effort incorporated tasks that have the potential for being both problematic (the lighthouse application) and insightful (the triangle task) and included many of the details of sequence and implementation. It was an advantage that each student's individual lesson plan addressed different aspects of the lesson-planning task and that each student had the opportunity to create his own image of what and how to teach prior to the group planning.

Implications and Future Research

A current challenge for mathematics teacher educators is to determine methods for effectively preparing prospective teachers to teach in ways they were never taught (Nicol, 1999). For this reason, some might argue that collaboration among prospective teachers, within the context of teaching tasks, could be detrimental by leading to additional support for traditional teaching strategies. However, the preliminary results of this study indicate differently. During the group planning session, there was evidence that individual images of traditional teaching strategies were challenged and possibly changed. We hypothesize that several factors significantly contributed to the effectiveness of the group planning session. In particular, the nature of the varying focuses of each prospective teacher when initially planning the lesson, the opportunity to create individual images of what and how to teach prior to group planning, and the willingness and ability to negotiate added to the effectiveness of this group.

Future research plans include observing these prospective teachers in practice and studying the work they produce throughout the remainder of their education program to more effectively understand their understanding of what and how to teach.

References

Berenson, S. B., Cavey, L. O., Clark, M. R., & Staley, K. (2001). *Adapting Pirie and Kieren's model of mathematical understanding to teacher preparation.* Paper presented at the Twenty-Fifth Conference of the International Group for the Psychology of Mathematics Education, Utrecht, Netherlands.

Berenson, S. B., & Cavey, L. O. (2000). *Thought and action in context: An emerging perspective of teacher education.* Paper presented at the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Tucson, AZ.

Davis, R. B. (1997). Alternative learning environments. *Journal of Mathematical Behavior,* 16 (2), 87-93.

Eisenhart, M. (1991). *Conceptual frameworks for research circa 1991.* Paper presented at the North American Chapter of Psychology of Mathematics Education, Blacksburg, VA.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Erlbaum.

Maher, C. (1998). Constructivism and constructivist teaching-can they co-exist? In O. Bjorkqvist (Ed.), *Mathematics teaching from a constructivist point of view. Reports from the faculty of education* (No. 3, pp. 29-42). Finland: Abo Akademi University.

National Council of Teachers of Mathematics. (2000). *Principles and standards for teaching school mathematics*. Reston, VA: The National Council of Teachers of Mathematics.

Nicol, C. (1999). Learning to teach mathematics: Questioning, listening, and responding. *Educational Studies in Mathematics, 37*, 45-66.

Oakley, J. A. (2000). *Voyage of discovery: What happens when inservice teachers explore teaching for understanding through collaborative investigation?* Paper presented at the American Educational Research Association, New Orleans, LA.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26*, 165-190.

Schoenfeld, A. (1999). Looking toward the 21st century: Challenges of educational theory and practice. *Educational Researcher, 28* (7), 4-14.

Sfard, A. (2000). Steering discourse between metaphors and rigor: Using focal analysis to investigate an emergence of mathematical objects. *Journal for Research in Mathematics Education, 31* (3), 296-327.

Simon, M. A., Tzur, R., Heinz, K., Kinzel, M., & Smith, M. S. (2000). Characterizing a perspective underlying the practice of mathematics teachers in transition. *Journal for Research in Mathematics Education, 31* (5), 579-601.

Steele, D. F. (1994). *Helping prospective teachers confront their conceptions about mathematics and mathematics teaching and learning.* Paper presented at the American Educational Research Association, New Orleans, LA.

LEARNING MATHEMATICS TEACHING

Brent Davis
Darren Stanley
University of Alberta

The Research Frame

In this paper, we discuss structures and strategies that are intended to support educators' efforts to teach mathematics differently—and, specifically, in ways that are informed by recent discussions in cognitive science, cultural studies, the history and philosophy of mathematics, and mathematics itself.

Conceptually, the work is oriented by accounts of cognition that are variously described as enactivist, ecological, embodied, and/or complex. Key influences to the work include Abram (1996), Bateson (1972, 1979), Lakoff (1987; Lakoff & Johnson, 1999; Lakoff & Núñez, 2000), and Varela (1999; Varela, Thompson, & Rosch, 1991). Among the important conceptual moves made within these discourses are (i) enlargements of the notions of *cognition* and *cognizing system/agent*, and (ii) conflations of knowledge, activity, and identity.

On the first point, in this writing, *cognition* is understood as an adaptive, evolutionary process through which a complex agent maintains an adequate or viable fit within a dynamic, evolving context. So framed, cognition is not seen to be located *in a body*, but as a process that defines a body. Cognition, that is, is understood as a global, coordinated or consensual activity of interacting agents that gives rise to some sort of coherent unity. Moreover, such coherence is not understood in terms of equilibrium. Rather, occurring as it does in an evolving context, cognition is more a matter of ongoing effective response to unrelenting disequilibrium.

The cognizing agent or unity need not be what is typically understood as a physical or biological body. Rather, the process of identifying these sorts of systems or bodies often involves a certain arbitrariness, as their boundaries tend to be diffuse, they might be nested in one another, and their activities can be mutually constitutive. Any complex, evolving form is a candidate for the title "cognitive system"—including, for example, a social corpus, a body of knowledge, the body politic, a species, or a planetary body. One might move in the direction of the microscopic as well. Various bodily subsystems, and subsystems of subsystems can be seen to obey the same complex dynamic and to be similarly organized as networks within networks. The brain, for instance, can be described in terms of systems embedded in systems, as neurons come together into minicolumns, minicolumns into macrocolumns, macrocolumns into cortical areas, cortical areas into Brodman Areas, and Brodman Areas into the cerebral hemispheres. (See Calvin, 1996.) Once again, cognition is not seen to occur or to emerge at any one of these levels in particular. Cognition, rather, is understood as the adaptive dynamic that occurs at and across each level—and, in fact, as the interactive dynamics that gives rise to the possibility of more complex activity at more global levels of (self-)organization.

Learning, in this frame, is about appropriate transformation—which is a statement about the physical character of a cognitive system. Upon learning, an agent's patterns of activity and its associations—internal and external, with and in other systems—undergo physical change. Put differently, learning is understood to affect not just the agent under scrutiny, but the entire web of being. From this it follows that *what one knows, what one does,* and *who (or what) one is* cannot be pried apart.

In the context of school mathematics, this conception of cognition prompts us, as best we are able, to attend simultaneously to a range of nested and co-implicated forms/events. For instance, as researchers we are keenly interested in the bodily basis of personal mathematics understandings and the recursive processes through which such understandings are elaborated, but we are also interested in the emergence of the consensual nature of the classroom microculture and the complex processes through which norms of activity are established and enforced. At the same time, we are attentive to the social context of the school and we try to be mindful of some of the cultural mythologies around formal education. As a strategy to bring focus to such simultaneous interests, we have chosen to concentrate on the teacher. We ask about, for example, the role of the teacher in framing students' conceptions of appropriate mathematical activity—and how, in turn, such conceptions figure into individual sense-making. And we also seek to understand the extent to which a teacher might hope to affect the various (and, oftentimes, competing) social projects that are at work in the classroom.

The Research Focus

In brief, we are interested in matters of how mathematics teachers' understandings of their roles—vis-à-vis their students, mathematics, society, the biosphere, and so on—are acted out in their classrooms.

We are not simply interested in polling opinions or measuring effects, however. Rather, following Varela (1999), we try to be attentive to the moral and ethical dimensions of any collective activity. We work from the premise that any such action, no matter how benignly intended, contributes to the phenomenon under investigation. For that reason, our research projects also involve deliberate challenges to teacher thinking.

This emphasis is mainly manifest in a focus in a specific category of activity: vocabulary. We are interested in habits of speaking and the ways that those habits support and compel other habits of acting. On that count, our investigations are oriented by Rorty's (1989) suggestion that "to change how we talk is to change what ... we are" (p. 20). As researchers, that is, we inquire into and aim to affect the ways that mathematics teachers talk about what they do and who they are.

Of importance, in this project, *language* is not understood principally in terms of a means to explicitly represent reality. Rather, the implicit or the tacit are what we find of greatest interest. We understand vocabulary more as a means of affecting reality than of capturing a presumed-to-be-language-independent-world. That is, we understand language to be knitted through our realities, not separate from them.

There are particular challenges that go along with the attempt to study such a fluid phenomenon as vocabulary. One cannot simply isolate single words or concepts, as these exist only in intricate webs of association. To press on one knot or to tug at one strand is to affect the texture of the entire weave—and so, one's focus of study is a constantly shifting target. We attempt to highlight some of this difficulty through one of the examples presented, below.

Before getting to that example, however, we feel it necessary to comment on the status of this writing. Consistent with the assertion that language is not principally a means of representing reality, the nature and status of a 'research report' must be correspondingly reframed. In particular, we are prompted to acknowledge that every observation is an interpretation. Observations, that is, are *partial*—in the double sense of fragmented part and of being screened by conscious and not-so-conscious biases. So understood, we believe there to be an imperative to be as explicit as possible about the convictions that give shape to our claims—a need that might involve a certain transgression of genres. In this project, a research report can very quickly begin to resemble a work of theory.

This point was not lost on a reviewer of our original PMENA proposal. Upon reading our 3-page submission, one person asked why this work was not submitted in the category of 'theoretical paper.' Even closer to home for us, we were recently compelled to revise a submission (for the study described below) to our university's Ethics Review Committee. Our original description of that investigation as "an unfolding conversation with teachers around transparent assumptions and enacted knowledge" did not fit with the committee's sense of legitimate inquiry. We were required to rewrite the application in terms of empirical data gathering techniques—in effect, to specify what we were doing in terms of what we were *not* doing. We were *not* videotaping, audio-recording, interviewing, transcribing, artifact gathering, and so on.

Somewhat paradoxically, this situation arose not because we had proposed something radically different, but because we had framed it a vocabulary that was too familiar. For instance, to the Committee's ears, our mention of "conversation" was heard as "interview," which meant that our proposal should have included details on specific protocols, management of the data, and so on.

We might have elected to work with the Committee to establish new ways of talking about research. We decided, however, to take the expeditious route and to avoid debates around the epistemological status of interpretive research. But our attitudes and interests are very different in our work with mathematics teachers. With them, engagements are all about cutting through the webs of association and assumption that give rise to such breaches in communication, as the following example is intended to illustrate.

The Research Context

Over the past decade, this program of research has been organized around a series of intertwining and overlapping micro-studies. These projects vary in duration, in context, in size, and in the ranges of interest, expertise, and experience among participants. One project, for instance, unfolded over three years and involved the entire staff of a small rural elementary school. Another study, of two-year duration, involved most of the teachers in an inner-city Toronto elementary school and was developed around a university program for graduate credit. Another—the one that is used as the focus of this report—began in the past year, and involves all of the mathematics teachers of a large middle school in a Los Angeles exburb. Others have involved cohorts of preservice teachers enrolled in special sections of courses on mathematics teaching. One such project, currently ongoing, involves preservice teacher candidates, practicing teachers, parents, and some community organizations.

Each project is (or was) structured around two main components, (i) researcher-led seminars and (ii) extended collaborative investigations. The seminars are structured principally around issues of learning and are explicitly focused on examinations of how emergent insights into cognition might inform mathematics teaching. Each seminar is also intended to provide a forum for participants to talk about their 'theories' of cognition—although moments of explicit theorizing tend to be rare. Most assertions about the natures of learning and thinking are embedded in incidental images and metaphors which are invoked in descriptions of classroom dynamics, rationales for mathematics teaching, educational dilemmas, and so on.

As for the extended collaborative aspects of the projects, these are generally framed by the specific preoccupations and worries of participants. Some have been driven by interests in specific curriculum topics, others by concerns over the diversities of abilities within single classrooms, others by desires to improve student performances on tests, and, most often, by individuals' hopes to be better mathematics teachers. Given such diversities of interest, the sorts of research interventions have varied considerably across settings, ranging from regular reading groups to intense joint teaching efforts.

For the purposes of brevity and focus, we report here on just one of these microstudies. Located in a Los Angeles exburb, this project was initiated by a group of middle school mathematics teachers, none of whom have any background in formal mathematics beyond their high school experiences. The student population of their school is predominantly Hispanic, mostly the children of first- or second-generation immigrant farm workers.

The collective project began with a day-long orienting session, during which the specifics of the teachers' invitation for our participation became clearer. Consistent with their colleagues in many other North American school jurisdictions, these teachers felt themselves trapped between two irreconcilable imperatives. On one hand, they sought to better understand the reasons behind and the practical implications of a district-mandated, experience-based, and group-oriented program of studies. On the other hand, this program was made to occur in the shadow of state-administered, performance-based, and individually-written examinations. In this conflicted space, many of the issues that frame popular debate in mathematics education seemed to be represented: rote mastery versus conceptual understanding, back-to-basics versus problem-solving, concrete versus abstract, and so on.

For our part, the opening day-long seminar was focused on rational number concepts (which was to be the focus of in-class instruction for the subsequent two weeks) and structured around such classroom-appropriate activities as paper-folding and fraction kits (see Kieren, Davis, & Mason, 1996). These activities were not simply intended as illustrative examples of the sorts of things one might do in a middle school classroom. From our perspective, the main reason for their introduction was to develop a context to announce some of the intertwining themes of our research. Most important, in their presentation, we endeavored to enact a fluid pedagogy—one that was attentive to happenstance and surprise, yet one that did not surrender the intended outcomes to accident. This emergent lesson dynamic, we hoped, could be used as a reference point to talk about other aspects of learning and teaching that obey a similar evolvement. Specifically, we wanted to create a context to introduce the complex and contingent natures of personal interpretation, collective knowledge, and social circumstance.

In the two weeks that followed this opening day, we worked in the teachers' classrooms in different capacities. For one block every day—at a time that all the participating teachers were able to come to the same classroom—we assumed the main teaching responsibilities. In a few others, we co-taught. In others, we participated as assistants and critical observers. Each day included two debriefing sessions, at lunch time and after school.

Structuring Problems

From the start, and across the various contexts of our interactions, we noticed a pronounced tendency among these teachers to describe their efforts in terms drawn from (or, at least, strongly influenced by) constructivist discourses. In the day-long orienting session, for instance, every one of them described their practices as "constructivist." As well, such assertions as "Learners construct their own understandings" and "Teachers have to provide learners with the tools to build on their own ideas" were spoken more than once. However, to our hearing, the teachers' uses of constructivist vocabulary were well removed from the epistemological developments that have driven the rise of constructivism to prominence.

On this issue, it was clear to us that ours and the teachers' understandings of constructivism varied dramatically—and were more or less split along the lines described by von Glasersfeld (1989) in his distinction of 'radical' and 'trivial' accounts. At first we paid relatively little attention to the issue, in large part because we had expected it to be the case. Our initial attentions were, rather, focused on the pre-selected topic of popular beliefs around the nature of mathematical knowledge.

A few days into project, however, we were compelled to address the matter of constructivist terminology more directly. One of the Grade 7 teachers, Lori, opened our daily debriefing meeting with an exasperated, "Why aren't they getting this?" Earlier, she had attempted to duplicate a lesson, taught by one of us, that she had observed the day before in a different classroom. To her surprise and frustration, despite her faithful adherence to the observed structure, her students simply did not demonstrate the sorts of insights and understandings that she had noticed in the earlier class.

We had a response. Although Lori's lesson plan followed a nearly identical sequence of activities, we suggested, the pedagogical structures differed significantly between the two classes. In the first instance, the teacher inserted himself into the action and allowed the lesson to flow around emergent insights, unexpected questions, and perceived needs. But that's not what Lori had observed. Rather, she had taken detailed notes on the sequence of events. As such, when it came to teach the lesson, she was only able to re-enact an explicit organization, not the implicit structure.

But this response seemed to fall mute, and prompted little response beyond Lori's defensive, "But I *did* follow the same structure!" Unsure how to proceed at that moment, we promised to return to the matter the next day after we had opportunity to think about it.

Somewhat ironically, as we left that de-briefing session, we heard ourselves ask precisely the same question of the teachers as Lori had asked of her students: "Why aren't they getting this?". It actually took us some time to realize that, as with our experience with the Ethics Review Committee, the problem lay not so much in the difference of represented perspectives but in similarity of transparent vocabularies. Specifically, our response had revolved around the phrase *lesson structure*, and we used it in a very different way that the teachers used it.

For the teachers, *structure* implied a rigid form, as in the 'structure of a building' or the imagined-to-be-pristine 'structure of mathematical knowledge.' The phrase 'lesson structure' was thus heard as 'lesson plan,' which in turn prompted associations of linear trajectories, pre-specified itineraries, and the like—a conception that diverts the teachers' attentions onto their own activities and away from student sense-making. Hence, when Lori observed the structure of the first lesson, she observed a sequence of discrete events.

Our intention with the use of the word *structure*, however, was to describe something more participatory and happenstantial. That is. we intended *structure* in much the way it is used in biology, ecology, and complexity theory, in reference to the ways that emergent forms collect together their histories of interaction. So framed, a 'lesson structure' is something that can only be described after-the-fact, when the unfolding has unfolded.

Prepared with this new insight, the next day we attempted to shift the conversation to varied meanings of *structure, construction*, and associated notions. We hoped that this topic might provide an effective entry point to a broader discussion of the complex, fitness-seeking, adaptive, evolutionary dynamic that, following recent developments in cognitive studies, we believe to be at work in events of learning. We had already introduced this idea of 'evolutionary drift' (Varela, Thompson, & Rosch, 1991) during the first day of the collaboration. It was used then as a means of describing and of noting the overlaps of such events as an individual's learning of a new idea, the unfolding of a lesson, the emergence of cultural knowledge, and the evolution of a species.

The de-briefing discussion, however, didn't go there. Quite the contrary, things quickly bogged down around the very notions—i.e., *structure* and *construction*—that we thought would enable a conversation. We and the teachers simply could not find a way of impressing the import of our thinking on one another. And, once again, the difficulty seemed to be all about a similarity of vocabulary.

This is not to say that the teachers could not hear that we were concerned about the usage of such terms as *structure* and *construct*. They caught that point. But we were completely unsuccessful in our efforts to communicate the problem with such terminology. Midway through the discussion, in an attempt to help things along, Lori asked if what we were talking about had anything to do with a paper she had read some years earlier during a district-sponsored inservice. Retrieving a copy from her filing cabinet, she opened it and read from a well marked paragraph:

> *Scaffolds* allow students a *framework on which* they can begin to *build their own knowledge* and provide help in *organizing their thinking*, with the goal of gradually *removing the scaffolding* and allowing full *ownership of the constructed knowledge*. ... [E]ven when the *proper balance* is struck by creating an *appropriate scaffolding*, there is a danger that it might become *a permanent fixture*. (Williams & Baxter, 1996, p. 23, emphasis added)

Contrary to Lori's intention, this quote was a conversation-stopper. For our part, we were unprepared and unable to respond to it in a manner that we felt would be meaningful to our collaborators. In fact, we were torn. On one hand, we might have pointed out that the authors of this citation did indeed seem to draw on the same learning theories that have influenced our thinking. On the other hand, we were troubled that the authors seemed to be entangled in their own web of literalized metaphors, unaware that they had translated fluid and contingent notions into hard and fast objects. In either case, there was a clear problem: The images and metaphors that had been adopted and adapted by constructivists to redescribe cognition are easily misconstrued. Even worse, deployed as they are in a context where cognition is overwhelmingly understood in terms of building an inner model of and outer reality, these images and metaphors actually seem *to compel* misconstruction.

What do you do, in a project oriented by an interest in 'how we speak,' when your collaborators make fluent use of the terminology to defend the very practices you had hoped problematize through that vocabulary? Our slow recognition of this dilemma prompted us to recall a line from Rorty (1999): "Inquiry that does not achieve coordination of behavior is not inquiry but simply wordplay" (p. xxiii).

This was the frame of our evening's discussion as we met to debrief the day's debriefing sessions. This time, however, we were not focused on the issue of clear definition, as we had been the day previous. Rather, we considered the possibility of introducing an entirely different vocabulary, one that did not rely on such overly determined notions as *structure* and *construction*. As it felt at that moment, we faced the prospect of having to start from scratch to invent or to highjack language to better communicate with these teachers.

We settled on a sort of compromise. Our response, we decided, would use the word *structure* as a sort of hinge. We would draw on some previous work into the history of the term (Davis, Sumara, & Luce-Kapler, 2000; Steffe & Kieren, 1994) to highlight contrasts and contradictions in current usages. Specifically, we decided to focus on a sort of bifurcation in popular meaning that occurred some centuries ago, at a time when attitudes toward architecture in the Western world underwent a dramatic shift. Early in the Industrial Revolution, buildings (and, with them, 'structures') came to be framed more in terms of premeditated, permanent, deliberate products than as contingent, evolving projects.

Our presentation the next day began with the point that *structure* shares its roots with such terms as *strew* and *construe*. When first used in English, according to the *Oxford English Dictionary,* the word was used to describe how things spread out or piled up in ways that cannot be predetermined, but that are not necessarily completely random either. *Structure*, that is, seems to have entered the language as a means to label complex unfoldings and emergent orders. This meaning is preserved in the biologist's and ecologist's use of the term. Such phrases as "the structure of an organism" or "the structure of an ecosphere" are references to the complex histories and dynamic characters of organic forms. *Structure*, in such cases, is both caused and accidental, both familiar and unique.

When the notion was first applied to buildings some centuries ago, it was at a time when such forms were not thought to be static or predetermined. Rather, they unfolded over years and decades as parts were added, destroyed, or otherwise altered. One built according to need, opportunity, or whimsy. The resulting edifices were thus not seen as permanent, but as evolving. That is, they were structures in the biological sense of ever-evolving forms. But the project of building changed in a newly industrialized and urbanized world. Further supported by scientifically reframed notions of predetermination (which had been prompted in large part by the demonstrated predictive power of Newton's mechanics), the popular definition of *structure* shifted toward the anticipated and fixed edifice.

These almost contrary meanings of *structure* and *construction* have confounded educational discourse since they were first taken up in earnest. As Steffe and Kieren (1994) point out, the terms had a conflicted introduction to discussions of mathematics education. From the beginning, in the early 1960s, their divergent meanings were conflated, owing to an inappropriate association of Piaget's notion of genetic structure and the formalist idea of the structure of mathematical knowledge. The former drew on the biologist's understanding of the term, the latter on the modern architect's.

We brought these ideas to the next day's discussion, during which we actually drew a chart to point to the point of bifurcation in the history of the word. In the process, we added a few terms that we felt to reflect the contrasts in meaning. On one side of the chart, for instance, we listed the words *strew* and *construe*, along with such terms and phrases as *emergent form*, *unfolding*, *dynamic*, *organic*, and *contingent*. On the other, we listed such related words as *construct*, *instruct*, and *destruct*, along with *scaffolds*, *frameworks*, *foundations*, *build*, *permanent fixture*, and other terms that had come up in the previous day's attempt to distinguish between our and the teachers' uses of these vocabularies.

It was a turning point—not only in the establishment of better communications between us and our collaborators, but also in terms of the teachers' appreciations for our preoccupation with habits of speech.

Almost without discernible rupture, the teachers adopted the more biological sense of *structure* to describe their actions and intentions in the classroom. The event might be described as a *collective accommodation* to the new use of a term — as opposed to the manner in which the term had previously been *collectively assimilated* into established patterns of thinking and acting. This is not to say, however, that there was some sort of grand epiphany. Our collaborators did not suddenly develop an ability to identify and critique their assumptions about how people learn, about what mathematics is, about why it is taught, and so on. The event was simpler, more subtle. People began to talk differently, but not unproblematically—as was indicated in a statement made by Lori: "Oh. So the lesson you structured really wasn't at all like the lesson I structured, even though they had the same structure."

Changing Lessons

Somewhat to our surprise, then, this important moment in the collective project was not around a deep insight into how people learn, but around a specific and almost mundane aspect of the teacher's role: the preparation of a lesson.

This realization actually came as a surprise to us. As noted above, we had framed our 'intervention' as an attempt to uncover and challenge teachers' beliefs about the nature of mathematics and the processes of learning. In effect, that is, we had ignored a lesson of our own theorizing—specifically, that such conceptual matters cannot often be front-loaded. In the same way that infants learn language as others assist in the interpretation of immediate experience, in the same way that children's mathematics learning is enabled if concepts are used to interpret events that have already occurred, the interpretive possibilities of we adults are anchored to experiential realities. Our accidental mooring of the notion of 'structure' to the activity of 'lesson planning' was powerful for this reason.

In terms of the research focus on vocabulary, in fact, the event has been of tremendous significance within the collective projects. In particular, it served as an occasion for teachers to unite in a critique of 'behavioral objectives,' 'anticipatory sets,' prescriptive lesson plan templates, and other technocratic impulses. Such constructs, they agreed, had little to do with the complexities of their tasks. Yet, to that point, they lacked a means of effective critique. The terminology that went along with a more organic sense of *structure* was clearly an enabler in this project.

For our part, we attempted to further the discussion by interjecting alternative metaphors for the task of lesson planning. In particular, using our own teaching interventions as illustrative examples, we offered two key notions: First, we suggested that lesson plans might be thought of as 'thought experiments'—that is, as occasions of thinking through some of the possibilities for particular activities with particular learners in particular contexts. The key quality of lesson planning, in this sense, is that it should support considerations of the dynamic and complex possibilities that might arise.

Second, we offered the notion of 'liberating constraints'—an idea that acknowledges the necessary act of imposing boundaries on student activity, but that seeks to enable possibilities within those limits. We offered the analogies of a script for a stage play, the legal and ethical codes that structure social interactions, and the environment inhabited by a species. In each of these cases, the creativity and originality of the performance, the culture, or the emergent ecology arises as a result of (and not in spite of) the defining constraints.

Both of these ideas prompted considerable discussion. In fact, they have become focal points in teachers' inquiries into possibilities for their classrooms. In our regular email correspondences with them, for example, lesson ideas are no longer assessed strictly according to the intended outcomes. Together, we also wonder about the range of mathematical ideas that might be addressed through specific activities—that is, as we have come to call it, the 'all-at-onceness' of these activities.

Of course, there continue to be some rather formidable obstacles to such discussions. In particular, at the moment of this writing (late in the school year), the looming presence of state-imposed standardized testing has taken center-stage. As researchers, we have been unable to offer an alternative frame for these tests. They remain 'constraining constraints,' limiters to innovation.

Given their relative fixedness on the landscape of teacher activity, though, with the teachers we have decided to make a concerted effort to talk differently about these examinations. Recently, for example, we've begun to explore the interpretive possibilities of casting such tests as opportunities for expanding *foregrounds*, rather than merely as events of measuring *backgrounds*. That is, the teachers have begun to make more concerted efforts to represent standardized examinations as opportunities for individuals rather than simply as obstacles—in effect, to push the phenomenon of the fixed examination into the space of emergent possibility, of *structure*, in the biological sense.

This shift in vocabulary seems to be having some effect, especially in the co-implicated spaces of teachers' and students' attitudes. As one teacher commented in a recent message, "I think that some of the kids are actually looking forward to writing the test!"

We do not mean to represent such shifts in attitude (toward standardized tests) as a major achievement. In fact, in many ways, we find them troubling. Regardless, however, they are good examples of the ways that deliberate attendance to how we talk can contribute to the possibilities for what we do and who we are as mathematics educators. At the very least, they serve as reminders that learning about mathematics teaching is never merely about deepened understandings of mathematical concepts or broadened insights into processes of cognition. Learning mathematics teaching is a complex phenomenon that stands in need of a complexified vocabulary.

References

Abram, D. (1996). *The spell of the sensuous: Perception and language in a more than human world*. New York: Pantheon Books.

Bateson, G. (1972). *Steps to an ecology of* mind. New York: Ballantine.

Bateson, G. (1979). *Mind and nature: A necessary unity.* New York: E. P. Dutton.

Calvin, W. H. (1996). *How brains think: A new scientific understanding of living systems.* New York: Doubleday Anchor.

Davis, B. (1996). *Teaching mathematics: Toward a sound alternative.* New York: Garland.

Davis, B., Sumara, D., & Luce-Kapler, R. (2000). *Engaging minds: Learning and teaching in a complex world*. Mahwah, NJ: Erlbaum.

Kieren, T. E., Davis, B., & Mason, R. T. (1996). Fraction flags: Learning from children to help children learn. *Mathematics Teaching in the Middle School, 2* (1), 14-19.

Lakoff, G. (1987). *Women, fire, and dangerous things: What categories reveal about the mind.* Chicago: University of Chicago Press.

Lakoff, G., & Johnson, M. (1999). *Philosophy in the flesh: The embodied mind and its challenge to Western thought*. New York: Basic Books.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Rorty, R. (1989). *Contingency, irony, and solidarity*. Cambridge, UK: Cambridge University Press.

Rorty, R. (1999). *Philosophy and social hope*. Toronto: Penguin.

Steffe, L. P., & Kieren, T. E. (1994). Radical constructivism and mathematics education. *Journal for Research in Mathematics Education, 25* (6), 711–733.

Varela, F. (1999). *Ethical know-how: Action, wisdom, and cognition*. Stanford, CA: Stanford University Press.

Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge, MA: MIT Press.

Williams, S. R., & Baxter, J. A. (1996). Dilemmas of discourse-oriented teaching in one middle school mathematics classroom. *The Elementary School Journal, 97* (1), 21–38.

CONCEPTIONS OF RATIO IN A CLASS OF PRESERVICE AND PRACTICING TEACHERS

Karen Heinz
Texas A&M University-Corpus Christi
karen.heinz@mail.tamucc.edu

Abstract: In this paper I characterize conceptions of ratio underlying preservice and practicing teachers' approaches to ratio tasks and discuss how those conceptions are related developmentally. I conducted a retrospective analysis of a 20-week whole-class teaching experiment in which the goal was to promote the teachers' conceptions of ratio. I formulated a construct—the identical groups conception—as a way to characterize the conception of ratio that seemed to be underlying the teachers' work. The identical groups conception is limited because not all ratio situations conform to the identical groups structure and hence the identical groups conception is not sufficient for making sense of all ratio situations. I contrast the identical groups conception with a more advanced conception of ratio that includes the understanding that a ratio is a single, *intensive* quantity that measures the multiplicative *relationship* between two quantities.

The development of the ability to reason multiplicatively is fundamental to students' mathematical growth (Heller, Post, Behr, & Lesh, 1990; Hoyles & Noss, 1989). It is necessary that teachers, those responsible for fostering students' growth, have at a minimum the understandings of the mathematics concepts that they are to teach (Simon & Blume, 1994b; Sowder et al., 1998; Thompson & Thompson, 1996). Therefore, it is crucial that teachers have an understanding of how to reason in multiplicative situations which includes an understanding of ratio (Post, Harel, Behr, & Lesh, 1988; Sowder et al., 1998). However, research indicates that teachers' abilities to reason multiplicatively are inadequate (Harel & Behr, 1995; Harel, Behr, Post, & Lesh, 1994; Simon & Blume, 1994b; Sowder et al., 1998), and that they have a difficult time acquiring an understanding of multiplicative relationships at the depth required for effective instruction (Simon & Blume, 1994b; Sowder, 1995). Efforts aimed at promoting development of teachers' understanding of multiplicative situations can be informed by research studies that investigate teachers' developing conceptions of ratio. In an attempt to contribute to the growing body of research focusing on teachers' understandings of ratio (cf. Harel & Behr, 1995; Post et al., 1988; Simon & Blume, 1994a, 1994b), I conducted a retrospective analysis of a whole-class teaching experiment involving preservice and practicing teachers. In this paper, I analyze those teachers' solutions to ratio tasks, present a hypothesis of the conception underlying their work, and contrast that conception of ratio with a conception of ratio as an intensive quantity (Schwartz, 1988), a measure of the multiplicative *relationship* between two quantities.

Conceptual Framework on Ratio

When making hypotheses regarding teachers' conceptions of ratio, it is useful to consider the constructs of extensive and intensive quantities introduced by Schwartz (1988) and the construct of ratio-as-measure introduced by Simon and Blume (1994b). An *extensive quantity* can be obtained from direct measurement—it indicates how much, or the extent, of a given quality of an object (e.g., 4 feet, 7 oranges). In contrast, an *intensive quantity* is a "statement of a relationship between quantities" (Schwartz, 1988, p. 43). An intensive quantity is derived from multiplicatively comparing two other quantities and as such quantifies a particular quality (e.g., speed as miles per hour).

Simon and Blume (1994b) used the term *ratio-as-measure* to refer to a ratio that is a measure of a particular attribute. For example, if the length of the base of a given ramp is 2 units and if the height is 3 units, then the ratio of 3 to 2, or 1.5, is a measure of the slope of not only that particular ramp, but of all ramps of that same slope. In general, the ratio of the height of a ramp to the length of the base of the ramp is a measure of a particular attribute (slope), and that attribute remains invariant for all ramps represented by a particular ratio. Below, I discuss an example of a ratio that created a perturbation for the teachers because it could not be thought of as a ratio-as-measure.

Method

This study was part of a larger research project, the Mathematics Teacher Development (MTD) Project. The overall goal of the MTD Project was to promote and study elementary mathematics teacher development. As part of the MTD Project, the participants—a combined group of preservice and practicing teachers—were engaged in a five-semester series of whole-class teaching experiments (Cobb, 2000). This study was a retrospective analysis of the ratio unit that spanned two semesters of that whole-class teaching experiment.

I conducted a line-by-line analysis of the videotapes of those classes along with full transcripts with the goal of investigating, characterizing, and contrasting the developing conceptions underlying the teachers' work on ratio tasks. I kept notes of key observations, current hypotheses, and remaining questions that I then referred to during subsequent analysis. As I proceeded with the data analysis, I reviewed the hypotheses generated up to that point and examined them in light of the new data. I then either revised or rejected existing hypotheses and generated new hypotheses as appropriate to account for all the data. When the hypotheses about the teachers' conceptions or the relationships between them seemed to hold up as compelling explanations for the data, I stated those hypotheses as claims. I then synthesized those claims into models of teachers' understandings.

Conceptions of Ratio Underlying Teachers' Work on Ratio Tasks

In this section I analyze data that seem to indicate that the teachers had a partial understanding of ratio, that is, the teachers' understandings enabled them to appropriately make sense of some ratio situations and yet their conceptions of ratio proved insufficient for making sense of other ratio situations. I postulate a new construct as a way to characterize this partial understanding of ratio. I also present and discuss data from classes in which teachers seemed to make a conceptual advance. Analyses of these data provide further insight into the more limited conception of ratio and how it differs from a more advanced conception.

As part of their work during the ratio unit, the teachers were asked to justify, in writing, their claim that a ratio was an appropriate way to measure "lemoniness," the relative strength of the lemon taste of two mixtures each containing specified amounts of lemon juice and lime juice. Sally's response was representative of many of the teachers' responses:

> We have been trying to determine if there is a great deal of lemon juice or a little bit of lemon juice. To make this type of determination, we also need to know about the amount of lime juice. We need to compare the amount of lemon juice to lime juice. By doing this, we end up looking at how much lemon juice we have compared to lime juice. A ratio allows us to mathematically set up the numbers in a way to do this.

In general, there were two related themes that ran throughout the teachers' responses. First, the teachers focused on the fact that one needed to know the amount of lemon juice *and* the amount of lime juice (or the amount of lemon juice and the total volume) in order to know the lemoniness of a particular mixture. Second, and more specifically, the teachers focused on the need to compare the concentration of lemon juice in each mixture because they understood that it was the particular combination of lemon juice and lime juice (i.e., concentration) that determined a particular lemon-lime flavor. The teachers justified that a ratio was appropriate for use because (a) a ratio allowed them to simultaneously take the two quantities of juices into account, and (b) a ratio is an expression of a specific combination of lemon and lime cubes that defines a particular lemon-lime flavor.

Another characteristic that figured into the teachers' decision to choose ratios as a way to compare the lemoniness of two mixtures is that equivalent ratios represent mixtures that are the same with respect to lemoniness, the quality of interest. When asked to state how they knew that equivalent fractions represent mixtures that taste the same, Ivy responded using a particular example:

> The 6/10, the 9/15 are just groupings of the 3/5. So you have the 3/5 taste and you put two groupings of the 3/5 or two sets of the 3/5 together and you get 6/10 but it's still [the same taste].

The teachers understood that if one started with a particular group of lemon and lime cubes and then either broke that group into smaller, identical groups, or accumulated multiple copies of that original group, then the lemoniness of the mixture created by each of those groups of cubes was the same. Furthermore, the teachers understood that if one wrote the two quantities in each of those groups in fraction form, then all of those fractions would be equivalent. Thus, they concluded that equivalent fractions represented mixtures that were the same with respect to lemoniness, the quality of interest. In other words, from the teachers' perspective, there was a match between the physical situation and the mathematical model of ratio with respect to invariance in the quality of interest—mixtures that had the same flavor were represented by equivalent ratios.

Based on these and other data, I formulated a construct—the identical groups conception—as a way to characterize the conception of ratio that seemed to be underlying the teachers' work. The identical groups conception includes the understanding that (a) in some situations the quality of interest is created by associating (grouping) two quantities, (b) the quality of interest as determined by the particular amounts of those two quantities remains invariant when that original group is partitioned into identical groups or when groups identical to that original group are collected, and (c) ratio is the mathematical model that labels and organizes that system of identical groups.

The teachers' responses to other ratio problems (e.g., the Field Watering Problem) provided further insight into their conceptions of ratio. "An irrigation tank holds water for watering 6 sports fields for 3 hours each. If the water in the tank were used for 10 fields, for how long could these fields be watered?" The teachers first solved this problem by dividing the product of 6 and 3 by 10 to conclude that each field could be watered for 1.8 hours. After further discussion, the teachers noted that the two ratios in the proportion $F_1:F_2 = T_2:T_1$ were equal (F and T represent the number of fields and the number of hours, respectively, in each situation). However, that proportion had no meaning to them. As Nevil stated,

> It's pretty nonsensical that we have any kinds of fields equaling any kinds of hours per field. . . . The way that I checked numbers is I divided the top one by the bottom one, and got .6. And I don't know *what*, .6 *what* [Nevil's emphases].

After much discussion and work on related tasks, some of the teachers developed an understanding of the meaning of the $F_1:F_2$ ratio. Ivy shared her newfound understanding. "So it's not representing really like the fields divided by the fields but rather the amount to which the grass of space is enlarged from the first time you water to the second time you water."

I propose that the key understanding that the teachers had to develop to make sense of the ratios in $F_1:F_2 = T_2:T_1$ was that a ratio is a measure of a *relationship*, that is, a ratio is not (simply) an association of two extensive quantities that together define a quality of interest. Rather a ratio is an intensive quantity (Schwartz, 1988)—a measure of the multiplicative relationship between two quantities. This particular understanding of ratio is not characteristic of an identical groups conception of ratio—it is a conceptual advance beyond the identical groups conception.

Discussion

Research indicates that incorrect addition strategies do not seem to disappear with maturation—a significant portion of high school students and adults continue to use an incorrect addition strategy in some situations that require multiplicative reasoning although they have the ability to reason multiplicatively as evidenced by their performance on other ratio tasks (Hart, 1988; Lesh, Post, & Behr, 1988; Simon & Blume, 1994b). The purpose of this section is to discuss the identical groups conception of ratio as distinct from a conception of ratio as an intensive quantity.

The addition strategy is appropriate for use in situations in which the quality of interest is determined by the amount that one quantity is in excess of another quantity. The development of an identical groups conception enables students to solve a wider range of ratio problems than an additive conception because the identical groups conception is the basis upon which students understand that some qualities of interest are determined by a particular combination of two quantities. That is, they understand that invariance in the quality of interest is maintained not when the same amount is added to or subtracted from two quantities but rather when an original group is partitioned into smaller, identical groups or when groups that are the same with respect to the quality of interest are combined. Thus, initially, covariation is perceived largely in terms of a process of a coordinated incrementing, decrementing, or partitioning of two quantities.

The identical groups conception is limited because not all ratio situations conform to the identical groups structure (e.g., the ratios in the field watering proportion). Therefore, students with only an identical groups conception of ratio, regardless of how proficient they become in maneuvering within the identical groups structure, will not be able to recognize ratio as an appropriate mathematical model for situations that require an understanding that a ratio is a quantification of the multiplicative relationship between two quantities, that is, that a ratio is an intensive quantity (Schwartz, 1988)—a single quantity that measures a *relationship*.

The development of the conception of ratio as a quantity subsumes and extends (i.e., supersedes) the identical groups conception. All of the understandings of ratio that are part of the identical groups conception of ratio (e.g., collections of an original group form new groups that are the same as the original group with respect to the quality of interest, equivalent ratios represent groups that are the same with respect to the quality of interest) are still available and seen as appropriate by those with a conception of ratio as a quantity. The key difference is that once students develop a conception of ratio as a quantity, their identical groups conception of ratio is transformed and restructured to include an understanding that ratio situations are those in which the relevant relationship is the invariant multiplicative relationship that exists between the two relevant quantities.

References

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307-333). Mahwah, NJ: Erlbaum.

Harel, G., & Behr, M. (1995). Teachers' solutions for multiplicative problems. *Hiroshima Journal of Mathematics Education*, *3*, 31-51.

Harel, G., Behr, M., Post, T., & Lesh, R. (1994). The impact of the number type on the solution of multiplication and division problems: Further investigations. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 363-384). Albany, NY: SUNY Press.

Hart, K. (1988). Ratio and proportion. In M. Behr & J. Hiebert (Eds.), *Number concepts and operations in the middle grades* (pp. 198-219). Reston, VA: National Council of Teachers of Mathematics.

Heller, P. M., Post, T., Behr, M., & Lesh, R. (1990). Qualitative and numerical reasoning about fractions and rates by seventh- and eighth-grade students. *Journal for Research in Mathematics Education, 21*, 388-402.

Hoyles, C., & Noss, R. (1989). The computer as a catalyst in children's proportion strategies. *Journal of Mathematical Behavior, 8*, 53-75.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93-118). Reston, VA: National Council of Teachers of Mathematics.

Post, T., Harel, G., Behr, M., & Lesh, R. (1988). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 177-198). Madison, WI: Center for Education Research.

Schwartz, J. (1988). Intensive quantity and referent transforming arithmetic operations. In M. Behr & J. Hiebert (Eds.), *Number concepts and operations in the middle grades* (pp. 41-52). Reston, VA: National Council of Teachers of Mathematics.

Simon, M. A., & Blume, G. W. (1994a). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education, 25*, 472-494.

Simon, M. A., & Blume, G. W. (1994b). Mathematical modeling as a component of understanding ratio-as-measure: A study of prospective elementary teachers. *Journal of Mathematical Behavior*, *13*, 183-197.

Sowder, J. (1995). Continuing the mathematical preparation of middle-grade teachers: An introduction. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a* *foundation for teaching mathematics in the middle grades* (pp. 1-11). Albany, NY: SUNY Press.

Sowder, J., Armstrong, B., Lamon, S., Simon, M., Sowder, L., & Thompson, A. (1998). Educating teachers to teach multiplicative structures in the middle grades. *Journal of Mathematics Teacher Education*, *1*, 127-155.

Thompson, A. G., & Thompson, P. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, *27*, 2-24.

USING METHODS QUESTIONS TO PROMOTE MATHEMATICAL UNDERSTANDING IN A PROBABILITY COURSE FOR TEACHERS

Todd M. Johnson
Eastern Washington University
Todd.Johnson@mail.ewu.edu

Abstract. In this study, having preservice teachers write about misconceptions that teachers might encounter when working with students was used successfully to develop preservice elementary teachers understanding of probability.

The Problem

Many preservice elementary teachers are unable to apply the content taught in mathematics courses for elementary teachers. Preservice elementary teachers in a mathematics methods class I taught claimed they retained little of the content taught in the prerequisite mathematics courses for elementary teachers. The following samples from the pretest given in the mathematics methods class are specific examples of the problem.

> *Example One*: When asked to identify the rational numbers in a list of numbers, 18 of the 21 methods students identified $\pi/2$ as a rational number.
>
> *Example Two*: When asked the following question 13 of the 21 methods students answered the question incorrectly. George bought his new computer at a 12% discount. He paid $700. How many dollars did he save by buying it at discount?
>
> *Example Three*: When asked to explain which is larger 0.41 or 0.411, 15 of 21 methods students explained why 0.41 was larger than 0.411. In addition to the expected incorrect explanations based on the idea that thousandths are smaller than hundredths, students argued that 0.411 was smaller than 0.41 because (a) 0.411 was closer to 1 or (b) .411 was farther to the left of zero.

Preservice teachers' inability to retain and apply mathematics taught in mathematics classes may be explained in part by how the mathematics was taught. Mathematical knowledge is not abstractable from one situation (i.e., a content course) and easily transferable to different situations (i.e., a methods course, an elementary classroom). Knowledge is situated in the context in which it is developed and used (Brown, Collins, & Duguid, 1989; Lave, 1991). Unfortunately, when preservice teachers learn abstract concepts independent from the situations in which the concepts are used, applicable knowledge is not developed. One alternative to traditional teacher education would be to develop understanding through situated use.

Incorporating the use of Professional Development Schools in teacher education programs can be thought of as one attempt to develop the knowledge of preservice teachers in the context in which they will use the knowledge. Unfortunately, the use of Professional Development Schools is not an option for many teacher education programs due to a large enrollment of preservice teachers, the limited number of K-12 schools near the university, and/or bureaucratic constraints. Without access to Professional Development Schools, teacher education programs must create other means to develop in preservice teachers knowledge that is useable. This paper reports the use of misconceptions that teachers might encounter when working with students as a meaningful context to promote preservice teachers' understanding of probability.

Perspective

Based on the belief that writing has a special capacity for promoting learning by forcing personal involvement with the material and the integration of ideas, teachers have been encouraged to have students write in all content areas (Emig, 1977). However, the privileged status writing has been given as a medium for mediating thinking and promoting learning has been challenged. Smagorinsky (1995) argues that writing like any tool can provide a learner the potential to construct meaning through mediated thought and activity; however, writing can only serve a mediational purpose when the confluence of writer and situation makes it valuable as an appropriate means of mediation. In this study writing about misconceptions that teachers might encounter when working with students was considered as an appropriate tool to promote learning in a probability class for preservice teachers. Having preservice teachers write about misconceptions that teachers might encounter when working with students was intended an *authentic task*--tasks that would facilitate the construct of knowledge through mediated thought and activity in a meaningful context (Newmann, Secada, & Wehlage, 1995).

Subjects

Two intact classes of probability for undergraduates preparing to be elementary teachers with a specialization in mathematics participated in the study. A professor awarded tenure for his exemplary teaching and his research of the teaching and learning of probability taught the control class. The author of this paper taught the experimental class.

Class Assignments

Investigating Probability and Statistics (Jones, Rich, Thornton & Day, 1996) was the textbook used in the control class and in the experimental class. Assignments in the experimental class included having students write complete answers to probability problems. Complete answers were defined in the experimental class as answers that were mathematically correct, justified mathematically, and included a student error that teachers might encounter when working with students. Homework assigned in the experimental class included problems used in educational research (e.g., Kahneman & Tversky, 1972; Fischbein & Schnarch, 1997) to assess students' understandings of probability.

Emphasis was placed on identifying errors that are based on common misconceptions described in educational research. Two such misconceptions are *representativenes*--the belief that a sample should reflect the distribution of the parent population or mirror the process that generated the random event—and *availability*—estimating the likelihood of an event based on how easy it is to recall a specific instance of the event (Shaughnessy, 1992).

Sample Homework Question. In a lotto game, one has to choose 6 numbers from a total of 40. Pat has chosen 1, 2, 3, 4, 5, and 6. Robin has chosen 39, 1, 17, 33, 8, and 27. Who has a greater chance of winning?

Preservice Teacher:	They have the same chance of winning. There is only one way to pick the numbers {1, 2, 3, 4, 5, 6}. There is only one way to pick the numbers {1, 8, 17, 27, 33, 39}. The probability that either set of numbers is chosen is the same. Some people may think the numbers {1, 8, 17, 27, 33, 39} are more likely because the chance of choosing unconsecutive numbers is greater than the probability of choosing consecutive numbers.

Sample Homework Question. When tossing a fair coin, there are two possible outcomes: either heads or tails. Chris flipped a fair coin three times and in all cases heads came up. Chris intends to flip the coin again. What is the chance of getting heads a fourth time?

Preservice Teacher:	The chance of getting a heads on the next flip is 1/2, because the only two possible outcomes are to get a head or a tail. Since each of these two outcomes has an equal chance of happening, the chance that one of them would happen (heads) would be one of two, or 1/2. A student might have interpreted the question as, "What are the chances of flipping the coin four times and getting a heads every time?" then the answer would be $(1/2)^4 = 1/16$.

Sample Homework Question. Is the likelihood of getting at least two heads when tossing three coins smaller than or greater than the likelihood of getting at least 200 heads when tossing 300 coins.

Preservice Teacher: They both have the same chance at getting at least 2/3 of the coins to land heads. The probability of getting heads is the same, 1/2. So each time you flip, neither way has a greater chance of getting 2/3 heads. After reviewing my answer, I noticed my first explanation is wrong. It is more likely to get at least 2/3 heads with 3 coins. The more trials you have the less chance you have of getting at least 2/3 heads. This is because the more trials you have (the more times you flip the coin) the closer you will come to the theoretical probability of 1/2 of the coins landing heads. I used an example with 6 flips instead of 300 so I could compare easier. If you add up the probability of having at least 2/3 heads for 3 flips, you get 3/8 + 1/8 = 4/8 or 32/64. If you add up the probability of having at least 2/3 heads for 6 flips, you get 15/64 + 6/64 + 1/64 = 22/64. 32/64 > 22/64.

In the control class, students were also expected to write answers to probability questions that included mathematical justifications for their answers. However, students in the control class were not expected to describe misconceptions that teachers might encounter when working with students.

Instrumentation

The pretest and the posttest were tasks developed and used by Jones, et al. (1997) to assess students' understanding of probability. The pretest was administered during the first day of the control class and the first day of the experimental class. The posttest was conducted during the last day of each class.

Pretest Task. Sailboats numbered 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 are going to race. When you roll two dice, the sailboat whose number matches the number rolled moves forward one space. (e.g., If a sum of three is rolled, sailboat number 3 moves forward one space. If a sum of eight is rolled, sailboat number 8 moves forward one space.) The first sailboat to move 12 spaces wins the race. Which sailboat do you think will win the race? Explain your answer. (Van Zoest & Walker, 1997)

Preservice Teacher: Sailboat #7. There are 6 different combinations of dice rolls that equal 7: 6 and 1, 1 and 6, 2 and 5, 5 and 2, 3 and 4, 4 and 3. All of the other sailboats have less than 6 number combinations. Even numbers have an odd number of combinations because the same 2 odd numbers can be added together to get the number, such as 6 can have 3 and 3. The same numbers both ways.

Posttest Task. You enter the following maze (Figure 1) and flip a coin at each junction. If the coin lands heads you turn left. If the coin lands tails, you turn right. You can never turn back. From which exit are you most likely to leave, or are they equally likely? (Jones, Langrall, Thornton, & Mogill, 1997)

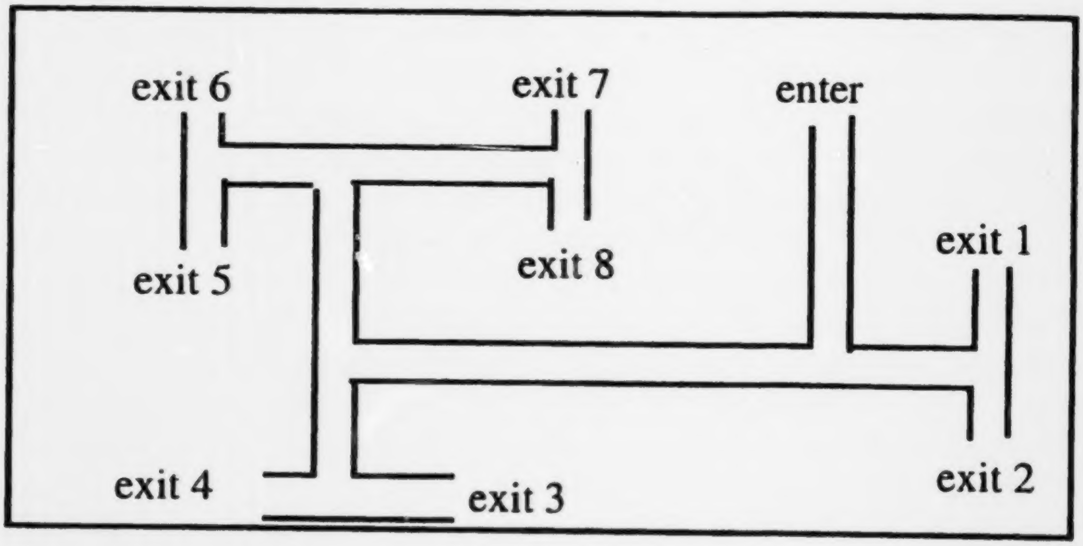


Figure 1. Posttest task.

Preservice Teacher: You are most likely to leave through exit 1 or exit 2. Both have a 25% chance because it only takes 2 flips to leave through these exits. Exit 3 and exit 4 both have a 1/8 chance. Exit 5, exit 6, exit 7, and exit 8 all have 1/16 chance. A person may think exit 2 and exit 5 have the best chance because they have an equal number of heads and tails flipped in order to exit there.

Results

The control group did better than the experimental group on the pretest. In the control group 14 of 23 students gave the correct answer—Sailboat 7. Nine of the students in the control group incorrectly identified Sailboat 6, Sailboat 7 and Sailboat 8 as the most likely boats to win the race. In the experimental class 8 of 19 students gave the correct answer—Sailboat 7. Six of the students in the experimental class incorrectly identified Sailboat 6, Sailboat 7 and Sailboat 8 as the most likely sailboats to win the race. In the experimental class five students reported unique incorrect answers. A Chi-square test for homogeneity on pretest responses for the control group and the experimental group was statistically significant (10.56 > 9.12 Chi-square df = 2, alpha = .01).

Although the control group did better than the experimental group on the pretest, the experimental group scored better than the control group on the posttest. During the experiment, two students in the control group withdrew from the course after receiving low quiz scores. Of the remaining 21 students, 15 students correctly identified Exit 1 and Exit 2 as the most likely exits. Six students in the control group reported other exits were most likely. All 19 students enrolled in the experimental course correctly identified Exit 1 and Exit 2 as the most likely exits. A Chi-square test for homogeneity on posttest responses for the control group and the experimental group was statistically significant (7.5 > 6.63 Chi-square df = 1, alpha = .01).

Discussion

As expected, answers by preservice teachers in the experimental class included misconceptions that teachers might encounter when working with students and answers by preservice teachers in the control class did not include misconceptions that teachers might encounter when working with students. Preservice teachers provided answers that met class norms. However, in addition to providing misconceptions that teachers might encounter when working with students, the experimental group did significantly better than the control group on the posttest despite the fact that the experimental group did significantly poorer than the control group on the pretest.

One explanation for the improvement exhibited by the experimental group is that having preservice teachers write about misconceptions teachers might encounter when working with students was a meaningful context for preservice teachers to study probability. Writing about probability problems in a meaningful context allowed for mediated thought and action. Teachers of other mathematical content courses for preservice teachers may be interested in having preservice teachers write about misconceptions that teachers might encounter when working with students. Writing about misconceptions that teachers might encounter when working with students could be a meaningful context to facilitate preservice teachers' understanding of the mathematical content.

This study suggests that having preservice teachers write about misconceptions that teachers might encounter when working with students may assist preservice teachers' understanding of mathematical content. However, teacher educators need to remember preservice teachers may not be accustomed to writing in mathematics classes. In addition, preservice educators who have become efficient at performing roles in traditional classrooms may have trouble understanding experimental instructional practices such as writing in a mathematics course. Therefore, teacher educators who intend to have preservice teachers write in a mathematics course need to prepare the preservice teachers for writing in class. Sharing one's rationale for having the preservice teachers write in a mathematics class is one means to motivate writing.

Related Work

Currently I am teaching mathematics methods courses for elementary teachers. I am investigating the use of writing and the following question format to develop preservice teachers' content knowledge and usable pedagogical content knowledge.

Sample Problem. *Directions*: For the following question if the student's response is correct, explain how the student may have gotten the answer. If the response is incorrect, identify:

a) The correct answer with mathematical justification.

b) A key mathematical concept that the student may have misunderstood.

c) What a teacher should do to help the student develop understanding of the key mathematical concept.

Question: All families of four children in a city were surveyed. In 16 of these families, the exact order of births of boys and girls was Girl, Boy, Girl, Boy. What is your estimate of the number of families surveyed in which the exact order of birth was Girl, Girl, Girl, Girl?

Student Response: The answer is 1.

Preservice Teacher: a) There should be about 16 families with exactly 4 girls. The probability of a family with 4 children having a Girl, a Boy, a Girl, and then a Boy is the same as the probability of a family with 4 children having 4 girls. The probability of each specific sequence of children is the same (1/2)^2 or 1/16.

b) The student may have confused that the probability of having 2 boys and 2 girls is greater than the probability of having 4 girls with the idea that each specific outcome of four children is equally likely.

c) I would have the student make a tree diagram to see that there are 6 ways to have 2 boys and 2 girls. 1 - G, B, G, B; 2 - B, G, B, G; 3 - B, B, G, G; 4 - G, G, B, B; 5 - G, B, B, G; and 6 - B, G, G, B. Each specific outcome happens 1/16 of the time which is the same as the probability of a family with 4 children having 4 girls.

References

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, January, 32-42.

Emig, J. (1977). Writing as a mode of learning. *College Composition and Communication*, *28*, 122-128.

Fischbein, E., & Schnarch, D. (1997). The evaluation with age of probabilistic, intuitively based misconceptions. *Journal for Research in Mathematics Education*, *28*, (1) 96-105.

Jones, G. A., Rich, B., Thornton, C. A., & Day, R. (1996). *Investigating probability and statistics*. Menlo Park, CA: Addison-Wesley.

Jones, G. A., Langrall, C. W., Thornton, C. A., & Mogill, A. T. (1997). A framework for assessing young children's thinking in probability. *Educational Studies in Mathematics*, *32*, 101-125.

Kahneman, D., & Tversky, A. (1972). Subjective probability: A judgment of representativeness. *Cognitive Psychology*, *3*, 430-454.

Lave, J. (1991). *Cognition in practice*. Cambridge: Cambridge University Press.

Newmann, F. M., Secada, W. G., & Wehlage, G. G. (1995). *A guide to authentic instruction and assessment: vision, standards and scoring*. Madison: WI. Wisconsin Center for Educational Research.

Shaughnessy, J. M. (1992). Research in probability and statistics: Reflections and directions. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 465-494). New York: Macmillan.

Smagorinsky P. (1995). Constructing meaning in the disciplines: Reconceptualizing writing across the curriculum as composing across the curriculum. *American Journal of Education*. February, 160-185.

Van Zoest, L. R., & Walker, R. K. (1997). Racing to understand probability. *Mathematics Teaching in the Middle School*, *3*, 162-170.

BUILDING ON EACH OTHER'S IDEAS: K-3 TEACHERS' MATHEMATICS LEARNING

Ok-Kyeong Kim
University of Missouri- Columbia
ok182@mizzou.edu

Joy Whitenack
University of Missouri- Columbia
whitenackj@missouri.edu

The current mathematics reform movement requires forms of mathematics teaching that are different from traditional ways (National Council of Teachers of Mathematics, 1989, 1991, 2000). For new forms of teaching mathematics, teachers should not only know pedagogical issues related to the teaching of mathematics but also have a deep understanding of the important mathematics they teach. To understand students' ways of doing mathematics, to use appropriate mathematical representations, to facilitate students' discussions, to work with students as a guide or co-worker, and to encourage students' autonomy in learning mathematics, teachers should understand key pieces of mathematical knowledge and connect them in meaningful ways. In doing so, teachers need to have opportunities to solve problems and to explore mathematical ideas. In addition, they need to have opportunities to discuss the problems and their problem solving in a way that their students do in their classroom. Sharing each other's problem solving strategies is one way of building an understanding of mathematical ideas. While expressing one's ideas encourages one to reflect and clarify one's own thinking, listening to others' ideas inspires new ways of thinking and reasoning with mathematical ideas.

This report will describe individual and collective teachers' mathematics learning through math share as they participate in a K-3 teacher professional development project during the 2000-01 academic year. The project was a collaborative effort between a university and a school district. The project provided opportunities for 36 K-3 teachers not only to develop and refine their expertise as teachers of mathematics but also to explore mathematical topics that underpin the mathematics their students learn.

Theoretical Framework

Educational researchers need an appropriate theoretical framework to look at and interpret the process of the teachers' professional development (Simon, 2000). This report is part of a study that describes individual teachers' changes in their knowledge of mathematics and mathematics teaching through the teacher professional development project. The changes occur in individuals' minds, yet are situated in a communal context. Individuals develop the culture of their community and the community constraints the individuals' development. On the one hand, individuals (teachers and teacher educators) co-construct normative ways to participate in the professional development activities. On the other hand, the community in which the participants construct constrains the types of professional development opportunities. Thus, individuals cannot be considered separately from their social practices in a community. As such, we need to look at not only individual teachers' professional development but also those social interactions that provide opportunities for individual teachers to change their practices. Thus, the theoretical lenses that we use to observe and interpret teachers' learning through professional development activities must consider both individual and collective perspectives. Using the coordinated perspective looking at both individual and collective aspects, this report provides an account of how individual teachers develop their understanding of mathematical ideas and how the teachers build shared understandings of mathematical ideas and the teaching of mathematics. Eventually, this approach will clarify how interactions among the teachers influence individual teachers' mathematical thinking and ways of teaching mathematics.

Research Design

This report is an account of individual teachers' understandings of mathematics and the teachers' shared understandings of the important mathematical ideas and mathematics pedagogy. To capture the shared understandings, ethnographic techniques were employed. The approach is most appropriate because every group that is together for a period of time with common goals evolves into a culture. Patton (1990) states that ethnography whose central idea is culture has become an approach to program evaluation because "programs develop cultures" and "the program's culture can be thought of as part of the program's treatment"(p. 68). The shared understandings are part of culture that the teachers developed through project activities for a year. By looking at the shared understandings, this report can provide the characteristics of the project activities that potential led to teachers' professional development. As such, this study primarily employed ethnographic methods, such as participant observation and intensive fieldwork, that yielded a prolonged, systematic, in-depth study.

Data Collection and Analysis

Using ethnographic methods, this study primarily employed observations to collect data. We also collected and analyzed other types of data including brief reflections after each class (quick checks) and the teachers' by-weekly written reflections.

The analysis was begun with an open coding technique and was continued to compare data and categories and rearrange categories to make connections with respect to core categories (Strauss & Corbin, 1990). Based on the theoretical framework of the study, the units of analysis were not only individual teacher teachers' understanding of mathematics and pedagogy but also collective teachers' interactions. Once we analyzed individual teachers' conceptions, we made connections among them in order to find shared understandings of mathematical ideas and for the teaching of mathematics.

Participant observations

One of the researchers made participant observations of teachers as they engaged in course activities such as math problem solving and discussions regarding mathematics pedagogy. The researcher took field notes to document the teachers' participation during the class meeting times. To supplement the field notes, each activity was audio-taped. Field notes and transcripts of audio-tapes of course observations were analyzed to look at individual and collective teachers' understandings of mathematics and mathematics pedagogy.

Quick checks and reflections

We collected project teachers' written reflections and quick checks. During the course period, the teachers wrote by-weekly reflections related to issues that surfaced during class discussions about teaching children mathematics and their own problem solving. The teachers also wrote brief reflections (quick checks) after each class related to issues raised during the class. The analysis of the reflections leaded to explanations of how each teacher thought about their problem solving, their understanding of the mathematics, and important issues related to teaching mathematics. As a result of the analysis, common themes emerged across individual responses.

Results

The analysis of data revealed that teachers' understanding of mathematics was prompted by working with others and sharing each other's ideas. The Math Share time, which was a follow-up whole class discussion occurred after they solved problems, gave the teachers opportunities to build their understanding of mathematics based on each other's ideas about mathematical topics. Often teachers needed more time than planned to solve problems on their own, or to make sure what others thought about the problems. Even though the course instructors did not directly teach a certain concept or strategy, in most cases the teachers came up with other possible ways of solving problems when they engaged in math share. Whereas a few of the same teachers often provided an idea, different teachers played an important role during discussions.

An example

One day the teachers solved the following problem: "Would you rather work seven days at $20 per day or be paid $2 for the first day and have your salary double every day for a week?" The problem seemed easy for them and every teacher got the right answer. The more important, significant event, however, occurred during the Math Share. Typically, the teachers solved the problem by computing each salary and then and comparing the two total salaries. While talking about the problem during the Math Share, the teachers began to explore the relationships between the two different number sequences: 20, 20, 20, 20, 20, 20, 20, … and 2, 4, 8, 16, 32, 64, 128…. Initially, most of the teachers had no idea how to compare the two sequences.

One teacher suggested graphing the information. This suggestion instigated further discussion. Finally, two different types of graphs were visible on the board that showed the dramatic difference after the fifth day. As the discussion continued, another teacher shared how she represented the total salary in the second case: n+2n+4n+8n+16n+32n+64n+..., where n=1st day salary. The equation encouraged the other teachers to think about how to represent the number sequence. They examined the number sequence in several ways. One teacher (Teacher A) shared her equation that represented the sequence: $2(2^n)-2$. This formula made other teachers raise a question: "Where does that come from?" Each teacher started to think about what each number represented. The teachers decided the formula was derived by doubling the previous term and noticing the difference between the n-th term and the sum up to the (n–1)th term, which was 2. According to Teacher A, each day the total salary was double of the total salary the day before minus 2. At this point in the discussion, teachers wanted to explore what would happen with 3's (they investigated two different cases – doubling of each day starting with 3 and tripling of each day starting with 3). Even though the teachers could not find an appropriate mathematical representation for each case due to the lack of time, most of them were excited with their explorations of number patterns. One teacher wrote in her quick check that "the problem seemed to come together too easily. ... I really enjoyed when the problem branched out and went deeper and challenged my thinking." Some teachers reflected that they started to really understand the mathematics.

The Math Share gave teachers opportunities to share their own ideas and to listen to others' ideas. In particular, math share made unconfident teachers relax and listen to important ideas, and finally build their own understanding of mathematical ideas. Through the Math Share time, the teachers not only had opportunities to build their own knowledge of mathematics, but also could relate their experiences to think about children's ways of solving problems. In particular, during discussions they realized the importance of clarifying, challenging, and stretching each other's thinking and thought about how to help their students build on each other's ideas. Consequently, the teachers' math share activities made them prepare for a better teaching of mathematics in their classroom.

Conclusion

With an example of teachers' mathematics learning, this report addresses the processes of individual and collective teachers' professional development for the teaching of mathematics. In addition, this study explores the relationship between individual teachers' mathematical exploration and mathematics teaching and collective teachers' shared understandings of mathematical ideas and mathematics teaching as their culture occurred during the project. As we continued to analyze data, we began to confirm the effectiveness of the methods of the program. In turn, theoretically we clarified the usefulness of using the coordinated perspective to understand and account for individual and collective teacher professional development.

References

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Patton, M. Q. (1990). *Qualitative evaluation and research methods* (2nd ed.). Newbury Park, CA: Sage.

Simon, M. A. (2000). Research on the development of mathematics teachers: The teacher development experiment. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 335-364). Mahwah, NJ: Erlbaum.

Strauss, A. L., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory, procedures and techniques*. Newbury Park, CA: Sage.

812

QUESTIONING STRATEGIES IN TWO MIDDLE SCHOOL MATHEMATICS TEACHERS

Karen Koellner Clark
Georgia State University
Kkoellner@gsu.edu

Roberta Y. Schorr
Rutgers University
Schorr@rci.rutgers.edu

Abstract: This study investigated the use of questioning strategies by in-service teachers. We were interested in learning more about how teachers recognize the need to develop beyond their typical style of questioning toward questioning strategies that are more focused on having students describe, explain, and justify their thinking. Results indicate that some of the catalysts for these transitions occurred when the teachers were (a) actively focusing on the questions they were asking as they implemented problem activities in their own classrooms, (b) listening to how students were thinking about mathematical ideas as they solved the problems, (c) writing daily reflections about their questioning, (d) examining and reacting to their own and their colleagues questioning as they viewed video clips, and (e) collaboratively identifying effective questions for particular situations. Results suggest that all teachers changed their questioning strategies. As one would expect, they changed in different ways and to different extents. We believe that this research has important implications for researchers and teacher educators who are interested in helping teachers implement reform oriented instructional practices that promote thoughtful classroom discourse.

Introduction and Theoretical Framework

Current reform efforts in mathematics education continue to emphasize discussion and mathematical communication in the mathematics classroom (NCTM, 2000). This study investigates the role of teacher questioning in encouraging substantive discourse and creating a classroom environment where students develop logical and persuasive predictions, explanations, and justifications; analyze or critique solutions and, consider the ways in which they could refine, revise, and assess their own work. In this paper, we will be considering the types of questions that teachers pose to students, and how that develops over time. In order to consider this, we begin by exploring some of the different types of questions that teachers tend to ask. For example, Hiebert and Wearne (1993) refer to four different types of questions that teachers tend to pose:

1. Recall—questions in this category tend to focus on asking students to recite previously learned or currently available facts; prescribe previously learned rules, or recall a previously discussed topic.
2. Describe strategy—questions in this category tend to focus on asking students to tell how they solved a problem, or describing another way to solve the same problem.

3. Generate problems—questions in this category tend to focus on having students create a story to match a number sentence or create a problem to fit given constraints.
4. Examine underlying features—questions in this category focus on explaining why a procedure is chosen or why it works, or considering the nature of a problem or a solution strategy.

Hiebert and Wearne (1993) do not suggest that classroom questions can always be categorized in an unambiguous manner. Indeed, questions that are framed in a particular manner or style can lead to entirely different types of discussions depending upon the circumstances. Therefore, in this research, we were not exclusively interested in the specific questions that were posed. Rather, we were interested in the cognitive activity that was elicited as a result of the questions and how teachers began to use questions as means by which to stimulate rich classroom discussion.

Our analysis of how teachers progressed was considered in terms of "models" and "modeling cycles" (Koellner-Clark & Lesh, in press; Schorr & Lesh, in press). Briefly, a model can be considered to be a way to describe, explain, construct or manipulate a complex series of experiences. Models help us to organize relevant information and consider meaningful patterns that can be used to interpret or reinterpret hypotheses about given situations or events, generate explanations of how information is related, and make decisions about how and when to use selected cues and information. Models tend to develop in stages where early models are often fuzzy or distorted versions of later, more advanced models. To this end, instead of classifying questions, we will examine how teachers recognize the need to develop effective questioning strategies and the ways they revise, refine and modify their strategies.

Guiding Questions

We will examine two key questions: how is it that teachers recognize the need to develop beyond their typical style of questioning, and how is it that they are able to move toward questioning in ways that are progressively "better". We will do this by focusing on two middle school teachers who have been part of this project for the past year. For the purposes of this research, we have collected data regarding the types of questions that the teachers use, and interpreted them in the context of models and modeling approach.

Methods and Procedures

This study represents one component of a multi-faceted teacher development project. The goals of the larger project were to:

- Provide teachers with the knowledge and ability to create classroom environments where students can build concepts and skills as they engage in meaningful, compelling, and challenging problem-solving explorations.

- Help teachers to become more comfortable with new materials, standards-based curricula, and teaching strategies.
- Help students gain knowledge and skills as they work on complex and challenging problem solving activities.

In order to help teachers make the transition from teaching mathematics in traditional ways that emphasize the execution of procedures and rote memorization, mathematics education researchers met regularly with teachers—both in the context of workshops and in the context of their own classrooms. During these meetings, opportunities were provided for teachers to: deepen their own understanding of the mathematics they were expected to teach, develop insight into the ways in which students build these ideas; and, consider the pedagogical implications of teaching mathematics in new ways. As mathematics educators worked with teachers and students, they attempted to provide opportunities for the students to develop logical and persuasive predictions, explanations, and justifications; analyze or critique their own solution, or the solution of others; and, consider the ways in which they could refine, revise, monitor, and assess their own work. Teachers were encouraged to consider the mathematical discourse that was elicited, and consider how they might begin to use new methods of questioning to encourage students to, for example, defend and justify their solutions, pose interesting hypotheses, and share strategies.

The data for the study include: (a) transcripts of the questions that teachers asked during implementation of problem solving activities, (b) transcripts from four semi-structured interview sessions, (c) transcripts from informal questions in regards to their perception of their own questioning as well as their reaction upon examination to actual video-footage of their questioning techniques, (d) transcripts from teacher workshop sessions, (d) teachers' daily reflections on their questioning techniques , and (e) researcher field notes taken while working with teachers in classrooms and workshop sessions.

Each workshop session was videotaped and audio-taped. The audio-tapes were fully transcribed and the videotapes were used for teachers to view and respond to their questioning techniques. From the first day of data collection, we analyzed all observation data for the purpose of identifying patterns or themes that might appear to be emerging. We used the process of going through the entire data set line by line to confirm and substantiate, or disconfirm any identified patterns. By constantly reviewing the data, we were able to form a coordinated analysis of the patterns that emerged.

The results will be discussed in terms of two abbreviated case studies of middle school mathematics teachers: Beth and Gene. Beth had taught fifth grade mathematics for six years. She was conscientious and thoughtful in her reflections of posing questions to further student learning in her classroom. Gene had taught fifth grade for ten years. He was very goal oriented in his approach to teaching mathematics in that he set long term and short term goals and conscientiously reflected on how and if he was meeting these goals. Gene set questioning as a long-term goal for the year, which caused to him transparently analyze the questions he posed throughout this study.

Results

Overall results indicated that the teachers were able to glean a deeper understanding of the relevant questions to ask by engaging in the following interrelated activities: (a) focusing on the types of questions they asked when implementing mathematical problems, (b) viewing videotapes of their own teaching and their colleagues teaching and examining and discussing why particular questions were asked, and (c) listening and trying to understand their students thinking. Their modeling cycles appeared to progress from naïve conceptual understandings to more complex understandings over time. This was illustrated in that initially, the questions that they posed generally did not produce thoughtful classroom discourse. As they focused on the questions they asked and their students' thinking, classroom discourse became more meaningful. For example, initially, the teachers did not ask students to explain or justify their thinking. While they may have asked students to talk about their solutions, generally speaking, student responses were not substantive. Student responses did not go beyond simplistic explanations of why they may have chosen a particular strategy or procedure. As the study progressed, the teachers were able reinterpret their own thinking and refine their questioning techniques accordingly. When this happened, students' responses became more substantive. This occurred as teachers transitioned from asking more recall questions to more thought provoking questions. Our data also suggest that the teachers also deepened their own understanding of the middle school mathematics concepts they taught and their awareness of how their students thought about those concepts.

To illustrate these results we will use one problem that both Gene and Beth implemented in their classroom near the end of our study. This problem and implementation was discussed in their workshop group where they provided video clips of their implementation and provided their own analysis of their questioning. Albeit this does not show the entire growth of questioning throughout the study it does provide data to illustrate their depth of questioning at this point in the study.

For this particular workshop the teachers had implemented a problem called *River Watch* in their classrooms. Beth and Gene brought in video clips to the workshop to discuss their questioning and implementation of the problem with their peers. The river watch problem provided their fifth grade students with an article that explained how macro invertebrates, fish species, and chemical analysis data could help analyze the health of different parts of a river. Using data interpretation techniques students were to write a letter explaining how others could interpret the health of the river using the method they created.

Initially Beth shared a list of questions with the workshop group that she had written down while viewing the videotape of her teaching (see Figure 1). The list did not include all of the questions that she had asked during implementation rather they were questions that she referred to as “thinking questions.”

> “I think these questions that I asked my students helped them to think deeper. That is why they are labeled “deep thinking questions.” For example, the first question (Are the number of bugs and fish enough to determine the health of the river?) was posed to a group of students who weren’t using all of the information provided. I thought this might make them take a look at the data and incorporate more of the variables to determine the health of the river. You can see I added the statement, “think a little deeper” because I used that specifically to refocus their attention on the work that had done. I really wanted them to think deeper.” (Workshop group transcript, March, 2000).

Her questions indeed made her students rethink their solutions throughout the period as illustrated on the video clips she shared in the discussion group. For example, one group showed Beth a line graph they created to illustrate the health of the river. She asked them, “What does the line graph show?” Is that what you were trying to say using this graph?” This showed growth in her ways of addressing students as in previous observations she had kids explain their graphs but mainly addressed the construction and aesthetics of the graph, instead of the mathematical meaning the graph portrayed. In this case, the group grappled with this question and decided that their line graph was limiting.

1. Are the number of bugs and Fish enough to determine the health of the river?
2. How will you show what you just said?
3. Are there any other factors that determine the health of the river?
4. Think a little deeper.
5. What does a line graph show? [student suggested]
6. Is that the best way to show the health of the river?
7. How will that show the health of the river?
8. Try to find all of the factors that are important.
9. Why do high numbers of species make the river healthy? [spurred on by student’s response]
10. Can the answer be an equation? Can an equation help you make sense of your picture?
11. What is the goal of this project? [redirect]
12. What are you trying to show [in this graph]?

Figure 1. Beth’s list of questions.

"We need to show more than just the chemicals at each site," suggested one student.

Another student added, "Yes, we should make a graph for bugs and fish too for each site." The students then made three graphs per site.

A short while later, Beth came back to the group. She asked the group, "Can you explain what the graphs tell the reader?" The students were able to explain each graph per site but they had difficulty comparing their graphs because they didn't use the same increments on their x and y axes.

One student suggested that they use a double line graph to show the bugs and fish at each site. The group agreed that if they did this for each site they could then look at the two quantities compared to each other.

Beth persisted with another question, "Are the number of bugs and fish enough to determine the health of the river?" Beth's questioning was critical in the continual thinking of this group. She was able to ask questions so that the students explained their products. As they explained their ideas they continually revised and refined their thinking as they clarified their own ideas. This continual use of effective questioning illustrated mathematical growth in her students' ideas. By the end of the period the group had decided to make a "quadruple" line graph to show the chemical data of each site on one graph, the macroinvertebrate data from each site on one graph, and fish data from each on one graph using different colored lines to show the different sites. This was a significant advancement in that they were able to compare multiple sets of data among each site and then compare data across sites by comparing all five graphs. Beth then asked them, "What do your graphs show?" This gave the students a chance to (a) verbally express the meaning of their graphs using mathematical comparisons, and (b) how they determined the health of the river using the graphs.

Beth had an even deeper insight into the mathematics this group used to make their graphs as she shared with the workshop group that they first used ratios to make sense of the data but realized that a more complex weighting system was necessary to really explain how to determine the health of the river. The group thought her understanding of the student strategies was quite impressive as she identified several student attempts on the video tape that were naïve and explained why she asked particular questions to move them along in their mathematical thinking.

Her colleagues in the workshop noted that Beth redirected her students without leading them in any direction except "using all of the data or incorporating all of the factors in their solutions." The main challenge the group posed to Beth involved her questioning as the groups shared their products with the class, the culminating activity. Beth let the students explain their solutions but she did not actively use questioning techniques to ensure that the other groups were listening, able to follow others explanations, or to help clarify ideas.

It was clear that Beth focused on good use of questioning during implementation of this problem. She asked higher-level thinking questions that helped students think deeper and moved them along in solving a difficult real life problem. Second, the video clip she shared provided the workshop group with a window into Beth's pedagogical decisions. Her peers were able to praise her when she was quick on her feet but also give her suggestions. The whole group learned from sharing the video clips with each other. The group offered suggestions as to how a question could have been asked more effectively when necessary.

The Case of Gene

Gene also taught fifth grade mathematics. Both Gene and Beth thought that this problem was difficult for their students. However, they both agreed that it "stretched their students to their limits" and challenged them in their implementation of such a difficult problem. To this end, they were both surprised at how well their students were able to tackle and solve the problem. Gene took more time in the introduction of the problem to explain vocabulary and make sure his students were able to make sense of the problem and the data. Thus, the first ten questions on his list reflect the comprehension of the problem while the latter questions addressed particular groups' strategies and solutions (see Figure 2).

One teacher commented on his use of yes and no questions, stating that they did not elicit mathematical ideas nor did he really know if the students understood the

1. Does everyone understand the reading?
2. What fish are they referring to?
3. What do the chemicals have to do with anything?
4. What do you need to know to solve the problem?
5. What are macro-invertebrates?
6. What do you need to find out?
7. How many sites are we looking at that are healthy?
8. What strategies do you think you are going to use?
9. What does a healthy river have?
10. How are you going to determine if the river is clean?
11. Do you understand the problem?
12. Can you prove that part of the river is clean?
13. Can you explain your graph to me?
14. What do your graphs show?
15. What interpretation can you make from your charts?
16. What point are you trying to make?
17. Why did you decide that site B was the worst? Can you justify your reason?
18. Why are you focusing on the oxygen statistic?

Figure 2. Gene's list of questions.

problem (see question 1 on Figure 2). This was typical throughout the study. Initially in the beginning of the study, most teachers asked these types of questions frequently. However, as the study went on these types of questions were not frequently asked in video clips so when they were, teachers pointed them out to each other immediately.

Gene typically asked direct recall questions in the beginning his class. When asked about this by his peers he responded, "I ask comprehension questions in the beginning to assess their understanding of the reading as well as the problem they are expected to answer. Once I know that everyone is on the right track then I ask harder and harder questions." (Workshop Discussion Transcript, March, 2000).

Throughout the study Gene conscientiously made decisions that he felt were best for his students. If he was not comfortable letting his students try a difficult problem, he modified the problem to the level where he felt all of his students would be able to be successful. The workshop group asked him to consider "letting his students try" instead of always "helping them so much." One peer told Gene, "You are guiding your students too much. You need to let them make sense of the problem and see what they will come up." Obviously, this is a subtle matter of opinion, yet it was brought on each occasion that Gene shared a video tape of his teaching in his classroom.

The workshop group viewed Gene implementing the River Watch problem. He explained that the first video clip showed him "thoroughly explaining the problem." Again, the workshop group challenged Gene's initial use of whole class discussion of the problem and short reading. They wondered if the students might have been able to understand without such a detailed discussion. Gene's response,

> "I wanted to make sure that the students understood the problem. This was the hardest problem that I had given my students this year however, it showed me that they can think and work through problems using all of the strategies we had discussed this year. I am not sure how they would have done if I had not "taught" so much in the whole class discussion that you just saw but I wasn't comfortable doing it another way at the time. But I will do that for the next problem that I video tape, I won't teach so much and let the students take a stab at it. You know what you guys, I am going to video tape that and bring it in so you can see what happens." (Workshop Discussion Transcript, March, 2000)

This was typical when teachers gave each other suggestions on pedagogical techniques. The teachers were quite specific in their suggestions and critiques for each other. This seemed to be a critical facet as teachers considered peer feedback constructively. For example, one of the teachers pointed out to Gene that most of his students raised their hand when he posed the question, "What do we need to find out?" Gene's response to the critique was, "I see what you mean, they really did know what the problem asked. I guess I need to trust that they will comprehend the question when reading." (Workshop Transcript, March, 2000.)

It appeared that when implementing this particular problem and typical of many of the problems Gene implemented with his class, he started by asking direct recall questions and as he was more comfortable with his students solving the problem he asked higher level questions. He posed questions to find out why his students used particular strategies (i.e., using graphs, charts), his students interpretation of their work, as well as questions that helped students justify why their solutions were accurate.

Discussion and Implications

Focusing our teachers' attention to their questioning strategies was an effective activity to promote deeper thought regarding their pedagogy and content knowledge. The study format was particularly effective for helping teachers to make fundamental changes in their ways of thinking about teaching mathematics. They consistently considered their current practice and made pedagogical changes that impacted the level of mathematics learning that took place in their classrooms.

All of the teachers participating in the study made changes to their practice aligned with reform oriented practice. We saw the level of discourse substantively increase due to the questions the teachers asked. To this end, the reasoning that underlyed the pedagogical decisions they made became quite sophisticated and was usually related to how their students thought about particular mathematical concepts or their understanding of the mathematics they were teaching.

The types of questions the teachers asked became more sophisticated. Although Gene asked direct recall questions or comprehension questions initially, his questioning was more focused on the nature of the student thinking and their interpretation of the mathematics. This was salient across participants. Collectively they were able to discuss and construct effective questions for particular situations based on the mathematical ideas being generated at a particular point and time.

We feel this study has important implications for teacher development in general. More research is needed to identify the catalysts that aid teachers to transition from more traditional modes of teaching to more reform oriented approaches.

References

Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal, 30*, 393-425.

Koellner-Clark, K., & Lesh, R. (in press). A modeling approach to describe teacher knowledge. In R. Lesh (Ed.), *Beyond constructivism: A models and modeling perspective*. Hillsdale, NJ: Erlbaum.

National Council of Teachers of Mathematics. (2000). *Principles & standards for school mathematics*. Reston, VA: The Council.

Schorr, R. Y., & Lesh, R. (in press). A modeling approach for providing on-the-job teacher development. In R. Lesh (Ed.), *Beyond constructivism: A models and modeling perspective*. Hillsdale, NJ: Erlbaum.

WHAT DO YOU EXPECT? HOW LEARNING EXPECTATIONS INFLUENCE PRESERVICE TEACHERS IN A FIELD EXPERIENCE

LouAnn H. Lovin
James Madison University
lovinla@jmu.edu

Abstract: Because gaining experience in the classroom is considered vitally important by classroom and preservice teachers as well as by researchers and teacher educators, teacher education programs are including more practicum experiences for preservice teachers. To gain more insight into preservice teachers' experiences during practicum settings, this paper considers two elementary preservice teachers' experiences during a four-week field placement early in their teacher education program. Although the preservice teachers described similar field settings, they experienced these field settings very differently. The epistemological reflection model posited by Baxter-Magolda was used to better understand the preservice teachers' experiences. The model describes four qualitatively different ways of knowing that lead to specific expectations of the learner, his/her peers, the instructor, how learning should be evaluated, and the nature of knowledge. Because these assumptions influence what students expect, different interpretations arise from different ways of knowing. Implications for teacher education are discussed.

Introduction

In recent years, experiential learning has become increasingly valued, not only in learning mathematics, but also in learning to teach mathematics. Gaining experience in the classroom is considered vitally important by classroom and preservice teachers as well as by researchers and teacher educators. In her review of the learning-to-teach literature, Kagan (1992) described several studies that suggested that in order for preservice teachers "to acquire useful knowledge of pupils, direct experience appears to be crucial, particularly extended opportunities to interact with and study pupils in systematic ways" (p. 142). Moreover, according to Russell (1988), preservice and first-year teachers in his study found that experience was a significant factor in understanding and applying theory and research to teaching. Thus, it appears that gaining experience in classrooms provides substance with which teachers can reflect about practice and make sense of theory. In fact, Dewey (1933) contended that "[N]o one can think about *anything* without experience and information about it" (p. 34).

Consequently, we have seen an increase in the number of practicum experiences required of preservice teachers before and during teacher education programs. But what goes on during these practicum experiences and what are our preservice teachers learning from them? We know that many preservice teachers consider their cooperating teachers to have the most significant influences on their teaching (e.g., Frykholm, 1996; Zeichner & Gore, 1990). One reason may be because classroom teachers are willing to address practical issues (e.g., Wilson, Anderson, Leatham, Lovin, and Sanchez, 1999). Frykholm (1996) also found that preservice teachers were searching for teaching models to emulate and the cooperating teacher provided such a model.

To gain more insight into preservice teachers' practicum experiences, this paper considers two elementary preservice teachers' experiences during a four-week field placement early in their teacher education program. This paper is based on part of a larger study, which examined how elementary preservice teachers defined and interpreted their experiences in a mathematics methods course and the concurrent field experiences. In particular, the experiences of two of the participants, Susan and Jackie, are highlighted because their cases were so dissimilar.

Theoretical Framework

The epistemological reflection model posited by Baxter-Magolda (1992) provided a tool to better understand the preservice teachers' experiences. The model describes four qualitatively different ways of knowing (absolute knowing, transitional knowing, independent knowing, and contextual knowing) that hinge on an individual's varying reliance on authority for knowing. Each way of knowing leads to specific expectations of the learner, his/her peers, the instructor, how learning should be evaluated, and the nature of knowledge. Because students interpret their educational experiences based on these assumptions, "different ways of knowing result in different interpretations" (p. 4). Baxter-Magolda argued that becoming aware of the set of assumptions someone is using can enhance one's understanding of that person's experience.

Absolute knowers view knowledge as certain and believe that there are absolute answers in all areas of knowledge. Because instructors (i.e., authorities) are thought to have all the answers, the role of an absolute knower as a learner is to obtain the knowledge from the authority. In fact, this type of knower places a great deal of responsibility for learning on the authority figure – either by keeping the student interested or by making acquisition easier. Transitional knowers accept the uncertainty of some knowledge but still believe certain knowledge exists in some areas. Since some knowledge is uncertain, learning becomes more complex which prompts students to value understanding over acquiring and remembering information. At this position, instructors are expected to use methods that emphasize understanding and application of knowledge. Independent knowers realize that most knowledge is uncertain and assume that knowledge is open to many interpretations. Consequently, instructors (i.e., authorities) are no longer the only source of knowledge. These knowers begin to view their own opinions as valid. Now the instructor is expected to provide an environment in which knowledge can be explored and both independent thinking and the exchange of opinions are encouraged. Open-mindedness and allowing everyone their own opinion are stressed and evaluating ideas is overlooked because ideas are thought to be equal in value. Thus, an "everything goes" approach is endorsed. Contextual knowers realize knowledge is uncertain, but understand that "some knowledge claims are better than others in a particular context" (Baxter-Magolda, 1992, p. 69). No longer is the "everything goes" approach endorsed. Contextual knowers, like independent knowers, think for themselves, but contextual knowers realize that they must evaluate ideas by considering context and how their ideas fit into the big picture. Knowledge from one area is insufficient – it needs to be related to knowledge in other areas in order to be useful. Thus, learning is transformed into integrating and applying knowledge as they think through problems in a context.

Methods and Data Sources

The qualitative case study method was chosen because it allowed the researcher to pursue each participant's perspective with an up-close view of the preservice teachers' experiences in the methods course and field experiences (LeCompte & Preissle, 1993; Merriam, 1998; Patton, 1990). Four semi-structured interviews (audiotaped and transcribed), course assignments, journal writings, several informal conversations, and observations of the participants during the on-campus classes and field experiences informed the case studies. The on-campus classes were audiotaped and field notes were taken to enhance data collection. Field notes were also taken during observations in the field. Data for the larger study were collected for one 15-week semester. During the four-week field experience, the preservice teachers spent all day in a classroom for four weeks. Informal conversations, journal entries, and reflection papers formed the greater part of the data gathered during the four weeks.

The research cycled between data gathering and analyzing, with the initial analysis phase informing subsequent data collection. In particular, data collection and analyses were guided by analytical induction techniques (LeCompte & Preissle, 1993; Patton, 1990). Analytic induction involves scanning the data for themes and relationships, and developing and modifying hypotheses on the basis of the data.

Results

Susan and Jackie described classroom teachers who expected them to interact with and teach mathematics to students throughout the field experience. Both also described teachers who interacted very little with them and offered little direction. Data gathered throughout the semester suggested that Susan was an absolute knower and that Jackie was an independent knower. Consequently, although they described similar field settings, they experienced these field settings very differently.

Susan: Just Tell Me What to Do

Susan's thinking that knowledge was absolute and came from an authority influenced how she interpreted interactions between herself and the classroom teacher. During the four-week field placement, the classroom teacher asked Susan to work daily on number recognition and counting with a small group of children in the hallway. This situation became a major focal point for Susan during the four-week field experience. Susan was upset because the classroom teacher was not telling her exactly what she should be doing with the children. When she described the situation, she said "[I] blindly taught a math group number recognition randomly throughout the four weeks" (Field Reflection Paper). She perceived that it was the classroom teacher's responsibility to provide precise instructions to her and she became frustrated when that did not happen. Consequently, she interpreted the situation to mean that the teacher did not want those children to "do well in math, to sort of prove a point" (Interview 3) - that these particular children could not learn mathematics. As a result Susan was adamant that she would teach mathematics to everyone when she became a classroom teacher.

Susan used the perspective of absolute knowing to make sense of her experiences during this field placement. Consequently, she expected to be told what to do with the students because she did not see her role as determining this information. Although the classroom teacher could have been providing opportunities for Susan to figure this out on her own, as an absolute knower, Susan was unable to interpret the situation in this way.

According to Susan, her university supervisors were critical of her teaching because she "missed teachable moments all the time.... I wouldn't ask the right questions.... But I just didn't know. I don't know what you ask" (Interview 3). Later she replied, "I don't know the questions to ask. One of the things that [the reading instructor] said to us was to ask them to predict. And so I did. And I could do that because somebody told me to do it" (Interview 4).

Susan's views as a learner seemed to strongly influence her views of herself as a teacher. As a learner, she expected to be told exactly what to do and as a teacher, she rarely asked questions, tending just to tell students what steps to do to accomplish the goals she had set. Susan thought she "would be a teller" (Interview 4) as a teacher because she could not think of pointed questions that would lead her students to the answers: "I can't ask questions. I'm not reflective and I have a hard time asking pointed questions. I will probably be a teller. That's probably not good, but..." (Interview 4). When I observed her during the four-week field experience, her interactions with the students were very directive and I was not convinced the students went away with much more than the idea that mathematics is something the teacher tells you to do. Susan's interactions with the children revolved around her giving mechanistic directions. She frequently became impatient with students and would simply repeat her directions when they failed to do as she expected.

During the four-week field experience, Susan was required to teach at least two mathematics lessons. Since the classroom teacher used centers in her classroom, Susan organized her lessons around centers. For one of the centers the teacher suggested that

Susan focus on estimation. When Susan continued to ask for advice, the teacher told her to put a varying number of manipulatives in ziploc bags and ask the children to estimate how many are in the bag. Susan pressed, "How high should I go?" (Interview 3) and the teacher told her to put no more than 15 in the bags. Susan doubted her teacher's direction, saying

> I don't know if that was easier or harder for them...I mean, 7, 8, 9, 10 would be sort of hard to judge the differences. When I think of estimating a number...it is not like I have to choose between 27 or 30. I don't know. I have never taught estimation. I don't know how you are supposed to. I think I would probably have broader ranges of numbers...but I don't know. She has done it for longer, so I assume she is right. (Interview 3)

Although Susan questioned her teacher's direction and felt uncomfortable using her idea about estimation, she assumed that the teacher was right and set up the center accordingly.

To Susan, neither herself nor her classmates were sources of knowledge. At various points in the semester Susan made comments like, "when I grow up..." (e.g., Interview 1; Interview 4), indicating that she saw herself as a child, not a responsible agent. When she thought a teacher was wrong, she would revert to thinking, "Well, maybe she knows something that I don't know, that I haven't learned yet, but I will learn it later. I sort of think that when I grow up I am going to know all this stuff" (Interview 1). However, right now Susan expected those people she saw as authorities (i.e., teachers) to tell her what to do and what to think. She had "complete total faith in what [her teachers] say....I have never had a teacher that I really, really doubted, that I really didn't trust" (Interview 1). She saw her teachers as authorities who dispensed knowledge.

Situations that required Susan to do more than regurgitate facts and ideas did not fit well with her notion that learning was about acquisition. She found it very difficult, if not impossible, to generate her own ideas, let alone recognize the benefits of this process. For example, during one of our phone conversations, Susan confessed that she had not written in her journal that week and wanted to know what I wanted her to write about. After I reminded her that these journal entries were what *she* wanted to write about, she still insisted I tell her what to write, explaining that writing more than what just happened was near impossible for her to do. Susan, as an absolute knower, assumed that knowledge did not come from within; instead, it came from someone else. Thus, situations that required her to expend energy and attention to think through ideas were pointless and unproductive to Susan.

Jackie: I'll Figure It Out On My Own

Because of the four-week field experience Jackie sensed she had a better idea of how to teach mathematics. Being able "to teach anything in general" (Interview 3) made it easier to teach a particular subject. She "was scared to death" (Interview 3) when she taught her first and even her second lesson. However, as she became more familiar with the students "the easier it got for me. They would even say stuff like, 'Oh, you sound like a teacher!'...Just being there...just being around the kids made it easier" (Interview 3). She described being involved in the classroom:

> I would just do pretty much anything in general, like I would give their timed tests and their spelling tests. A lot of times [the classroom teacher] would tell me what pages they were doing in social studies and be like, 'You can do it, go ahead and do it.' (Interview 3)

To Jackie simply being involved in classroom processes gave her confidence that she could teach any subject including mathematics.

During the four-week field experience, Jackie's classroom teacher used timed tests to compel students to learn their basic multiplication facts. Jackie recalled how "You'd get to one minute left, and they'd start scribbling and I could tell that the time limit was messing them up" (Interview 3). Some of the students would not even try:

> There were a few of them that it did [affect their attitudes] and you could tell that they didn't care and a lot of them would just start it and quit because they knew that they weren't going to finish so they didn't try. (Interview 4)

Although the classroom teacher considered this method a legitimate way to encourage students to learn their basic facts, Jackie rejected the idea. Jackie saw the time issue as having a detrimental effect on students' attitudes which confirmed for her the decision not to use timed tests when she became a classroom teacher.

Jackie indicated she had very little interaction with or direction from her classroom teacher. The first day in the field the teacher told Jackie, "I hated student teaching. I hated being watched. So, I don't want you to feel like I'm watching you" (Interview 3). Jackie recounted, "I would be teaching and she would be grading papers or doing something else" (Interview 3). She described a teacher who was very hands-off: "She wouldn't really hound me and watch everything I did" (Interview 3). Jackie also indicated that she "didn't really watch [the classroom teacher] teach. While she was teaching, I would be doing stuff with kids or doing stuff for her, like grading papers" (Interview 4).

While Susan wanted more interaction with the classroom teacher in the form of advice and ideas about teaching specific topics and critiques of her teaching, Jackie was very comfortable with her teacher's laissez-faire approach. Jackie indicated that the teacher did provide some suggestions for lesson plans, but that "I got to do whatever. She would pretty much give me free rein to do whatever I worked out" (Interview 3). As an independent knower, Jackie equally valued her own ideas and the ideas of others. While Jackie listened to the voice of authority (i.e., the methods instructor, her classroom teacher, the textbook, the teacher education program), she heard their ideas as suggestions, not dictates. She described the teacher education program as providing her with "[I]deas, like *if* you want to do this, then this would be a good thing for you to do....They really leave it up to us. They just give us a lot of background knowledge" (Interview 3).

Although Jackie described little interaction between herself and the classroom teacher, she indicated that she learned more from the four-week field experience than from the in-class portion of the methods course. In fact, she described the final reflection paper about the semester and the field experience reflection paper as "almost the same because...it was like I was writing the same things over and over, because a lot of the stuff I got out of the class was from the field experience" (Interview 4). In fact, she indicated that she did not use information from the methods course to help her teach during the four-week field experience. Thus, it was as if Jackie largely ignored the voice of authority because she was confident in her own ideas.

Discussion and Implications

When the preservice teachers talked about what they observed during the field experience, at times they offered reasons *they assumed* the teacher used to make decisions. When I inquired if they asked the teacher why she did certain things, in every instance the answer was "no." For example, Jackie discussed students' negative reactions to timed tests and wondered if the teacher used timed tests "just because the other teachers were doing it that way" (Interview 4). When I asked her why she did not ask the teacher she indicated that it never occurred to her to ask. Susan indicated she did not ask the classroom teacher questions because she perceived the teacher to be too busy to be bothered and since the teacher did not voluntarily offer information, Susan perceived that the teacher was not interested in helping her learn. Because the preservice teachers had little or no interaction with their classroom teachers, including no direct answers to most of their questions, they were left to come to their own conclusions about why the teachers did what they did.

Previous research suggests that preservice teachers develop views of teaching and learning by observing teacher behaviors during their own schooling (e.g., Holt-Reynolds, 1991; Lortie, 1975; Pulver, 1996). The findings of this study indicate that they are continuing to do so during field experiences. Given the little or no substantial interaction between the classroom and preservice teachers, the preservice teachers continued to use perspectives they entered the methods course with to make sense of their field experiences. It is logical to assume that until the perspectives they are using become useless in making sense of their worlds, they will continue to use them. Susan was already experiencing frustration because her absolute knowing perspective was becoming ineffective in helping her make sense of her teacher education program. Other perspectives such as Jackie's perspective that teaching is an "anything goes" endeavor need to become problematic for preservice teachers. Otherwise, the preservice teachers will continue to watch teachers' observable behaviors and infer what constitutes effective teaching.

By placing preservice teachers in practicum experiences, we are attempting to provide them with opportunities for experiential learning. This type of learning can be very effective. But simply placing preservice teachers in the practicum experiences and hoping for the best is obviously not enough, especially for absolute knowers like Susan. So, if preservice teachers are absolute, transitional, or independent knowers, how can we help them learn more effectively in a practicum setting? First, we need to recognize that the type of knower dictates the assumptions they bring to the settings and then we need to "play" to those assumptions to meet the "needs" of the knowers. For example, for absolute knowers, we may consider placing them with more directive cooperating teachers to meet their need to be told what to do. However, for all these types of knowers, validating preservice teachers' ideas is essential to promoting their belief that they are capable of constructing knowledge. Only after preservice teachers believe they have something of value to offer will they move toward more complex ways of knowing. Baxter-Magolda (1992) maintained that experiences that promoted complex thinking and a distinctive voice for her participants were relational in nature. In other words, the people were treated as if they had something of value to offer to the situation. All this requires a different approach during practicum settings than the ones preservice teachers generally encounter – one in which learning is viewed as "jointly constructing meaning" (Baxter-Magolda, 1992, p. 380) and one in which the classroom teacher actively embraces the role of teacher educator. Classroom teachers and teacher educators can encourage movement toward more complex ways of knowing by opening ideas to speculation by candidly discussing what has been seen in the classroom and what it might mean from different perspectives and in different contexts. By including preservice teachers in these discussions, and expecting and valuing their contributions, we can begin to establish relational aspects of knowing.

Partnerships between universities and schools already exist, but this study suggests important points of communication for these partnerships. First and foremost, knowing that the preservice teachers are continuing their apprenticeship of observation during early field experiences, partnerships can determine how to best promote explicit communication between classroom and preservice teachers. It appears that an important focal point for teacher educators and classroom teachers is determining preservice teachers' expectations in how they will learn from a field experience. These expectations appear to influence the preservice teachers' perception of what is expected of them and what they ultimately learn from the experience. By becoming aware of preservice teachers' assumptions about learning in the field we stand a better chance of connecting with them and ultimately, improving educational practice.

References

Baxter-Magolda, M. (1992). *Knowing and reasoning in college: Gender-related patterns in students' intellectual development.* San Francisco: Jossey-Bass.

Dewey, J. (1933). *How we think.* Boston: D. C. Heath.

Frykholm, J. (1996). Pre-service teachers in mathematics: Struggling with the Standards. *Teaching and Teacher Education, 12* (6), 665-681.

Kagan, D. (1992). Professional growth among preservice and beginning teachers. *Review of Educational Research, 62* (2), 129-169.

Holt-Reynolds, D. (1991). *The dialogues of teacher education: Entering and influencing preservice teachers' internal conversations.* Research Report no. 91-4. East Lansing, MI: National Center for Research on Teacher Learning.

LeCompte, M., & Preissle, J. (1993). *Ethnography and qualitative design in educational research* (2nd ed.). San Diego: Academic Press.

Lortie, D. (1975). *Schoolteacher.* Chicago: University of Chicago Press.

Merriam, S. (1988). *Case study research in education: A qualitative approach.* San Francisco: Jossey-Bass.

Patton, M. (1990). *Qualitative evaluation and research methods* (2nd ed.). Newbury Park, CA: Sage.

Pulver, P. (1996). *Preservice teachers' beliefs about the nature, teaching and learning of elementary mathematics, and the influence of previous experiences on these beliefs.* Unpublished dissertation, State University of New York at Buffalo, Buffalo, NY.

Russell, T. (1988). From pre-service teacher education to first year of teaching: A study of theory and practice. In J. Calderhead (Ed.), *Teachers' professional learning* (pp. 13-34). Philadelphia: Falmer Press.

Wilson, P., Anderson, D., Leatham, K., Lovin, L., & Sanchez, W. (1999). Giving voice to mentor teachers. *Proceedings of the Twenty-First Annual Meeting of the North American Chapter for the Psychology of Mathematics Education* (Vol. 2, pp. 811-817).

Zeichner, K., & Gore, J. (1990). Teacher socialization. In W. R. Houston, M. Haberman, & J. Sikula (Eds.), *Handbook of research on teacher education* (pp. 329-348). New York: Macmillan.

A CASE STUDY OF CHANGE IN TEACHING HIGH SCHOOL ALGEBRA

P. Michael Lutz
California State University, Bakersfield
plutz@csub.edu

Abstract: This is an evaluative, qualitative case study of a high school mathematics teacher who, with the researcher's assistance, attempted to reform his teaching practice. It is part of the author's dissertation research and focuses on the use of Clarke's (1993) work as a conceptual framework for studying reform in high school classrooms and on the use of non-routine problems as a focus of instruction. The use of Clarke's work in studying *high school* reform is significant as his work was based on elementary and middle school projects. The teacher used non-routine problems prior to the study but increased the problem solving benefits and mathematical content that he could derive from each activity.

During the past two decades, several documents calling for change in mathematics education have received national attention (Edwards, 1990; National Council of Teachers of Mathematics [NCTM], 1980, 1989, 1991, 1995, 2000; National Research Council [NRC], 1989, 1990; Stigler & Hiebert, 1999; U. S. Department of Education, 2000). These documents call for a new role for mathematics teachers as facilitators in a social environment instead of deliverers of information to passive learners. What challenges do high school teachers face as they accept the new role and endeavor to change both their philosophy of mathematics learning and their teaching practices? What conceptual framework should be used to study these changes? This study used Clarke's (1997) research as the framework for an evaluative, qualitative case study of a high school mathematics teacher, Mr. Evans[1], attempting to change his practice. Although Clarke's work was primarily based on elementary or middle school projects, it proved to be an effective tool for studying change in Mr. Evans's high school classroom. This paper is based on portions of the author's dissertation research and describes two issues: (a) the use of Clarke's framework to study changes in high school mathematics classrooms and (b) Mr. Evans's struggle to go beyond simply choosing good problems as he utilized non-routine problems in his class.

Conceptual Framework

"A fundamental problem in the study of mathematics teaching . . . is figuring out what to look at. In this sense, no research is free of a conceptual lens, the concepts that determine what a researcher attends to and sees" (Simon, 1997, p. 62). Simon's quote describes my situation as I designed this study. I was interested in teacher change and in investigating the factors that influence teachers as they attempt to create a contemporary classroom. However, I needed to determine what I was going to look for as I conducted my research. What are the significant characteristics that define a contemporary classroom?

I was still searching for a conceptual framework during the early stages of data collection. I planned for listening to be an important issue in my study, so I considered using Davis's (1996, 1997) work. Since Ernest (1989) described the relationship between teachers' practices and their knowledge, beliefs and attitudes of mathematics, I considered using his work. Neither Davis nor Ernest emphasized non-routine problems, and I knew Mr. Evans well enough to realize that to him the primary focus of change was the use of non-routine problems. I needed a framework that included non-routine problems as a focus. Fortunately, as I studied the literature on teacher change, I discovered that Clarke (1997) had developed the description of change that I needed.

Clarke studied seven noted examples of reform to find components that are common to this kind of classroom. The projects that Clarke analyzed were The Second Grade Mathematics Project (Cobb, Wood, & Yackel, 1990), The Whitnall High School Experiment (de Lange, van Reeuwijk, Burrill, & Romberg, 1993), Cognitively Guided Instruction (Fennema, Carpenter, & Peterson, 1989), the work of Lampert (1988), The Middle Grades Mathematics Project (1988), the work of Schoenfeld (1987), and The Reality in Mathematics Education Project (Stephens & Romberg, 1985). Although the examples are mostly elementary or middle school studies, I saw no characteristic that I thought would not apply to high school mathematics classrooms. In order for a characteristic to be included as one of the components in his framework, it had to appear in at least four of the seven projects or studies. He found six such components:

1. The use of non-routine problems as the starting-point and the focus of instruction.
2. The adaptation of materials and instruction according to local contexts and the teacher's knowledge of students' interests and needs.
3. The use of a variety of classroom organizational styles (individual, small-group, whole-class).
4. The development of a "mathematical discourse community" with the teacher as "fellow player" who values and builds upon students' solutions and methods.
5. The identification of and the focus on the big ideas of mathematics.
6. The use of informal assessment methods to inform instructional decisions.

In this paper, I concentrate on the first one, the use of non-routine problems as the starting-point and the focus of instruction.

Methodology

I selected Mr. Evans because he expressed an interest in participating in the research and in changing his classroom and because of the close proximity of his school to my residence. As a non-funded graduate student, I utilized a convenience sample.

I was an active participant and wrestled with the definition of my role throughout the study. I knew from the beginning that I did not want to be a "fly on the wall" and that I wanted to help Mr. Evans become a better teacher, but I was concerned that my active role might diminish the value that the research community placed on my study. I received early encouragement when I discovered Romagnano's dissertation (1991) and the positive attention that it has received. Chazan's work (2000) was inspirational as I concluded my study.

During the nine weeks that constituted the most intensive part of my data collection, I taught one of Mr. Evans's three classes and observed the other two. Likewise, he observed the class that I taught. I attempted to spend my time observing during Mr. Evans's classes, but I occasionally assisted students as they worked in groups, and there were occasions when Mr. Evans directed a comment or question to me during his class. I always responded.

I worked hard not to be seen as an expert who had all of the answers. Although I had 27 years of experience teaching high school mathematics, I did not feel like an expert. I had a lot of questions myself regarding reform classrooms. My goal was to be Mr. Evans's colleague, but he probably considered me to be more than that. I wanted to come as close as I could to being in his shoes so that I could more accurately describe his efforts. By teaching one class, I believe I gained a greater appreciation for what was happening to him than I would have gained by only observing. I did not study my classroom; Mr. Evans and his classroom were the focus of my study. However, by teaching one of his classes, I was able to bring a more informed perspective to our discussions and to my analysis of the data. Occurrences in each of our classes were topics of discussion during our preparation time. Each of us brought our own individual expertise and shortcomings to the planning sessions. Merriam (1998) states that the key concern in qualitative research "is understanding the phenomenon of interest from the participants' perspectives, not the researcher's" (p. 6). I believe that by being a participant observer, I was better able to understand Mr. Evans's struggles to change his practices.

There is no doubt that my presence greatly influenced what happened during the study. Mr. Evans and the students knew about my experience teaching high school mathematics and that I was in the final stage of earning a doctorate. My credentials implied that I was an expert. Mr. Evans often asked for my opinions, and I provided them. What I did in my class influenced what he did in his class. Wolcott (1995) explains that bias in participant observation is acceptable as long as the biases are

834

made explicit and used to give meaning to the study. I am attempting to make my biases explicit, and I believe they give meaning to the study.

Data collection occurred over approximately a nine-month period from June, 1997 through March, 1998. The most intensive data collection occurred between January 20, 1998 and March 27, 1998. That was the period of time that I taught one of the classes. I made numerous visits to Mr. Evans's classroom during the Fall and quickly noticed that he seemed committed to using non-routine problems as a vehicle for instruction. Yet, I was confused because it seemed as if he would move on to another activity before getting to a lot of the significant mathematics that I thought was in each problem. His struggle to go beyond simply choosing good problems became a focus of my study.

From January 20, 1998 through March 27, 1998, I spent the entire school day with Mr. Evans. Since he was teaching on a block schedule, he was assigned to teach three of the four 87-minute periods. I observed him teaching one section of Algebra One and one section of Advanced Algebra, and I taught one section of his Advanced Algebra. The students in the classes were scheduled into the sections using the normal school process. I took over all of his duties teaching the morning Advanced Algebra course, although Mr. Evans retained responsibility for it.

Yin (1994) gives three principles of data collection that are "extremely important for doing high-quality case studies" (p. 79). They are the following: (a) using multiple sources of evidence, (b) creating a case study database, and (c) maintaining a chain of evidence. "The need to use multiple sources of evidence [in case studies] far exceeds that in other research strategies" (p. 91). Data for this study include field notes in written or audiotape form, audiotapes of discussions between Mr. Evans and me, audiotapes of Mr. Evans's classroom, journal entries by Mr. Evans, and classroom artifacts. All of these were maintained in my case study database. By citing the appropriate evidence in the database and indicating the circumstances under which the evidence was collected, I provided for an external observer such as the reader "to follow the derivation of any evidence from initial research questions to ultimate case study conclusions" (Yin, 1994, p. 98). In this way I met Yin's condition of maintaining a chain of evidence.

The field notes, Mr. Evans's journal entries, and the transcriptions of all audiotapes were combined and sorted by date. These documents were coded for analysis using computer software (QSR NUD*IST2 4.0, 1997). The coding was based on Clarke's (1997) six components and the themes that emerged from my informal analysis during data collection. I organized the data chronologically, making separate documents for each day. There were 62 separate days in which I collected at least one piece of data. Once I completed the coding, I was able to use NUD*IST to scan all 31,181 lines of the 62 documents of transcribed data to generate reports for each code.

Data analysis during the data collection process might best be described as informal. I would reflect on the day's activities, read my field notes, and listen to audiotapes as I thought it seemed appropriate, but the formal coding of the data did not occur until after most of the data had been collected.

Although the computer was helpful in my data analysis, I used only a small part of its capability. I used its "code-and-retrieve" capabilities (Merriam, 1998) that involve labeling passages of text according to content and collecting similarly labeled passages. The codes (nodes) that I used were based on Clarke's (1997) six components, the results of the informal analysis during data collection, and on nodes that were added during the formal, computer aided process after data collection. The first time that I went through the data I coded according to Clarke's six components. Subsequent coding included categories of factors that influenced the change process and of themes that emerged from the data. For example, I added codes for students' resistance and for covering the curriculum as factors that influence the change process. I added mathematical content as an issue that emerged from the data. The software allowed me to quickly pull all data that were labeled with the same code. For example, I could bring into one report all data that had been labeled as mathematical content. I maintained printed copies of all 62 documents and always referred to them during my explorations. Also, I retained all of the audiotapes and referred to them in any cases where a key word or phrase might change the meaning of the data and, therefore, needed to be authenticated. As a regular part of my discussions with Mr. Evans, I shared my on-going analysis and sought his agreement with my interpretation of the data. This tactic strengthened the construct validity of the study (Yin, 1994).

Results

Clarke's Research

Clarke's work served well as the framework for my research. The subject of my study, Mr. Evans, had a goal to create a contemporary classroom, and I had a goal to study the creation of one. Mr. Evans agreed that the items on Clarke's list were consistent with his vision of reform, so the list provided him with a target for his efforts, a lens for me to look at his efforts, and major topics of discussion in my interviews with him. It allowed me to spread a broad net early in the study and to narrow the focus later in the study as interesting issues emerged from the data. I began my study by seeking data for all six of Clarke's characteristics. At that time, I did not know what topics would emerge as the most salient. However, by using his framework that was based on noted reform projects, I was confident that what I wanted to study would be included. Later, as the issue of non-routine problems emerged as a topic of interest, I was able to narrow my focus. It helped me "to focus attention on certain data and to ignore other data" (Yin, 1994, p. 104). When I reached the analysis stage of the study, Clarke's framework provided me with the initial categories for the coding process. As new concepts emerged, I was able to include them, but Clarke's work got me started.

Finally, when I began writing, the framework provided a way of organizing my thoughts and outlining my document. So, it provided Mr. Evans with a target for his efforts, a lens for me to look at his efforts, major topics of discussion in my interviews with him, structure for my data analysis, and structure for writing the results of the study. It is an effective framework for studying teacher change at any level. Researchers interested in studying change should consider using it.

Mr. Evans's Usage of Big Problems

Prior to this study, Mr. Evans used non-routine problems in his classes. Time to cover the curriculum prevented him from using them more often. However, he assigned the problems with regard for neither the mathematical content nor the relationship of the problem to the established curriculum. He believed that he should teach problem solving, specifically data analysis problems, and used the problems to provide those experiences for his students. However, he did what Lester (1994) describes as teaching *about* problem solving.

During the study, he and I discussed the importance of examining the mathematical content of a problem prior to using it in class. I used the work of Hiebert et al., (1997) to frame these discussions. Hiebert and his colleagues describe the mathematics that remains with students after completing an activity as *residue.* They are critical of activities that engage students but do not contain enough mathematics to justify the time that is spent. In Mr. Evans's situation, he selected problems that contained significant mathematics, but he tended to move on prior to deriving the mathematical content from the activity.

Although he selected challenging problems, he did not allow students to wrestle with them. Instead, he moved around the classroom delivering mini-lectures on the procedures that he thought students should use to solve the problem. I used the work of Stein et al., (2000) to focus our discussions on this practice. Stein and her colleagues theorize that assigning students cognitively demanding tasks may not result in the desired amount of student learning unless the tasks remain at a high cognitive level throughout the activity. Instructional tasks pass through three phases, and the cognitive level can deteriorate at any of the phases. Mr. Evans's tasks usually demanded a high level of cognitive demand as they were stated in the materials. However, they often deteriorated as he was setting them up or as the students were implementing them. The mini-lectures that Mr. Evans used during the early stages of the study reduced the cognitive level of the activities at the implementation stage.

During the study, Mr. Evans worked to use non-routine problems more effectively and efficiently. Through our discussions, he became more aware of the mathematical content of the activities that he had selected and worked to reduce the number of mini-lectures. By the beginning of the Spring Semester, one of his primary goals was to find or develop one big problem per chapter in the textbook that would cover all of the departmental proficiencies that were included in the given chapter. We did not achieve that objective, but we made progress.

A major concern for Mr. Evans was being sure that he "covered" the required curriculum. Every member of the mathematics department who was teaching a particular course gave the same final examination. Mr. Evans was worried that he would not be able to cover every topic that was included on the exam if he spent time on the big problems that he believed were important. Preston (1997) found the same concern among the teachers he studied who were attempting to use mathematics modeling projects in their classes. He describes teachers whom either focused on the curriculum while projects ran in the background or focused on projects while the curriculum ran in the background. Similarly, Mr. Evans tended to focus on non-routine problems while the curriculum ran in the background. However as the semester progressed, he felt pressure to sacrifice the non-routine problems and focus on the curriculum. He used the big problems extensively early in the semester but used them less as time passed. Eventually, we solved the problem by pulling more mathematics from his activities and by splitting the curriculum. Some days he lectured on the proficiencies that he was unable to cover using non-routine problems and some days he focused on the non-routine problems. This seemed to improve student attitudes, too. One of the obstacles that he encountered was difficulty "selling" the students on his classroom changes. By splitting the curriculum, during any given week each student was able to experience her or his preferred mode of instruction.

Conclusion

Although Clarke's (1993) work was based on elementary or middle school projects, it provided a framework for me to study a high school algebra teacher. It provided a research-based definition of reform that gave Mr. Evans a target for his efforts, a lens for me to look at his efforts, major topics of discussion in my interviews with him, structure for my data analysis, and structure for writing the results of the study. As I narrowed my focus, three other studies helped frame my work. Stein et al. (2000) describe the importance of teachers maintaining a high cognitive level for classroom tasks as they appear in the materials, as they are set up in class, and as they are implemented by the students. Their work provided a focus for Mr. Evans as he moved away from his mini-lectures, for us in our discussions, and for me as I wrote my report. Hiebert et al., (1997) provided similar structure in addressing the importance of mathematical content in classroom activities. They stress the importance of evaluating the mathematics that students gain from a given activity. By maximizing the mathematical content that came from each of his big problems, by becoming more aware of the importance of allowing his students to wrestle with the problems, and by splitting the curriculum, Mr. Evans was able to spend more time on the non-routine problems that he considered important and still have time to cover the prescribed curriculum. The process seemed to be working as he had completed more than half of the proficiencies in the school's curriculum guide by the time I left at mid-semester.

As researchers continue to study the nature of reform, it is important that they agree on the characteristics that define the contemporary classrooms that we hope to create. This study supports Clarke's (1997) work as a description of those classrooms. His work also justified Mr. Evans's desire to use non-routine problems. Then learning from other studies, Mr. Evans was able to increase the benefits his students derived from the problems.

References

Chazan, D. (2000). *Beyond formulas in mathematics and teaching: Dynamics of the high school algebra classroom (The series on school reform)*. New York: Teachers College Press.

Clarke, D. M. (1997). The changing role of the mathematics teacher. *Journal for Research in Mathematics Education, 28* (3), 278-308.

Cobb, P., Wood, T., & Yackel, T. (1990). Classrooms as learning environments for teachers and researchers. In R. Davis, C. Maher, & N. Noddings (Eds.), Constructivist views on the teaching and learning of mathematics. *Journal for Research in Mathematics Education, Monograph No. 4* (pp.125-146). Reston, VA: National Council of Teachers of Mathematics.

Davis, B. (1997). Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education, 28* (3), 355-376.

Davis, B. (1996). *Teaching mathematics: Toward a sound alternative,* New York: Garland. *Journal of Education for Teaching, 15* (1), 13-33.

de Lange, J., van Reeuwijk, M., Burrill, G., & Romberg, T. A. (1993). *Learning and testing mathematics in context: The case: Data visualization*. Scotts Valley, CA: Wings for Learning.

Edwards, E. L. Jr. (1990). *Algebra for everyone.* Reston, VA: National Council of Teachers of Mathematics.

Ernest, P. (1989). *The knowledge, beliefs and attitudes of the mathematics teacher: A model.*

Fennema, E., Carpenter, T. P., & Peterson, P. (1989). Teachers decision making and cognitively guided instruction: A new paradigm for curriculum development. In N. F. Ellerton & M. A. Clements (Eds.), *School mathematics: The challenge to change* (pp. 174-187). Geelong, Australia: Deakin University Press.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding.* Portsmouth, NH: Heinemann.

Lampert, M. (1988). *The teacher's role in reinventing the meaning of mathematical knowing in the classroom.* East Lansing, MI: The Institute for Research on Teaching, College of Education.

Lester, F. K. (1994). Musings about mathematical problem-solving research: 1970-1994. *Journal for Research in Mathematics Education, 25* (6), 660-675.

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. San Francisco, CA: Jossey-Bass.

Middle Grades Mathematics Project. (1988). *The Middle Grades Mathematics Project: Good mathematics -- taught well.* (Final report to the National Science Foundation). East Lansing, MI: Mathematics Department, Michigan State University.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1980). *An agenda for action: Recommendations for school mathematics of the 1980s.* Reston, VA: Author.

National Research Council. (1990). *Reshaping school mathematics: A framework for curriculum.* Washington, DC: National Academy Press.

National Research Council. (1989). *Everybody counts: A report on the future of mathematics education.* Washington, DC: National Academy Press.

Preston, R. V. (1997). *Mathematical modeling in the secondary school: Possibilities and constraints*. ERIC: Dissertation Abstracts.

QSR NUD*IST 4 [Computer software]. (1997). Melbourne, VIC, Australia: Qualitative Solutions & Research Pty Ltd.

Romagnano, L. S. (1991). Managing the dilemmas of change: A case study of two ninth-grade general mathematics teachers. *Dissertation Abstracts International.* (University Microfilms No. 9220447).

Schoenfeld, A. H. (1987). What's all this fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189-215). Hillsdale, NJ: Erlbaum.

Simon, M. A. (1997). Developing new models of mathematics teaching: An imperative for research on mathematics teacher development. In E. Fennema & B. S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 55-86). Mahwah, NJ: Erlbaum.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development.* New York: Teachers College Press.

Stephens, W. M., & Romberg, T. A. (1985, March). *Reconceptualizing the teacher's role.* Paper presented to SIG/RME at the annual meeting of the American Educational Research Association, Chicago.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap.* New York: Free Press.

U. S. Department of Education. (2000). *Before it's too late: A report to the nation from The National Commission on Mathematics and Science Teaching for the 21st Century.* Jessup, MD: Author.

Wolcott, H. F. (1995). *The art of fieldwork.* Walnut Creek, CA: AltaMira Press.

Yin, R. K. (1994). *Case study research: Second edition.* Thousand Oaks, CA: Sage.

Notes

[1] Mr. Evans is a pseudonym.

[2] NUD*IST stands for Non-numerical, Unstructured Data Indexing, Searching and Theorizing.

AN EXPLORATORY STUDY OF PROBLEM POSING FROM NUMERICAL EXPRESSIONS BY PRESERVICE ELEMENTARY TEACHERS

Armando M. Martinez-Cruz
California State University, Fullerton
amartinez-cruz@fullerton.edu

Jose N. Contreras
The University of Southern Mississippi
jose.Contreras@usm.edu

Abstract: Forty prospective elementary teachers were asked to create two word problems for each of nine numerical expressions involving four arithmetic operations (addition, subtraction, multiplication, and division). Students produced 442 problems and 7 statements. Three hundred seven problems (69.5%) were classified as appropriate problems (problems are real-world problems and they can be solved using the numerical expression given). Forty-three problems (9.7%) were classified as partial problems (either one of the following situations happen: (a) problems are incomplete, (b) they can be solved with an equivalent mathematical operation, or (c) are partially a real-world situation). Ninety-two problems (20.8%) were classified as inappropriate problems (either problems are not a real-world situation or they cannot be solved with the given statement). The 307 (out of 442) appropriate problems give about one appropriate problem per student for each expression. This situation is discouraging since the content involves basic operations. Results suggest the need for teachers to have the opportunity to write interesting and meaningful problems, reflect on them, and discuss the connections to real-world applications even when the content is elementary.

A central goal in current documents in mathematics education (e.g., NCTM, 1989, 2000) is that students become mathematical problem solvers. Mathematics teachers play an essential role in this enterprise (NCTM, 1991, 2000; Simon, 1998). Teachers are expected to model and emphasize "aspects of problem solving, including formulating and posing problems, solving problems using different strategies, verifying and interpreting results, and generalizing solutions" (NCTM, 1991, p. 95).

Problem formulation or problem posing is increasingly receiving attention from both a curricular and pedagogical perspective (NCTM, 2000). For instance the *Principles and Standards for School Mathematics* (NCTM, 2000) states that "students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking," (p. 52).

Teachers' adequacy to provide instruction, as recommended in current mathematics education documents has received greater attention since "the decisions that teachers make about problem-solving opportunities influence the depth and breadth of students' mathematics learning," (NCTM, 2000, p. 119). Still, there are many questions to be asked about teachers' abilities in problem solving and problem posing. This paper reports preliminary findings on the abilities of prospective elementary teachers to pose real-world problems from numerical expressions involving arithmetic operations (addition, subtraction, multiplication, and division) with real numbers.

Conceptual Perspective

Problem solving is an important goal, if not the most important goal, in mathematics education (NCTM, 2000). This central role of problem solving has attracted considerable research and curricular attention in the last twenty years. Even though problem posing is also considered an important activity in mathematics education (Brown & Walter, 1990; NCTM, 1989, 1991, 2000), it has received relatively little attention both from a research perspective and from a curricular perspective. Yet the relation between problem solving and problem posing is evident. One cannot solve a problem unless a problem has been posed or formulated.

The *Principles and Standards for School Mathematics* (NCTM, 2000) calls for an increased attention to problem posing. For instance, the document states that students "should have opportunities to formulate and refine problems because problems that occur in real settings do not often arrive neatly packaged" (p. 335). Mathematics learners can benefit from an orientation toward problem posing. As Moses and her colleagues state "we learn mathematics particularly well when we are actively engaged in creating not only the solution strategies but the problems that demand them" (Moses, Bjork, & Goldenberg, 1990, p. 90).

Teachers are the "essential ingredient" in problem posing (Moses, Bjork & Goldenberg, 1990, p. 86). The National Council of Teachers of Mathematics (NCTM, 1991) recommends that teachers model and emphasize aspects of problem solving and that "students should be given opportunities to formulate problems from given situations and create new problems by modifying the conditions of a given problem" (1991, p. 95).

Some researchers have investigated problems posed by children (e.g., Silverman et al., 1992), by prospective teachers (Contreras & Martínez-Cruz, 1999; Silver et al., 1996), and by inservice teachers (Silver et al., 1996). However, there are many questions still to be asked about mathematics teachers' abilities to pose problems in general, and by prospective elementary teachers in particular. This study is a contribution to a better understanding of the abilities of prospective elementary teachers to pose problems from numerical expressions involving four operations (addition, subtraction, multiplication and division) with real numbers.

Silver (1994) observed that "problem posing" applies in general to three distinct forms of mathematical cognitive activity: (a) a presolution posing, in which one generates original problems from a presented stimulus situation; (b) within-solution posing, in which one reformulates a problem as it is being solved; and (c) postsolution posing, in which one modifies the goals or conditions of an already solved problem to generate new problems. We look at the presolution-posing form in this study. Our decision to use four arithmetic operations with real numbers was based on the familiarity of the participants with these operations. Our interest to investigate word problems production emerged from the belief that word problems convey information on teachers' understanding of the mathematical operations used in the numerical expressions. This is similar to the idea used by Simon (1993), who studied the relationship between the operation proposed and a real situation. Simon asked participants to provide a real-world problem that can be solved using the numerical expression given. This information could be used to inform teacher education programs.

Methodology

Subjects

Data for this study were collected from two sections of a mathematics course for prospective elementary teachers (one male and 39 females) during Fall 2000. We anticipate to collect more data from prospective elementary teachers in the Fall 2001 to be included in our reporting session at the conference.

Procedures

A questionnaire with nine numerical expressions (table 1) was designed to obtain an exploratory view of prospective elementary teachers' abilities to pose problems from numerical expressions that use four operations (addition, subtraction, multiplication, and division) with real numbers. Expressions were designed using a combination of the operations and numbers (integers, fractions, and decimals). A rationale for using these operations and numbers was based on the idea that both were familiar to the participants. Pilot of the questionnaire made us conclude that asking the students to produce one problem was not a very informative. Hence we asked to provide two problems as when we ask the students to solve a problem in two different ways.

The questionnaire was given to students on Friday of the first week of classes as a homework assignment due the following Monday. Directions were read to students. A clarification on "real-world problems" was needed. We indicated that we wanted "word problems". However, we emphasize that "a verbalization of the statement" was not a word problem. Students were asked not to consult any person or book. If they could not think of an example, students were asked to indicate so. The questionnaire in Table 1 does not include the space provided for the answers.

Curricular context

The first chapter covered in this course refers to problem solving. About two sessions of the first week was all the experience that participants had on problem solving, and certainly no time had been devoted to problem posing at that time.

Table 1.

Directions: Below you are given several numeric statements. For each statement, give two real-world problems that can be solved using the numeric statement given. Problems must be DIFFERENT. In other words provide two different contexts, not just different words but still similar ideas. Do NOT just verbalize the statement given. In other words do not write something like "WHAT do you get if you multiply, 2*3?" for the statement "2*3".

1. $2.5 + 6 + 8 + 6.3$
2. $2*(14 - 2)$
3. $8.4*15$
4. $\frac{12 + 30}{14}$
5. $\frac{1}{4} * \frac{1}{3}$
6. $\frac{12}{\left(\frac{1}{4}\right)}$
7. $1.29*7.5$
8. $4*1.8*3.2$
9. $2500 - 8475$

Data analysis

A total of 449 problems and statements were created. We adapted portions of Silver and Cai's (1996) schema to analyze problems. First, problems were classified as mathematical problems, nonmathematical problems or statements. Four hundred forty two were accepted as mathematical problems and seven (1.56%) contain no problem to solve (statements). The next step was to classify the problem as appropriate, partial or inappropriate. A problem is appropriate if it is a real-world problem and if it can be solved using the operation proposed. A problem is partial when at least one of the following situations occurs: the problem is incomplete, the problem can be solved with an equivalent mathematical operation, or the problem is only partially a real-world situation. A problem is inappropriate if either the problem does not involve a real-world situation or the problem cannot be solved with the given expression. Three hundred seven problems (69.5%) were classified as appropriate, 43 problems (9.7%) as partial, and 92 problems as inappropriate (20.8%). Table 2 presents

Table 2.

	$\frac{1}{4}*\frac{1}{3}$	$\frac{12}{\left(\frac{1}{4}\right)}$	4*1.8*3.2
Appropriate	1/4 of our class likes pizza. Out of that 1/4 only 1/3 like anchovies on their pizza. What fraction of the class likes anchovies on their pizza?	How many quarters are in 12 dollars?	Bill owns a lawn service. Bill has 4 lawn mowers. He needed to fill them up with gas. Each tank can hold 3.2 gallons. Gas costs $1.80 per gallon. How much did Bill spend to fill all 4 gas tanks.
Partial	Kelly saves a 1/4 of her sandwich to feed 3 ducks at the pond. How much of her total sandwich does each duck get?	It's Sarah's turn to bring dessert to her weekly girlscout meetings. She decided to bake cookies. If there are 12 other girls in her troupe (including here) and each girl will have four cookies. How many cookies will Sarah need to bake?	Sparkly nailpolish is $1.80 for every 3.2 milliliters. If four girls each want to buy milliliter of sparkly nailpolish, how much will it cost total for all 4 girls?
Inappropriate	Chris spent 1/4 of his paycheck on ice cream, and 1/3 of his paycheck on candy. What is the total product of what he spends of his paycheck?	There were 12 concerts going on at the fair. 1/4 of the concerts were country music. How many concerts were not country music?	We went to the movies 4 times this week. Adult tickets are 3.20 and childrens are 1.80. How much did we spend on tickets.

some responses (sic) and their classification. The first row provides the numerical expression to which the problem refers.

Findings and Conclusions

As mentioned earlier 307 out of 449 problems were accepted as appropriate. This gives about one appropriate problem per student for each expression. This situation is discouraging. The 43 problems classified as partial problems suggest the need for some instructional interventions in teacher education programs. Specifically, the problems shown on the third row are not correctly stated or raise some questions about the units used, for instance. But all of these problems have some potential to be appropriate problems. Finally, the problems classified as inappropriate (92) suggest weaknesses on the participants' knowledge of the relationships between the operations and real-world applications. Indeed, the idea of asking the problem in reverse (i.e. asking for a real-world problem that can be solved using the numerical statement) demands a greater abstraction in the relationship between the operation proposed and a real-world situation (Simon, 1993).

The fact that the content deals with basic operations makes these results very disappointing. Several problems classified as inappropriate were merely a verbalization of the expression proposed or a verbalization without a meaningful context. The product and division of fractions, and the product of three decimals were the expressions for which the participants did not produce a problem or produced inappropriate problems.

These findings lead us to recommend the following. Teachers need to have the opportunity to write interesting and meaningful problems, reflect on them, and discuss the connections to real-world applications even when the content is elementary. "Students should have to consider each textually represented situation on its merit, and the adequacy of any mathematical model they propose. From an educational point of view, this means that careful attention has to be paid to the variety of examples to which students are exposed," (Greer, 1993, pp. 245-246). The phrasing of the problem must also be considered for it has been argued to be a motivational factor for the problem solver (Butts, 1980). Finally, similarities between the problems posed points to the experiences that these students have had in their mathematics classrooms. Research in this area is needed.

Teachers are the "essential ingredient" to foster problem posing (Moses, Bjork, & Goldenberg, 1990). Let's help them by engaging them in both problem solving and problem posing in teacher education programs.

References

Butts, T. (1980). Posing problems properly. In S. Krulik & R. Reys (Eds.), *Problem solving in school mathematics, 1980 yearbook* (pp. 23-33). Reston, VA: NCTM.

Contreras, J. & Martínez-Cruz, A. M. (1999). Examining what prospective secondary teachers bring to teacher education: A preliminary analysis of their initial problem-posing abilities within geometric tasks. In F. Hitt, & M. Santos, (Eds.), *Proceedings, of the Twenty-First Annual Meeting for the Psychology of Mathematics-North America* (pp. 413-420). Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Geer, B. (1993). The mathematical modeling perspective on wor(l)d problems. *Journal of Mathematical Behavior, 12*, 239-250.

Moses, B., Bjork, E., & Goldenberg, E. P. (1990). Beyond problem solving: Problem posing. In T. J. Cooney & C. R. Hirsch, (Eds.), *Teaching and learning mathematics in the 1990s. 1990 Yearbook* (pp. 82-91). Reston, VA: NCTM.

NCTM. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

NCTM. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

NCTM. (1998). *Principles and standards for school mathematics: Discussion draft.* Reston, VA: Author.

NCTM. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Silver, E. A. (1994). On mathematical problem solving. *For the Learning of Mathematics, 14* (1), 19-28.

Silver, E. A., Mamona-Downs, J., Leung, S. S., & Kenney, P. A. (1996). Posing mathematical problems: An exploratory study. *Journal for Research in Mathematics Education*, *27* (3), 293-309.

Silverman, F. L., Winograd, K., & Strohauer, D. (1992). Student-generated story problems. *Arithmetic Teacher*, *39*, 6-12.

Simon, M. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education*, *24* (3), 233-254.

Simon, M. (1998). Developing a conceptual framework for mathematics teaching. In S. B. Berenson et al., (Eds.), *Proceedings of the Twentieth Annual Meeting of the Psychology of Mathematics Education- North America* (pp. 107-111). Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

DEVELOPMENT OF A RUBRIC TO MEASURE MATHEMATICS TEACHER REFORM

Doug McDougall
OISE/University of Toronto
dmcdougall@oise.utoronto.ca

John A. Ross
OISE/University of Toronto
jross@oise.utoronto.ca

Ann Le Sage
OISE/University of Toronto
alesage@oise.utoronto.ca

Abstract: This paper describes the development of a rubric to measure mathematics teacher reform. Studies were undertaken to identify ten dimensions of mathematics reform and to develop a four-level rubric for each of the ten dimensions. This rubric will assist teachers, teacher educators, and principals in determining the degree to which teachers have progressed towards mathematics reform ideals. We believe that understanding teacher mathematical reform status will help develop appropriate professional development programs.

Introduction

Most mathematics education reform documents emphasize the need to change the environment of the mathematics classroom from one in which the teacher "transmits" knowledge to the students to one where teachers and students interact in a community of mathematical investigation and exploration. Implementation of mathematics education reform contributes to higher student achievement on traditional math tasks (Villasenor & Kepner, 1993) and problem-solving (Brenner et al., 1997; Cardelle-Elawar, 1995; Schoen et al., 1999). However, there are many challenges in creating this environment and in determining the extent to which teachers have progressed towards math reform ideals. An assumption for many reform movements has been that teachers will adapt to change and need only be instructed on the nature of these changes for high fidelity implementation to occur. However, we know that these changes are much more complex and difficult to measure than has been previously suggested.

Although rubrics have been suggested as a method of evaluation of student work, few systematic attempts to instruct teachers on the use of rubrics have been reported. Moreover, little is known about the effects of rubrics on teachers' movement toward the ideals of mathematics reform. In this research study, we define ten dimensions of teacher reform in mathematics as well as create and test a rubric for identifying teacher placement along each dimension of reform.

Theoretical Framework

There is evidence that implementation of reform practices is a very lengthy process in which multi-year timelines are appropriate (Edwards & Roberts, 1998; Hall, Alquist, Hendrickson & George, 1999). Some studies have shown that even teachers in districts with well-funded implementation projects may deviate from reform ideals after only one year (Reys, Reys, Barnes, Beem & Papick, 1997). Other studies described patterns of implementation. For example, Spillane and Zeuli (1999) identified three categories of teachers who believed they were high fidelity implementers of reform. These categories were (i) teachers who introduced peripheral reform (such as modest use of manipulatives) while maintaining existing practices; (ii) teachers who introduced rich tasks into their classrooms while maintaining existing patterns of discourse that converted these tasks in algorithmic practice; and (iii) teachers who used rich student tasks to generate rich discourse about students' mathematical thinking. McDougall, Lawson, Ross, Maclellan, Kajander & Scane (2000) and Ross, Hogaboam-Gray & McDougall (2000) also found this category scheme distinguished teachers within their sample population. Several recent studies have also identified differences between veterans and novices in change processes, describing differences in practices and beliefs as teachers internalized reform ideals (Bright, Bowman & Vace, 1998; Gabriele, Joram, Trafton, Thiessen, Rathmell & Leutzinger, 1999; Hall et al., 1999; Nolder & Johnson, 1995; Slavit, 1996).

Although time, teacher beliefs and experience are important factors in the implementation of reform practices, there is evidence to suggest that textbooks, which support math education reform also, contribute to change in teaching practice. Prawat and Jennings (1997) found that textbooks, which contained rich mathematical tasks, stimulated teachers to rethink how they taught. As teachers became more familiar with the new texts they internalized key ideas and were able to dip in and out of the texts, as needed. Remillard (2000) also found that textbooks oriented toward a constructivist approach to math teaching contributed to changes in the classroom. More specifically, a Junior teacher who was initially opposed to using complex, open-ended tasks was persuaded to change her practice when she saw students constructing sophisticated solutions and discussing them thoughtfully. However, this teacher and others in the study were unable to take full advantage of textbooks that supported reform when sections of the text conflicted with their beliefs about teaching. Remillard concluded that textbooks were most effective when they required teachers to exercise their professional discretion, thereby leaving important decisions for teachers to make. By the same token, reform-oriented textbooks were most effective when not used verbatim but rather as an overview of the domain and as a source of materials to build upon the teacher's own ideas.

The catalogue of barriers to reform is a lengthy one. Foremost, teachers must be agents of a change they did not experience as students (Anderson & Piazza, 1996). Moreover, the pedagogy is not only different but also more difficult to learn and develop. For example, in traditional mathematics, there is a generic script that guides each lesson through a manageable body of content. In addition, teachers, especially elementary generalists, tend to lack the disciplinary knowledge required to make full use of rich problems (Stein, Gover, & Henningsen, 1996; Spillane, 2000) and texts cannot prescribe universally applicable courses of action for those teachers to follow (Remillard, 2000). Also, some teacher beliefs about mathematics (a rigid set of algorithms, not understandable by most students, that must be approached in an inflexible sequence) conflict with reform conceptions of math as a fluid, dynamic set of conceptual tools that can be used by all (Gregg, 1995; Prawat & Jennings, 1997). It follows, therefore, that reform ideals often conflict with parental expectations about how math should be taught and assessed (Graue & Smith, 1996; Lehrer & Shumow, 1997). Reform conceptions of mathematics also conflict with mandated assessment programs that measure computational speed and accuracy (Firestone, Mayrowetz, & Fairman, 1998). Finally, Keiser and Lambdin (1998) found that student constructions took longer than lecture-recitations, novel problems increased class time taken for discussion of homework, and students with poor motor skills took longer to use manipulatives than anticipated. Even experienced and well-trained teachers encounter difficulty in implementing reform (Stein, Gover & Henningsen, 1996: Prawat & Jennings, 1997). Developing meaningful problems around which to organize their lessons can be substantial challenge. Thus, adoption of reform mathematics can leave teachers felling less efficacious because their contribution to student learning is less visible than in traditional teaching (Ross et al., 1997; Smith, 1996).

Design and Methodology

This is a continuing study based on the work with Grades 7 and 8 teachers (McDougall et al., 2000) and Grades 1 to 3 teachers (Ross et al., 2000) to investigate the characteristics of mathematics education reform. The rubric was developed in three studies involving samples of grade 1-3, grade 7-8, and grade 1-6 teachers. In each study we used a survey consisting of 20-30 Likert-style items measuring self-reported implementation of math reform to select teachers for observation. The selection survey was extensively pilot tested, validated through interviews and observations. The most recent version of the survey consists of 20 items that tap 10 dimensions of math teaching (described below). Teachers agree or disagree with each statement using a 6-point scale anchored by strongly agree and strongly disagree. In the field tests the instrument was internally consistent (alpha=.81). Evidence of its validity is presented in Ross, McDougall, and Hogaboam-Gray (in press).

After selection by the survey, we interviewed teachers about their implementation of each of the ten dimensions. In our most recent procedure, prior to the visits an observation-planning sheet was developed for each teacher. In the first column of the planning sheet we estimated the level of reform implementation based on a draft rubric for implementation of Primary mathematics developed in our first study (Ross, Hogaboam-Gray, & McDougall, forthcoming). We assigned levels of implementation based on a 1-4 scale (ranging from a traditional or direct instruction math program to one representing the highest ideals of math education reform that could be described

851

as constructivist teaching). We recorded the specific interview evidence used to assign the level and the level of confidence we felt in the assignment.

The observations followed procedures established by Simon and Tzur (1999). Teachers were interviewed before, during and after each math lesson to elicit the teacher's intentions and reflections on the lessons observed. It was anticipated that the teachers would add to the reform elements rather than replace traditional teaching (Garet & Mills, 1995). The observation planning guide for each teacher individualized the agenda for the observations. Field notes consisted of a narrative of instructional events, linked to the ten dimensions of reform, with special attention to issues emerging from the interview data for that teacher.

Results and Elements of the Rubric

Ten dimensions were identified (Ross et al., 2000) with each divided into four levels of reform within the rubric. The ten dimensions of math education reform (e.g., NCTM 1989; 1991; 2000) are (i) broader scope (e.g., multiple strands with increased attention on strands less commonly taught such as data management and probability, rather than an exclusive focus on numeration and operations) with access to all forms of mathematics for all students. (ii) Student tasks are complex, open-ended problems embedded in real life contexts; many of these problems do not afford a single solution. In traditional math, students work on routine applications of basic operations in decontextualized, single solution problems. Leighton, Rogers, and Maguire (1999) suggested that formal (traditional) tasks differ from informal (reform) tasks in that formal hold all relevant information required to solve the problem (whereas informal require the solver to bring knowledge to the problem). They suggest that traditional tasks are self-contained, provide a single correct answer, can be solved using conventional procedures, involve solutions that are unambiguous, entail topics that are of academic interest only and do not prepare students to solve real life problems. (iii) Instruction in reform classes focuses on the construction of mathematical ideas through student talk rather than transmission through presentation, practice, feedback, and remediation. (iv) The teacher's role in reform settings is that of co-learner and creator of a mathematical community rather than sole knowledge expert. (v) Mathematical problems are undertaken in reform classes with the aid of manipulatives and with ready access to mathematical tools (calculators and computers), support not present in traditional programs. (vi) In reform teaching the classroom is organized to promote student-student interaction, rather than to suppress it. (vii) Assessment in the reform class is authentic (i.e., analogous to tasks undertaken by professional mathematicians), integrated with everyday events, and taps a wide variety of abilities, in contrast with end of week and unit tests of near transfer that characterize assessment in traditional programs. (viii) The teacher's conception of mathematics in the reform class is that of a dynamic (i.e., changing) discipline rather than a fixed body of knowledge. (ix) Teachers in reform settings make the development of student self-confidence in mathematics as important as achievement, in contrast with the traditional view that achievement is the most important student outcome.

There is not sufficient space in this paper to display the rubric. The presentation at PME-NA 2001 will include a copy of the rubric and further areas of use. However, for illustrative purposes, one set of levels will be discussed. One of the dimensions of reform is the degree to which teachers promote student-student interactions in the learning of mathematics. In the traditional classroom (Level 1), students learn mathematics working individually, and student-student interactions are limited thus supporting the view that students are passive learners. In Level 2 implementation, the role of the student is to listen to teacher's explanation, and to participate in an equal amount of individual and group activities. However, the teacher assigns individual tasks that are completed in a group setting. There is some encouragement of help seeking and giving. In Level 3 implementation, the teacher assigns interdependent tasks, and students work cooperatively. The students are given leadership roles and are required to train peers. Although, students are active learners, they are not required to give explanations as part of the task.

By contrast, in the reform oriented classroom (Level 4), teachers create opportunities for students to learn from their peers through establishing mixed ability groups, assigning interdependent tasks which involve students cooperation, and requiring students to explain their mathematical ideas to peers. The teacher creates an environment that reflects respect for all student ideas and structures time to allow for sharing of these ideas within the mathematics program. Teachers who provide this form of student-student interaction are creating a reform-oriented mathematics classroom.

This is one of the ten dimensions of the rubric. We have found the rubric to be useful in situating a teacher along the spectrum of mathematics reform ideals. We are aware, however, that Mathematics reform is very difficult to implement and even teachers chosen as exemplars of reform practice regress from the ideal, displaying the height of reform one day but regressing to traditional methods the next (Senger, 1998). Thus, the rubric is meant as a tool to help mathematics educators and consultants identify the level of teacher reform to best develop professional development activities that will help teachers further implement mathematics reform ideals.

References

Anderson, D. S., & Piazza, J. A. (1996). Changing beliefs: Teaching and learning mathematics in constructivist preservice classrooms. *Action in Teacher Education, 17* (2), 51-62.

Brenner, M. E., Mayer, R. E., Moseley, B., Brar, T., Duran, R., Reed, B. S., & Webb, D. (1997). Learning by understanding: The role of multiple representations in learning algebra. *American Educational Research Journal, 34* (4), 663-689.

Bright, G. W., Bowman, A. H., & Vace, N. N. (1998). Reflections about listening and dialogue as evidence of teacher change. *Proceedings of the International*

Group of the Psychology of Mathematics Education-North America (Vol. 2, pp. 607-612). Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Cardelle-Elawar, M. (1995). Effects of metacognitive instruction on low achievers in mathematics problems. *Teaching and Teacher Education*, *11* (1), 81-95.

Edwards, T. G., & Roberts, S. K. (1998). Teacher change during an urban systemic initiative. *Proceedings of the International Group of the Psychology of Mathematics Education-North America* (Vol. 2, pp. 668-674). Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Firestone, W. A., Mayrowetz, D., & Fairman, J. (1998). Performance-based assessment and instructional change: The effects of testing in Maine and Maryland. *Educational Evaluation and Policy Analysis*, *20* (2), 95-113.

Gabriele, A. J., Joram, E., Trafton, P., Thiessen, D., Rathmell, E., & Leutzinger, L. (1999, April). *Traveling along the path of mathematics reform: Changes in teachers' source of efficacy*. Paper presented at the annual meeting of the American Educational Research Association, Montreal.

Garet, M. S., & Mills, V. (1995). Changes in teaching practices: The effects of the Curriculum and Evaluation Standards. *The Mathematics Teacher*, *88* (5), 380-388.

Graue, M. E., & Smith, S. Z. (1996). Parents and mathematics education reform. *Urban Education*, *30* (4), 395-421.

Gregg, J. (1995). The tensions and contradictions of the school mathematics traditions. *Journal for Research in Mathematics Education,* 26 (5), 442-466.

Hall, G. E., Alquist, A., Hendrickson, M., & George, A. A. (1999, April). *Bringing national standards for the teaching and learning of mathematics to the classroom: Implementation, student outcomes and system impacts.* Paper presented at the annual meeting of the American Educational Research Association, Montreal.

Keiser, J. M., & Lambdin, D. V. (1996). The clock is ticking: Time constraint issues in mathematics teaching reform. *Journal of Educational Research*, *90* (1), 23-31.

Lehrer, R., & Shumow, L. (1997). Aligning the construction zones of parents and teachers for mathematics reform. *Cognition and Instruction*, *15* (1), 41-83.

Leighton, J. P., Rogers, W. T., & Maguire, T. O. (1999). Assessment of student problem solving on ill defined tasks. *Alberta Journal of Educational Research*, *65* (4), 409-427.

McDougall, D.E, Lawson, A., Ross, J.A., Maclellan, J., Kajander, A. & Scane, J. (2000). *A study on the impact math implementation strategy for the Ontario mathematics curriculum grades 7 & 8* (p.182). Ontario Ministry of Education and Training.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics: An overview*. Reston, VA: National Council of Teachers of Mathematics.

Nolder, R., & Johnson, D. (1995). Professional development: Bringing teachers to the center of the stage. *Mathematics in School*, Jan, 32-36.

Prawat, R., & Jennings, N. (1997). Students as context in mathematics reform: The story of two upper elementary teachers. *Elementary School Journal, 97* (3), 251-270.

Remillard, J. (2000). Can curriculum materials support teachers' learning? Two fourth-grade teachers' use of a new textbook. *Elementary School Journal, 100* (4), 251-270.

Reys, B. J., Reys, R. E., Barnes, D., Beem, J., & Papick, I. (1997). Collaborative curriculum investigation as a vehicle for teacher enhancement and mathematics curriculum reform. *School Science and Mathematics, 87* (5), 253-259.

Ross, J. A., Hogaboam-Gray, A., & McDougall, D. (2000). *Elementary school reform and higher order reasoning in mathematics*. Final report of MET Transfer Grant.

Ross, J. A., McKeiver, S., & Hogaboam-Gray, A. (1997). Fluctuations in teacher efficacy during the implementation of destreaming. *Canadian Journal of Education, 22* (3), 283-296.

Schoen, H. L., Fey, J. T., Hirsch, C. R., & Coxford, A. F. (1999). Issues and options in the math wars. *Phi Delta Kappan, 80* (6), 444-453.

Senger, E. S. (1998). Examining teacher and classroom supports for student invention. *Proceedings of the International Group of the Psychology of Mathematics Education-North America,* (Vol. 2, 234-243). Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Simon, M. A., & Tzur, R. (1999). Explicating the teacher's perspective from the researchers' perspectives: Generating accounts of mathematics teachers' practice. *Journal for Research in Mathematics Education, 30* (3), 252-264.

Slavit, D. (1996). Graphing calculators in a 'hybrid' algebra II classroom. *For the Learning of Mathematics, 16* (1), 9-14.

Smith, J. P. (1996). Efficacy and teaching mathematics by telling: A challenge for reform. *Journal for Research in Mathematics Education, 27* (4), 387-402.

Spillane, J. P. (2000). A fifth-grade teacher's reconstruction of mathematics and literacy teaching: Exploring interactions among identity, learning, and subject matter. *Elementary School Journal, 100* (4), 307-330.

Spillane, J. P., & Zeuli, J. S. (1999). Reform and teaching: Exploring patterns of practice in the context of national and state mathematics reforms. *Educational Evaluation and Policy Analysis*, *21* (1), 1-27.

Stein, M., Gover, B., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33* (2), 455-488.

Villasenor, A., & Kepner, H. S. (1993). Arithmetic from a problem-solving perspective: An urban implementation. *Journal for Research in Mathematics Education*, *24* (1), 62-69.

WHAT MATHEMATICAL KNOWLEDGE DO PRESERVICE ELEMENTARY TEACHERS VALUE AND REMEMBER?

Mercedes A. McGowen
William Rainey Harper College

Gary E. Davis
Rutgers University,

Introduction

This study addresses the question of how pre-service elementary teachers can be induced to change their attitudes to mathematics. We know that pre-service elementary teachers want to get mathematical answers right. They want to know which formulas to use, and how to get the correct answer. We present transcriptions of student writing that illustrate and support this point. Changing what students *value* in mathematics is a much harder challenge than teaching them mathematical procedures and application of formulas. We need an antidote to a severely procedural orientation to mathematics focused on 'correct answers' that prospective teachers have learned to value above all. How can we explicitly emphasize connections, and assist students to construct relationships between parts of mathematics that they see as different?

Theoretical Perspective

We could approach the issue of pre-service teachers' attitudes about mathematics through the extensive literature on change, and teacher change in particular. However, our approach in this study is a different one. It is to focus on students' long-term declarative memories and the relation of these memories to individual student gain in test scores over the course. Thus we are attempting to build a bridge between what student teachers themselves recall as important in their learning of mathematics with what test scores tell us.

Declarative memories are those we are capable of expressing in words, drawings or gestures. They are to be distinguished from *implicit*, or non-declarative, memories that assist us to carry out routine procedures and habits (Squire, 1994). Long-term declarative memories are mediated by protein formation, following gene expression, to stimulate novel neuronal connections (Squire & Kandell, 2000). The relevance of this neurological fact is that long-term memory formation is an energetic, committed process for an individual. Long-term memories—certainly those sustained over a period of several months—are therefore a good indicator of what a student values. Using long-term declarative memories as an indicator of what pre-service teachers valued about the mathematics they engaged with, we focused on connections between combinatorial problems (Maher & Speiser, 1997) to promote the formation of such memories. The model of Davis, Hill & Smith (2000) emphasizes a role for a mathematics teacher in assisting students to turn implicit, procedural memories into declarative memories. We used that model to guide us in designing a series of interrelated combinatorial problems for the beginning of a 16-week mathematics content course for pre-service elementary teachers.

Episodic memory is the system of memory that allows us to explicitly recall events in time or place in which we were personally involved. (Tulving, 1983; Tulving & Craik, 2000). *Semantic memory* is the memory system that deals with our knowledge of facts and concepts, including names and terms of language. (Tulving, 1972, p. 386; Tulving, 1983; Tulving & Craik, 2000). *Explanative memory* is that part of declarative memory dealing with explanations for facts. Davis, Hill, Simpson, & Smith (2000) present a case that explanative memory is a separate memory system, linked to, but different from episodic or semantic memory. Most psychological studies of memory are oriented to memory for language. Studies of memory for mathematics are much less common. A semantic memory such as '*Paris is the capital of France*' has quite different content to one such as '*the square of the length of the hypotenuse in a right angled triangle is the sum of the squares of the lengths of the other two sides*'. The first is a linguistic convention, the other expresses a non-obvious fact. Note that this is the way *we* see these semantic facts: we do not claim that students see them similarly. In fact, they often interpret mathematical facts in a similar way to simple linguistic conventions.

There are, at least, the following distinctions in memory for mathematical facts:

1. Labels, customs, and conventions. For example: A prime number is a whole number with exactly 2 factors.
2. Things sensed, or done. For example: The proportion of prime numbers less than 500 is 19%.
3. Things believed. For example: There are infinitely many prime numbers.
4. Things explained. For example: A proof that there are infinitely many prime numbers.

We refer to these remembered facts as *semantic conventions, semantic actions*, *semantic beliefs*, and *semantic explanations*, respectively. Episodic memories and semantic actions—memories of things sensed or done in mathematical settings—are easily confounded. The reason, of course, is that a semantic action, by its very nature, involves memory of sensing or doing something. However, the critical difference is that one senses or does some non-trivial fact—calculating the number of primes less than 100, for example. The episode, remembered as such, is overlaid with an act or sense of a fact that goes beyond a mere episode. The distinction between episodic memories and semantic actions is one that distinguishes memory for mathematics from more everyday memories, as does, in part explanative memory—memory of mathematical explanations.

In this paper we will be concerned almost exclusively with semantic conventions—the linguistic conventions of mathematics. As we remarked, above, these are the simplest form of mathematical factual memory: the memories carrying the least mathematical information. They are non the less interesting. The main point of this paper is that the number of semantic conventions—memories of conventional mathematical language—a student has and can express at the end of a mathematics content course is an indicator of the degree of engagement with the mathematics content. That engagement, in turn, is indicative of a change in attitude as to what mathematics is about. Our principal aim in relating semantic conventions to student test scores is to generate hypotheses as to what motivates student change of attitude in mathematics classes, and to the nature of mathematics. Our hypothesis generation uses a combination of heuristic methods advocated by McGuire (1997) and is "shamelessly eclectic", in the spirit of Rossman & Wilson (1994), in that it mixes both qualitative and quantitative methods in formulating a relevant hypothesis about student growth.

Method

We set a number of connected combinatorial problems in the first 3 weeks of a 16 week course mathematics content course for pre-service elementary teachers. A detailed description of all the problems set in the first weeks of the course is seen at: www.soton.ac.uk/~plr199/algebra.html
These combinatorial problems were specifically designed to:

1. Be relatively straightforward to begin working on, given the students' backgrounds.
2. Not be susceptible to easy solution by known or remembered formulas.
3. Set up strong episodic memories as a result of students discussing their solutions in class.

For example, after students had attempted the problem of finding how many towers of heights 4 and 5 they could build using blocks of 1 or 2 colors, they were shown, and discussed, a video clip of four grade 4 students attempting the same problem. This problem and its connections with algebra, which we utilized, has been reported on by Maher (1998), Maher & Speiser (1997), Maher, C. A., & Martino, A. M. (1996).

Students worked on the problems in groups. We asked them to write reflectively after each of the combinatorial problem sessions, and re-writes were encouraged. After completing the sequence of problems, they explained connections between the problems as a homework exercise. Opportunities for making connections with their earlier work were provided during the semester in questions students hadn't seen previously on three group and two individual exams. Students also wrote mid-term and end-of course self-evaluations.

The prompts provided to students to generate their initial written statements included the following:

Reflections on Towers Building I

Reflect on building 4-high and 5-high towers .Complete the following:

1. When I first read this problem, I thought ....
2. I first attempted the problem by ....
3. After working with my group I realized ....
4. I know I have found all possible towers because ....

Reflections on Towers Building II

Write a brief paragraph describing your observations and reactions to the video of the "gang of four " students working the Towers problem.

Complete the following sentences:

1. Before seeing the video of the 4th graders working the Towers Problem, I thought....
2. After seeing the video, I ....
3. As a result of the class discussions, I now realize ....
4. I have the following additional comments and/or reflections....

These questions placed value on what students saw and thought, before and after various activities, what their reasons were for making various statements, and their reactions to the problems and issues raised by them. Other questions asked of students placed value on seeing and explaining connections between apparently different problems.

A statistic we focus on is the *individual gain factor*: *(final test% – initial test%)/ (100-initial test%)* (Davis & McGowen, 2001; McGowen & Davis, 2001; Hake, 1998). Individual gain factors are important in measuring student growth because they tell us what the student achieved in tests given what was possible for them to achieve. We also coded the written work of students for instances of statements of semantic conventions. One student's written work was omitted from the analysis: this student had recently suffered brain trauma due to a car accident and had some difficulty with verbal and written communication, and with memory.

Results

Students' written statements at the beginning of the course indicated wide-spread instrumental views of mathematics. Typical statements include:

- "Coming into this class, I was under the impression that finding a formula to solve a problem was, in reality, the answer to the problem."

- "All throughout school, we have been taught that mathematics is simply just plugging numbers into a learned equation. The teacher would just show us the equation dealing with what we were studying and we would complete the equation given different numbers because we were shown how to do it."
- "As long as I could remember I have seen math as getting the right answer, and that being the only answer."

Students' written comments and end of the semester interviews indicate that more than three-fourths of the students (15 of 19 or 79%) experienced changed attitudes towards mathematics and what it means to learn mathematics. The final write-up carried out individually by all students at week 16 included many examples of positive feelings about mathematics and the course. Typical of these comments are the following:

- "When our class finally concluded that the towers, tunnels, grids and Pascal's Triangle were all about 'choices', everything seemed to fall into place. ..."
- "... my perspective of mathematics changed over this semester ... learning that my mathematical understanding was instrumental and not relational. I had to re-learn basic math in order to eventually teach it to children."
- "Not only did we get the answers, we made connections with other ideas.... That is a true way of learning mathematics and what it means."
- "This class helped me realize you will get nowhere (*sic*) knowing equations ... you have to understand what you are doing. ... I have learned more in this class than any other class that I have ever taken."
- "I feel this was the most productive experience I have ever had in my educational career. ... I deeply feel that I will be a better educator because of it."
- " I have learned that mathematics is indeed a series of interrelated ideas. I was challenged to extract these connections from our daily work while acquiring new skills in mathematics."

Typical semantic conventions, relating to mathematics, in the student's written work include:

- "Prime numbers are counting numbers with exactly and only two different factors."
- "A bit is the smallest term of measurement."
- "Factors are numbers that can be multiplied together to get another number."
- "Triangular numbers are the cumulative sum of counting numbers."
- "... I discovered how each relates. Both present two choices, or a 'binomial'."

- "In our class discussion, we defined algorithm as a systematic procedure that one follows to find the answer."
- " The relationship between addition and subtraction is an inverse relationship and can be demonstrated with a four-fact family."

Individual student gain factors ranged from 0.14 through 0.91. The mean gain factor for the class was 0.56 with standard deviation of 0.19. Fourteen of the 19 students (74%) who completed the course had a gain factor of 0.5 or higher.

With the exception of the students with highest and lowest gain factors (0.91 & 0.14), the number of statements of mathematical convention generally increased with gain factor. We argue that the student, LJ, with the highest gain factor is an outlier in that LJ made very explicit statements about the way she approached the course. LJ was the only student to make remarks related to organization and we infer that this attitude played a role in her significant gain. Student NM, with the lowest gain factor (0.14), was also the student with the highest pre-test score: 77% (the pass mark was 80%). NM made many written statements about seeing the mathematics through the eyes of a child, and how that mathematics could be taught to a child. We infer that NM was reasonably satisfied with the level of mathematical understanding indicated by her test scores and concentrated, instead, on how that mathematics could be better taught to children.

A plot of $\log_2$ (# semantic conventions) versus gain factor gives a correlation with $r^2 = 0.70$ when the student with highest and lowest gain factors are excluded ($r^2 = 0.48$ when they are included). The number of semantic conventions did not correlate significantly with either pre- or post-test scores.

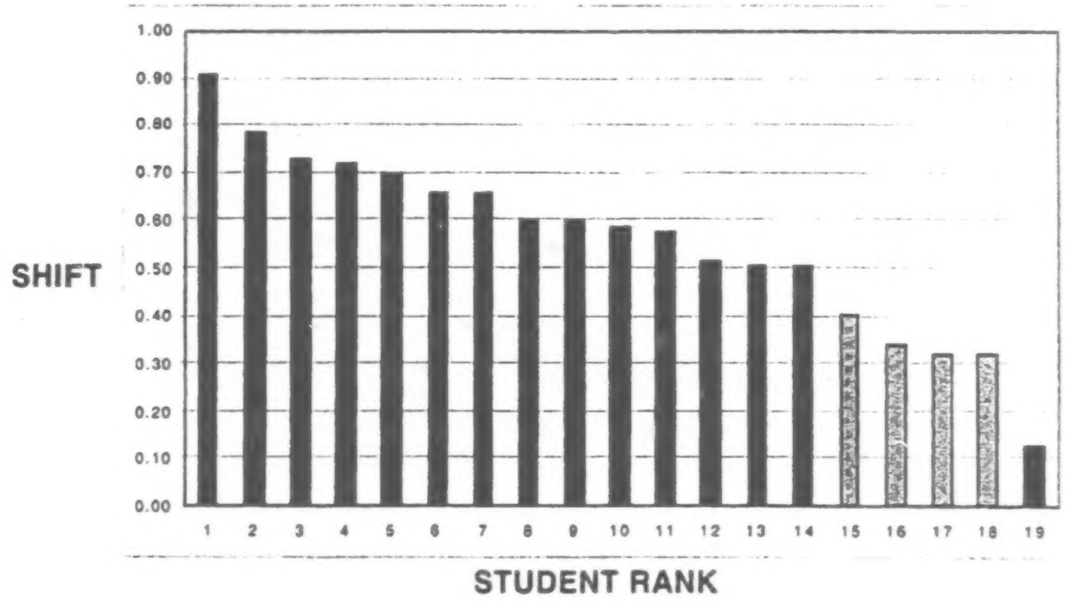


Figure 1. Student rank vs. shift factor.

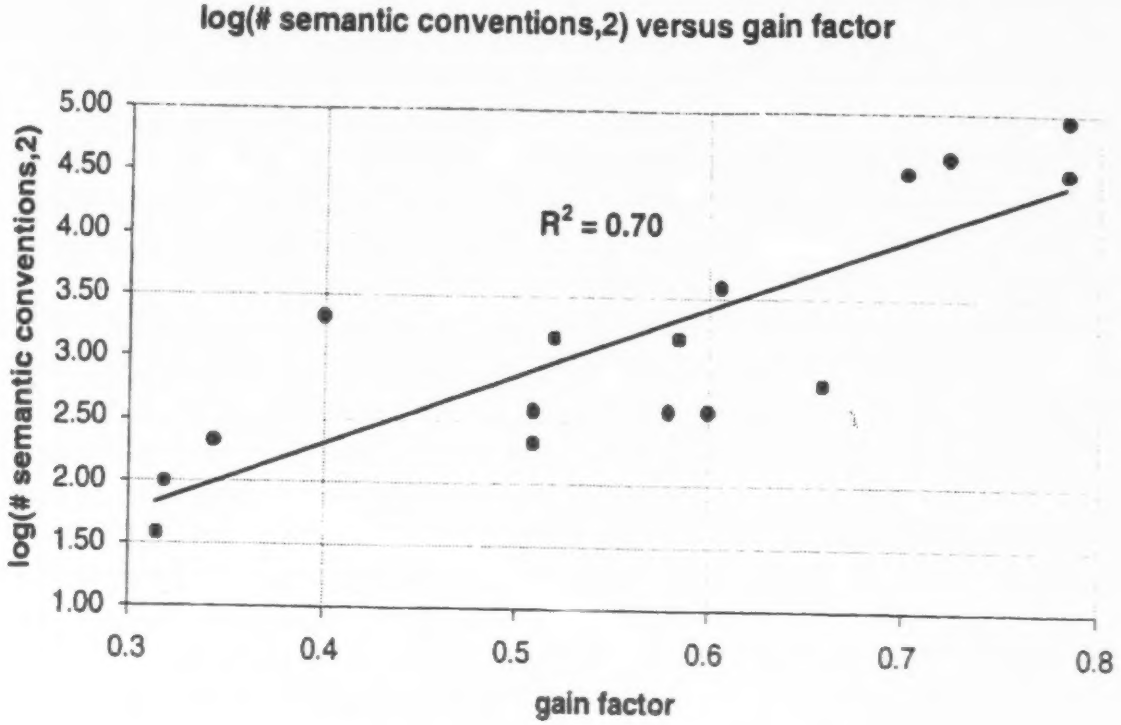


Figure 2. Semantic conventions vs. gain factor. Plot of the logarithm (base 2) of the number of semantic labels in the writing of 16 out of 19 students, versus the gain factor ($r^2 = 0.70$). The students with highest and lowest gain factors are omitted, as is one student who had difficulty with expression and memory due to brain trauma.

Discussion

On the basis of the data presented here we hypothesize that there are at least two distinct factors associated with a student obtaining a high gain factor (≥ 0.5). First there is a conscious determination to organize one's personal life and involvement in the course so as to give oneself the maximum chance of growth. This is illustrated by LJ, whose growth factor was the highest in the class at 0.91 (z-score 1.81). LJ wrote:

> "By immersing myself in the mathematics through group study sessions, timely arrival to class, and the completion of homework, I was able to give myself a solid foundation in order to delve deeper into mathematics. Throughout this semester, I have missed one class, and have been tardy (five-minutes) once."

Second, and not obvious until we counted occurrences of conventional mathematical terms in student writing, most of the students we studied who had above average gain factors used a number of conventional mathematical terms in their writings that increased exponentially with their gain factors. These two things taken together—the increase in gain factor correlated with an increase in the use of conventional mathematical terms—suggest to us that for these students mathematical growth was associated with an acceptance of and a long-term memory for the language of mathematics. It seemed to us that these students were "buying in" to a mathematical ethos, accepting that newly encountered, or revisited, mathematical words mean something other than an everyday connotation might suggest. Along with that change in attitude went a corresponding growth in mathematical achievement.

For most pre-service elementary teachers an important first step in coming to grips with mathematics, and developing mathematical maturity, is a decision to take the language of mathematics seriously, and to work on developing long-term memories for the words and terms they meet in content courses. The role of the instructor is paramount in setting up experiences for students that (a) establish strong episodic memories and (b) allow and encourage opportunities for students to express themselves using mathematical language.

We utilized a combination of methodologies—qualitative and quantitative—to formulate our main hypothesis, which is that, for a majority of students, higher gain factors are correlated exponentially with higher use of conventional mathematical terms. Further, both these factors are indicators of student growth in mathematics, accompanied with a changed attitude toward mathematics. Progress on this hypothesis would require a larger cohort of students, and pre- and post-administration of a mathematics attitude inventory such as that of Sandman (1980).

We also hypothesize that the structure and form of the early weeks of the course, focusing as it did on seeing connections and building strong episodic memories, assisted students in changing their attitude to the nature of mathematics. Written comments indicate that for approximately three-fourths of the class this was the case. There was little in the written comments of the remaining students, or in their individual gain factors, that indicate a long-term change in attitudes to mathematics or substantial increase in basic skills.

We believe, but do not provide evidence here for the belief, that it is through an emphasis on developing strong episodic memories of procedural aspects of mathematics that the implicit procedures gain a touch of "emotional color", and through discussion and reflection are shaped into long-term semantic memories. This was a guiding principle of the construction of the material for the course and the conduct of the sessions.

Finally, we hypothesize that it is an individual student's flexibility of thought in mathematical settings, that is largely a determinant of their mathematical growth in understanding and basic skills throughout the course. Krutetskii (1969), Dubrovina (1992), Shapiro (1992), and Gray & Tall (1994). Krutetskii (1969) and Shapiro (1992) characterize *flexible thinking* as reversibility: the establishment of two-way relationships indicated by an ability to "make the transition from a 'direct' association to its corresponding 'reverse' association" (Krutetskii, 1969, p. 50). Gray and Tall (1994) characterize flexible thinking in terms of an ability to move between interpreting notation as a process to do something (procedural) and as an object to think with and about (conceptual), depending upon the context. For the students reported on here, flexibility of thought encompasses both Krutetskii's and Gray & Tall's ideas as facets of a broader notion of flexibility (McGowen (1998). In addition to reversible associations, and proceptual thinking focused on the use of syntax to evoke both mental process and mental object, we are interested in connections between various representations of problem situation, which we refer to as conceptual. An inability to use syntax flexibly creates the *proceptual divide* (Gray & Tall, 1994) and is, in a broader sense, part of a *conceptual divide* (McGowen, 1998) in which flexibility is compounded by student difficulties in using and translating among various representational forms. Our hypothesis is that while flexibility is modifiable over the longer term it is probable that setting expectations for growth in flexible thinking at the beginning of a course plays a major role in determining how students grow mathematically.

References

Davis, G., Hill, D., & Smith, N. (2000) A memory-based model for aspects of mathematics teaching. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the Twenty-Fourth Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 225-232). Hiroshima, Japan: Hiroshima University.

Davis, G. E., Hill, D. J. W., Simpson, A. P., & Smith, N. C. Explanative memory in mathematics. Under revision for *Mathematical Thinking and Learning*.

Davis, G. E., & McGowen. (2001) Jennifer's journey: Seeing and remembering mathematical connections in a pre-service elementary teachers course. *Proceedings of the Twenty-Fifth Conference of the International Group of the Psychology of Mathematics Education*. Utrecht, Holland: Freudenthal Institute, Utrecht University.

Dubrovina, I. V. (1992). The nature of abilities in the primary school child. In J. Kilpatrick & I. Wirszup (Eds.), *Soviet studies in the psychology of learning and teaching mathematics* (Vol. 8, pp. 65-96). Chicago: University of Chicago Press.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25* (2), 116-140.

Hake, R. R. (1998). Interactive-engagement versus traditional methods: A six-thousand-student survey of mechanics test data for introductory physics courses. *American Journal of Physics, 66* (1), 64-74.

Krutetskii, V. A. (1969). An investigation of mathematical abilities in schoolchildren. In J. Kilpatrick & I. Wirszup (Eds.), *Soviet studies in the psychology of learning and teaching mathematics* (Vol. 1, pp. 5-57). Chicago: University of Chicago Press.

McGuire, W. J. (1997). Creative hypothesis generating in psychology: Some useful heuristics. *Annual Reviews in Psychology, 48*, 1–30.

Maher, C. A. (1998). *Can teachers help children make convincing arguments? A glimpse into the process.* Instituto de Educação Matemática, Rio de Janeiro, Brazil: Gráfica Botânica.

Maher, C., & Speiser, R. (1997). How far can you go with block towers? In E. Pehkonen (Ed.), *Proceedings of the Twenty-First Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 174-181). Helsinki: University of Helsinki.

Maher, C. A., & Martino, A. M. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27* (2), 194-214.

McGowen, M., & Davis, G. (2001). Changing pre-service elementary teachers' attitudes to algebra. *Proceedings of the 12th ICMI Study on the Future of the Teaching and Learning of Algebra Conference*, University of Melbourne, Australia.

McGowen, M. (1998) *Cognitive units, concept images, and cognitive collages: An examination of processes of knowledge construction* (Unpublished doctoral dissertation). University of Warwick, Coventry, U.K.

Rossman G. B., & Wilson B. L. (1994). Numbers and words revisited - being shamelessly eclectic. *Quality & Quantity, 28 (*3), 315-327.

Sandman, R. S. (1980). The mathematics attitude inventory: Instrument and user's manual. *Journal for Research in Mathematics Education, 11* (2*)*, 148-49.

Shapiro, S. I. (1992). A psychological analysis of the structure of mathematical abilities in grades 9 and 10. In J. Kilpatrick & I. Wirszup (Eds.), *Soviet studies in the psychology of learning and teaching mathematics* (Vol. 8, pp. 97-142). Chicago: University of Chicago Press.

Smith, N. C., Davis, G. E., & Hill, D. J. W. (in press) A classroom-experiment in mathematical memory. *The Journal of Mathematical Behavior*.

Squire, L. R. (1994). Declarative and non-declarative memory: Multiple brain systems supporting learning and memory. In D. L. Schacter & E. Tulving (Eds.), *Memory systems (*pp. 203-232). Cambridge, MA: MIT Press.

Squire, L. R., & Kandell, E. (2000). From molecules to mind. *Scientific American*.

Tulving, E. (1983). *Elements of episodic memory*. Oxford: Oxford University Press.

Tulving, E., & Craik, F. I. M. (Eds.). *The Oxford handbook of memory*. Oxford: Oxford University Press.

MENTORING STYLES IN MATHEMATICS: TWO CONTRASTING CASES

Blake E. Peterson
Brigham Young University
peterson@mathed.byu.edu

Steven R. Williams
Brigham Young University
williams@mathed.byu.edu

Abstract: This paper examines the conversations held between prospective secondary mathematics teachers and their cooperating teachers over the course of their student teaching experience. Particular attention is payed to the beliefs of the cooperating teachers, the shared views that develop, and the perceived relationships among views of mathematics, mentoring, and management evident in the conversations.

Theoretical Background

The Influence of Cooperating Teachers

The student teaching experience is consistently seen by prospective teachers as a critically important component of their teacher preparation programs. Certainly the cooperating teacher has a great influence on both the quality of this experience and the development of behaviors and beliefs in the student teacher (McIntyre, Byrd, & Fox, 1996), whether that influence is positive, negative, or neutral (Frykholm, 1996; Hawkey, 1998; Evertson & Smithey, 2000; Winitsky, Stoddart, & O'Keefe, 1992).

The influence that a cooperating teacher may have on a student teacher is clearly affected by the beliefs that each hold. Each pre-service teacher, for example, enters the student teaching experience with an image of what teaching is all about. Hawkey (1996), investigated the images of two pre-service history teachers and how these images changed over the course of the student teaching experience. One of the subjects had a clear image of herself as a teacher while the other struggled to articulate such an image. The student teacher who struggled simply adapted the image of the environment into which he was placed. Hawkey concludes that although the images are dynamic, the degree and way in which they change depends on how well formed the image is at the beginning of the student teaching experience.

Cooperating teachers also have beliefs that affect their relationship with student teachers. Martin (1997) indicates that mentor teachers approach the mentoring process in the same way they approach their teaching. Elliot and Calderhead (1993) found that "in many cases mentors indicated that they began to think about mentoring in the same terms as they thought about their own teaching" (p. 178). They also found that the mentors in their study believed that "learning to teach required students to experience a range of teaching environments from which they could select specific teaching events and 'absorb' them. (pg. 179)" They interpret this to mean that mentors in general saw learning to teach as simple as opposed to complex and multidimensional. Similarly, Haggerty (1995) found that mentors failed to recognize and address the complex nature of the decisions that teacher make, behaving instead as if it were not problematic for new teachers to implement good practice.

In our later analysis, we find a framework from Maynard and Furlong (1993) to be very useful. They identify five stages in the development of student teachers: Early Idealism, in which, prior to student teaching, the preservice teachers have a clear idea of the type of teacher that they want to be; Survival, in which the reality of the classroom causes the student teacher to move from being idealistic to seeking ways to survive; Recognizing difficulties, in which, the new teacher begins to recognize certain strengths and weaknesses and become concerned with the assessment part of the student teaching experience; Hitting the plateau, in which the student teacher has settled in and found some successful strategies for management and control, and have a tendency to stagnate because "they have found a way of teaching that seems to work and they are going to stick to it! (p. 72)"; and Moving on , a phase of development which may or may not occur, in which student teachers try new techniques, and are concerned about student learning and reflection on practice. Maynard and Furlong also describe three general models of mentoring: the *apprenticeship* model, the *competency* model and the *reflective* model. In the apprenticeship model, the student teacher simply seeks to emulate the experienced teacher's practices under his/her watchful eye. In the competencies model, the mentor has checklist of skills that he/she teaches to the novice teacher and assesses the mastery of them. The reflective model is more commonly known as the "reflective practitioner" model in which the novice teacher is trained to reflect on their own practice. Each of these models has strengths and weaknesses and each can be adapted to different phases of the student teachers stages of development. For instance, mastering a list of specified skills can be a good thing but it can also lead a new teacher to hit a plateau in which they feel they have mastered teaching (see also Elliot and Calderhead, 1993).

Interactions Between Student Teachers and Cooperating Teachers

Whatever influence cooperating teachers enjoy is mediated through the interactions and conversations they have with their student teachers. As recently as 1998, Hawkey commented on the "relative lack of studies which focus on observation and analysis of mentoring interactions" (p. 657). The work that has been done on the discussions held between cooperating teachers and their student teachers paints a somewhat consistent picture. Cooperating teachers tend to dominate the discussion (O'Neal & Edwards, 1983; Ben-Peretz & Rumney, 1991), which focused on classroom events and activities, specific teaching events, and the methods and materials of teaching (O'Neal & Edwards, 1983). Talk also tends to be evaluative or descriptive but not necessarily prescriptive – little attention is focused on reflection or on alternative instructional approaches (Ben-Peretz & Rumney, 1991). Tabachnick, Popkewitz,

& Zeichner (1979) found that classroom managment, procedural issues, and directions were the primary foci of cooperating teachers in their interactions with student teachers, and that what was taught and the purposes for teaching it were seldom addressed.

Borko & Mayfield (1995) conducted a detailed study of the conferences held between middle school mathematics student teachers and both cooperating teachers and university supervisors. They identified four domains of teacher knowledge that were addressed in eight or more of the conferences: pedagogy, students, math-specific pedagogy, and mathematics. They note that general pedagogical issues were discussed in eight of the nine conferences and account for most of the specific suggestions offered to student teachers. Students were discussed in all nine conferences, but primarily in ways that were concerned with lesson flow rather than focusing on student understanding. Mathematical pedagogy was also discussed in each of the nine conferences, but mainly at the level of general strategies rather than specific representations for particular mathematical concepts. Mathematics was rarely discussed in any but a superficial manner in any of the conferences.

Beliefs About Mathematics and Its Teaching

Although it is true that the central concerns of student teachers are those of survival in the classroom (management, discipline, presentation), it seems strange that so little mathematical discussion occurs among student-teacher/cooperating-teacher pairs. This is particularly unfortunate in light of Ball's (1991) findings on the mathematical preparation of teachers. She found three assumptions that dominated teacher preparation:

> (1) that traditional school mathematics content is not difficult, (2) that precollege education provides teachers with much of what they need to know about mathematics, and (3) that majoring in mathematics ensures subject matter knowledge (p. 449).

In reality, none of these assumptions hold up under scrutiny. Still, they form a core of the culture of teaching mathematics. Ma (1999), in comparing U.S. and Chinese teachers, observed

> Although the U.S. teachers were concerned with teaching for conceptual understanding, their responses reflected a view common in the United States —that elementary mathematics is "basic," an arbitrary collection of facts and rules in which doing mathematics means following set procedures step-by-step to arrive at answers (p. 123).

Stigler & Hiebert (1999) point out that "the typical U.S. lesson is consistent with the belief that school mathematics is a set of procedures" (p. 89). They go on to note that when teachers were asked that main thing they wanted students to learn from a given lesson, "61% of U.S. teachers described skills they wanted their students to learn" (p. 89).

This underlying, cultural view of school mathematics surely has implications for how student teachers learn to teach mathematics. Reuben Hersh observes,

> One's conceptions of what mathematics *is* affects one's conception of how it should be presented. One's manner of presenting it is an indication of what one believes to be most essential in it. . . .The issue, then, is not, What is the best way to teach? but, *What is mathematics really about?* (1986, p. 13).

If mathematical knowledge is taken for granted, and is never a focus of reflection in the student teaching experience, student teachers will settle into teaching practices that largely perpetuate the cultural view. They can become skillful motivators and managers – attempting to "jazz up" their lesson to make them interesting, or breaking tasks into manageable pieces and giving practice (cf. Stigler and Hiebert 1999), but will probably not change the basic character of what they teach.

It is important, then, to understand the nature of the conversations held between cooperating teachers and student. This project was undertaken to better understand (1) the major conversational topics between secondary mathematics student teachers and their cooperating teachers, (2) the nature of the mathematical discussions held between secondary mathematics student teachers and their cooperating teachers, and (3) the ways in which discussions of pedagogy, mathematical content knowledge, and pedagogical content knowledge were related.

Methods and Data Sources

Participants and Setting

Data reported here are from a larger study of eight secondary mathematics student teachers from a large private university, and their cooperating teacher from surrounding public schools.In an initial attempt to understanding the nature of interactions between student teachers and their cooperating teachers, a questionnaire was prepared asking about the amount of time cooperating teachers planned to spend meeting formally with their student teachers, and what percent of the time would be spend discussing management, mathematics, and pedagogy. It was distributed to all cooperating teachers who had agreed to accept a student teacher during the coming Fall Semester. From the 25 who responded, we chose 8, attempting to balance the sample on level of school (high school vs middle school), anticipated amount of contact time (more than one formal meeting per week vs less than one), and anticipated amount of time spend on mathematical topics (more than 15% vs. less than 15%). Each of the 8 teachers and their student teachers agreed to participate in the study, and were given a small stipend.

Data Sources

The corpus of data collected for this study consisted of two interviews for each participant, self-reports of contacts collected randomly through the semester, and audiotaped recordings of selected conversations.

Midterm and End of Term Interviews

Each participant was interviewed at roughly the middle of the semester. Interviews were conducted either at the schools or during a mid-semester meeting of the student teachers. These interviews were transcribed for analysis.

Selected Recorded Conversations

Each pair of participants were given a hand-held cassette recorder and a supply of tapes. They were asked to record any conversations they held that they expected to be over10 minutes in duration. A total of 42 conversations, ranging from 1 per pair to 9 per pair, were recorded. These conversations were transcribed for analysis.

Data Analysis

Analysis of the interviews and recorded conversations proceeded in three phases. In the first phase, project staff read the interview data and attempted to build a preliminary description of the mentor/protege relationship for each pair. That is, we initially approached the data as if we were conducting eight case studies. As we discussed our preliminary descriptions of each pair, common themes began to emerge. A second pass through the interview data was made to look for confirming and disconfirming evidence for the themes. In the second phase, project staff examined the recorded conversations with the goal of finding confirming and disconfirming evidence for both our initial descriptions of each pair, as well as the themes that emerged in the first stage. In the third phase, two pairs were chosen to highlight specific themes relating to mentoring style, beliefs about mathematics and teaching, and their effects on the student teaching experience.

Results and Analysis

Perhaps the best entry into understanding the experiences of these two dyads is by way of the central conversational themes that developed over the course of the semester. These themes were expressed both by the cooperating teacher and by the student teacher as major areas of discussion, and kept appearing as central ideas in their conversations.

Mr. B and Blake

For Mr. B and Blake, the theme can be described easily: Management is the central issue in teaching, and all else – including mathematics and how to teach it, is easy to learn once the management issues are resolved. A number of statements from both Mr. B and Blake make the identification of this central theme clear. In one interview, Blake said, "We spend most of our time talking about management issues," which was borne out by analysis of their recorded conversations. Roughly 77% of the coded text units in Blake's and Mr. B's conversations dealt with management. Indeed, when asked what he hoped his student teachers took away from their experience, Mr. B replied in one sentence: "The ability to manage kids without conflict." The importance of classroom management to this pair is clear from other of Mr. B's comments: "In the junior high level, the whole name of the game is classroom management. If you never get beyond that, you never teach. Teaching strategies and different lessons are easy to learn and you can pick a lot of that up from watching other teachers or workshops." "Until you have control of the kids there is no teaching that goes on." "You can't teach if you don't have control." Blake echoes these concerns: "Classroom management is something in itself that needs to be established and once that's established other things can build off of it."

One effect of this focus on management is the relegation of mathematical and pedagogical issues to secondary status. As Blake said, "To be honest, we didn't talk a lot about how to teach certain subjects." Mr. B said, "I still don't think that the challenge of teaching mathematics, at least at the junior high level, is mathematics. It's not having a better understanding of what we need to teach. I think it's understanding how to control them, and I feel that's the key to helping the mathematics at this level in the junior high." Whether Blake was influenced by this strong belief, or whether he had simply found a soul mate in Mr. B, it was clear that Blake felt the same way. He felt that because he taught comparatively low level mathematics (Algebra and Pre-Algebra), he wasn't learning anything about mathematics, nor did he and Mr. B need to discuss it. "Especially since we teach pre-algebra and algebra, we rarely talk about mathematics content." Earlier, he said he felt "like a mathematical king." Analysis of their conversations indicate that fewer than six percent of their utterances were about mathematics, or about how to teach a mathematical topic.

Besides mathematics and management, Mr. B's core beliefs also seemed to affect his mentoring style. He described his role at the first of the semester as "one of almost dominance, 'OK, you're going to do it my way.'" He valued above all Blake's willingness to "listen and try" what he suggested. Later, Mr. B characterized his role as evolving "more into the role of a coach than a dictator or the one that was in control of everything." This echoes, in a sense, Mr. B's basic model of teaching: once control has been established, then things can be done to soften the feel of the class, to make connections, to go beyond the basics. But, both in his mentoring, and in his teaching, basic skills come first: "But the core, the heart of what they learn, is not done with manipulatives or calculators. The core is traditional instruction and letting them practice it enough so that they internalize it." This was, in essence, Mr. B's approach with Blake.

Mr. T and Tara

For Mr. T and Tara, two themes emerged. In many of their conversations, Mr. T would guide Tara to realize that the students were better behaved when the students were actively involved in the learning process. They also spent a significant amount of time discussing the underlying concepts of sixth-grade mathematics such as the meanings of the various operations on whole numbers as well as how those meanings applied to operations on fractions.

Active Participation. In the interviews, both Tara and Mr. T commented that active participation was a common discussion point. In the midterm interview Tara said "He's [Mr. T] real big on active participation. If they have something to do, if they're doing things along with you. Have them sketching the things you're talking about. The more that the students are doing, the better behaved they're going to be because they'll have something to keep them busy." In the final interview when Tara was asked what she and Mr. T had discussed in terms of classroom management she responded, "Active participation. He's a real promoter of, if you can keep the students busy, if they have things that they're doing while you're up there lecturing or whatever, then classroom management is going to be reduced by half just because they have something to do." Similarly when Mr. T was asked about specific points that the had discussed with Tara, he responded, "One I keep going over and over again is, I refer to it as "active participation." Others might call it staying on task. I tell her that it really doesn't matter as much what you do in the class (it does matter, but that is not the important thing). The important thing is what the kid does. [It doesn't help] if she gives the best presentation and does a super job, and all the kids sit there and do is watch it." He followed this up weeks later with the comment, "The one thing that I think I've worked on and we've stressed and she's worked on and we've done together is being aware that it doesn't matter nearly as much what she does and how neat a presentation she gives, but what the kids actually do. If they're just sitting there, listening and watching or maybe just sitting there, it's not the same as for them to be actively participating in what's happening." From these comments it can be seen that Mr. T viewed active participation not only as a method of management but as a way to help students gain greater understanding.

Sixth Grade Mathematics. The second common theme evident in the interviews and recorded conversations was the discussion of mathematics at the sixth grade level. In the midterm interview when Tara was asked what she had learned about mathematics since she started teaching and what their mathematical discussions have focused on, she responded, "I've learned a lot about the basics. Division. What is it? What exactly ...When you say 1/2 divided by 1/3, what exactly are you saying? What exactly does it mean? And how to show it? He's real big with manipulatives. How to show it with a picture. It's amazing. I mean I've never..I didn't even ... I mean if I thought about it, yea ... But he has a way to show it every time." "What exactly the basics are. What exactly does this mean? Three times 4, what does it mean? Three groups of four. A lot of that. A lot of conceptual teaching. 'Conceptually what does this mean?'" In the final interview she repeated similar comments. When asked what strengths she had gained, she said, "In terms of math I definitely gained strengths also in just the basics of math: add, subtract, multiply, divide. What it means. What it means to add fractions. A lot of visual concepts.” When asked what mathematics she had learned, she mentioned “Also multiplication and division, what it actually means, how to show it with pictures. Like, there’s kind of two different ways that we look at division: how many groups can it be divided into and sharing and grouping. What you’re actually saying when you’re saying ‘4 divided by 2.’”

Mr. T made similar comments about the mathematics that they had discussed, learned and focused on during their conversations. He said, “I’d start with maybe whole numbers, and showing them what timesing [multiplying] whole numbers like three times five means five groups of threes and three groups of five. Three times by 1/5 means 3 groups of 1/5 and 1/5 of 3. I’ve done the same with division. Too often with division, the kids don’t know why they invert and they don’t know even what it means when it comes to story problems. So we’ve talked about having them actually draw pictures and then find how many fifths there are in it.”

It is worth pausing here to contrast views of management and mathematics between our two pairs. For Mr. T and Tara, management tended to be a by-product of students actively participating in the learning process, rather than as a pre-requisite for learning. Moreover, Tara clearly felt she had learned mathematics during her experience. This learning did not occur as a result of conversations whose aim was her learning new mathematics; indeed less than 1% of the coded text utterances were focused on mathematics. However, 22% were focused on teaching mathematics, and from this, Tara felt she learned a great deal. Mr. T recalls: “She’s come up several times and said she learned things she hadn’t realized before and it kind of surprised me because this is a sixth grade classroom and she’s a math major. I haven’t had that much advanced math so it surprised me.”

It is also the case, as with Mr. B, that Mr. T’s beliefs affected his mentoring style. His core belief that students had to be actively involved in learning was reflected strongly in his interactions with Tara. Mr. T described a typical session with Tara in this way:

> We’ll sit down at night and the first thing we always do is to talk about how things went during the day. I get her to try to evaluate what went well and what didn’t go well. She really knows better than me. For her to reflect on it with and come up what she feels really went well and then I ask her ‘what could have gone different?,’ ‘what would you do different?,’ ‘what would you continue to do?’ I try to get her to reflect on her own teaching and then she usually wants to know, of course, what I think, so then I will start with the positive and say ‘well, here’s what I liked and this is really good, this is really good, and this is an area where we need to look at some things.’ Then from the experience I’ve had we look at the lesson she’s going to do the next day and I try to show her some methods that have worked for me and let her choose.”

The statement is an accurate description of what he actually did in their recorded conversations. Tara also made a comment that substantiated this approach to mentoring by saying, "We review a lot, which is really nice. We talk about how I did, what did I do that was good, what could have been better."

Some of Mr. T's typical questions to Tara are illustrated in the statements below:

- "how would you suggest that could have gone better?"
- "what could have been done differently?"
- "You said you felt better about the 3rd period than the 2nd period, Why?" After some discussion he followed that up with "So what will you do tomorrow, do you think, with that period different than the 3rd?"
- "What do you suppose could be some reasons for classes going better today as far as behavior and so on?"
- "Is there anything else on today's lesson that you want to talk about?"

As can be seen from these questions, Mr T would help Tara reflect on how her actions affected the classroom climate. He would also ask questions to help her reflect on how the tasks that she chose for her students to do affected the climate. Mr. T's general mentoring style was one of watching and monitoring closely but asking questions in a way that would enable Tara to reflect and solve the problem for herself. It was only after he had really pushed her to think for herself that he would describe some of the things that he had done in the past and let her choose. In fact, his description of his role as a cooperating teacher was very consistent with what he actually did, and was also consistent with his beliefs in the importance of his students' (and Tara's) active participation in the learning process.

A Final Comparison – The Role of Classroom Activities

One way to see the differences that the two belief systems discussed above make in classroom life is by looking at how each of the two pairs dealt with classroom activities – classroom segments that used visualization, manipulatives, calculators, or richer tasks that were the norm. Mr. T and Tara would use these in an effort to provide various paths for learning; as Mr T said, presenting things "not only visually, but working sometimes with hands-on [materials]. The kids learn math in different ways – math is often so abstract that they need something more concrete." Activities for Mr. T and Tara also helped reach their goal of active participation: "Math isn't a spectator sport, they don't learn that way. . . .make sure that every one of them is doing something and not just watching because you don't get it. You don't get that watching someone else do it."

By contrast, Mr. B and Blake used activities to "soften" the feel of the classroom. As Mr. B explained, "I use the manipulatives and calculators more as a break or diversion to the routine of learning mathematics and try to make connections with them. But the core, the heart of what they learn, is not done with manipulatives or calculators. The core is traditional instruction and letting them practice it enough so that they internalize it." Blake describes their use of activities as "softening the feel" of the classroom. Mr. B commented that after establishing control of the classroom, "you can bring in the activities and the kids feel like they forget about the negativeness they felt about learning mathematics. All of a sudden the kids like you and they like math and the next day you can go back to math and everything is back in focus." For Mr. B and Blake, the participation of students in activities had mainly an affective result, with mathematical connections being a possible by-product.

Summary

The two belief systems represented by Mr. T and Mr. B had real effects on the student teaching experience of their student teachers. Although it is possible that Tara and Blake were serendipitously placed with teachers whose beliefs matched their own, it is more likely that Mr. T and Mr. B each had a profound affect on their student teacher. Tara likely left the experience with the belief that mathematics needed to be critically examined from the point of view of the learner, and that this process was the foundation of teaching. Blake likely left the experience with the belief that classroom control was the foundation of teaching, and that examining either his own mathematical understanding, or his students' mathematical understanding, was secondary. At the core of these differences was the role the cooperating teachers gave to mathematics –whether the understanding of mathematics essentially drove instruction, or was secondary to management issues.

In terms of the classifications given by Maynard and Furlong (1993), we feel that Mr. B's mentoring style was competency-based and was able to take Blake as far as the plateau, but provided no real impetus for moving beyond. By contrast, we feel that Mr. T's reflective mentoring of Tara gave her a model for Moving on–the ability to reflect on her practice, and to analyze mathematics from the point of view of the learner.

References

Ball, D. (1991). Research on teaching mathematics: Making subject-matter knowledge part of the equation. In J. Brophy (Ed.), *Advances in research on teaching* (Vol. 2, pp. 1-48). Greenwich, CT: JAI Press.

Ben-Peretz, M., & Rumney, S. (1991). Professional thinking in guided practice. *Teaching and Teacher Education*, *7* (5/6), 517-530.

Borko, H., & Mayfield, V. (1995). The roles of the cooperating teacher and university supervisor in learning to teach. *Teaching and Teacher Education*, *11* (5), 501-518.

Elliot, B., & Calderhead J. (1993). Mentoring for teacher development: Possiblilites and caveats. In D. McIntyre, H. Hagger, & M. Wilkin (Eds.), *Mentoring: Perspectives on school-based teacher education* (pp. 166-189). London: Kogan Page.

Evertson, C. M., & Smithey, M. W. (2000). Mentoring effects on proteges' classroom practice: An experimental field study. *The Journal of Educational Research, 93* (5), 294-304.

Frykholm, J. A. (1996). Pre-service teachers in mathematics: Struggling with the standards. *Teaching and Teacher Education*, *12* (6), 665-81.

Haggerty, L. (1995). The use of content analysis to explore conversations between school teacher mentors and student teachers. *British Educational Research Journal*, *21* (2), 183-197.

Hawkey, K. (1998). Mentor pedagogy and student teacher professional development: A study of two mentoring relationships. *Teaching and Teacher Education, 14* (6), 657-670.

Hawkey, K. (1996). Image and the pressure to conform in learning to teach. *Teaching and Teacher Education*, *12* (1), 99-108.

Hersh, R. (1986). Some proposals for reviving the philosophy of mathematics. In T. Tymoczko (Ed.), *New directions in the philosophy of mathematics* (pp. 9-28). Boston: Birkhäuser.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Erlbaum.

Martin, D. (1997). Mentoring in one's own classroom: An exploratory study of contexts. *Teaching and Teacher Education, 13* (2), 183-197.

Maynard, T., & Furlong J. (1993). Learning to teach and models of mentoring. In D. McIntyre, H. Hagger & M. Wilkin (Eds.), *Mentoring: Perspectives on school-based teacher education* (pp. 69-85). London: Kogan Page.

McIntyre, J., Byrd, D. M., & Foxx, S. M. (1996). Field and laboratory experiences. In J. Sikula, T. J. Buttery, & E. Guyton (Eds.), *Handbook of research on teacher education*, (2nd ed., pp. 171-193). New York: Macmillan.

O'Neal, S., & Edwards, S (1983). *The supervision of student teaching.* Paper presented at the annual meeting of the American Educational Research Association, Montreal, Canada, April 11-15, 1983.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* New York: Free Press.

Tabachnick, B. R., Popkewitz, T. S., & Zeichner, K. M. (1979). Teacher education and the professional perspectives of student teachers. *Interchange on Educational Policy, 10* (4), 12-29.

Winitsky, N., Stoddart, T., & O'Keefe, P. (1992). Great expectations: Emergent professional development schools. *Journal of Teacher Education, 43* (1), 3-18.

PRE-SERVICE ELEMENTARY TEACHERS AND PLACE VALUE: WRITTEN ASSESSMENT USING A DIGIT-CORRESPONDENCE TASK

Sharon H. Ross
California State University, Chico

Abstract: In this investigation, 254 preservice elementary school teachers were asked to explain the meanings of the individual digits for a randomly arranged collection of 25 pennies. Approximately half the undergraduates in a mathematics course for preservice elementary teachers were unable to explain that, for a collection of 25 pennies, the "2" in "25" represented twenty of the pennies and the "5" represented 5 pennies. Descriptions are provided of the differences and similarities in the responses of those who had completed a course which included numeration as a topic for study and those who had not completed the course. Results are compared with those from experienced elementary teachers. Implications for instruction and future research are discussed.

Background

To assess place-value understanding, the teacher used a straightforward digit-correspondence task. During math class the teacher asked the student to determine "how many" in a collection of 25 randomly arranged pennies and then to write the number ("25"). The teacher then circled the "5" and asked "What does this part of your 25 have to do with how many pennies you have?" The student wrote "It means five groups of five."

This response is typical of "wrong" answers given by third, fourth, or fifth graders (author, 1989, 1990, 1999). However, the subject in this vignette is a sophomore in college, enrolled in a mathematics content course for future elementary school teachers.

In elementary school mathematics, children have traditionally found place value difficult to learn and teachers have found it difficult to teach. Assessment of place-value understanding has also been difficult, since understanding requires a coordination of socially transmitted knowledge and individually constructed knowledge of both additive and multiplicative relationships.

What is place value? Our numeration system is characterized by the following four mathematical properties:

1. Additive property. The quantity represented by the whole numeral is the sum of the values represented by the individual digits.
2. Positional property. The quantities represented by the individual digits are determined by the position they hold in the whole numeral.

3. Base-ten property. The values of the positions increase in powers of ten from right to left.
4. Multiplicative property. The value of an individual digit is found by multiplying the face value of the digit by the value assigned to its position.

An understanding of place-value numeration can be held at many levels: it may include, for example, an understanding of "tens and ones," of computational procedures, of decimal numerals, binary numerals, or of scientific notation.

The National Council of Teachers of Mathematics (NCTM) Principles and Standards for School Mathematics call for the development of "initial understandings" in the primary grades (NCTM, 2000). But studies of young children have suggested that, limited by their developing conceptualization of the concept of number and numerical class inclusion, their understanding of place value may be problematic in the early grades (P. Cobb, & G. Wheatley, 1988; C. Kamii, & L. Joseph, 1988; E. Labinowicz, 1989; S. Ross, 1986; Steffe & Cobb, 1988). Irregularities in English number words, as contrasted with the linguistic support of number words inherent in many Asian languages, also contribute to the challenges young children face in constructing knowledge of our numeration system in the early grades (Fuson & Kwon, 1992; I. Miura, & Y. Okamoto, 1989).

The data reported in this exploratory study was collected as part of a larger study to examine differences in the explanations of "tens and ones" among students in grades 4-12, university students, and teachers. The learning and teaching of early place-value concepts has been studied under a variety of conditions and with a variety of instruments and designs, but we know little about the place value knowledge of older students or adults, including teachers, or their ways of communicating their understandings.

Purpose

This paper focuses on the conceptual understandings of place value numeration revealed by the pre-service elementary teacher subjects' written responses to a digit-correspondence task.

In many research studies of the mathematical thinking of elementary school students, digit-correspondence tasks have been used in semi-structured interview settings. In digit-correspondence tasks, students are asked to construct meaning for the individual digits in a multi-digit numeral by matching the digits to quantities in a collection of objects. As measured by such tasks, even in the fourth and fifth grades no more than half the students demonstrate an understanding that the "5" in "25" represents five of the objects and the "2" the remaining 20 objects (M. Kamii, 1982; Ross, 1989; Ross, 1990). By collecting and examining digit-correspondence data from 237 preservice teachers and 17 experienced teachers, we hoped to help to inform the search for the best ways to facilitate student understanding of base-ten numeration.

Research Questions

1. How well do preservice elementary teachers understand place value as assessed by examining written responses to a double-digit correspondence task?
2. How do the responses of preservice teachers who have completed a curriculum unit about numeration at the university level compare to those of students who have not yet studied numeration at the university level?
3. Are subjects' explanations dependent on the two-digit number that is used in the digit-correspondence prompt?

Sample

Data was collected from 254 preservice elementary teachers at a California State University campus that prepares a large number of teachers. In 2000-2001 1400 undergraduates majored in Liberal Studies, most hoping to become credentialed to teach multiple subjects in grades K-6. Two 3-semester-hour mathematics courses are required in the major. The content of the first course (MATH 50A) includes the structure of the real number system, numeration, whole number operations and algorithms, rational number operations, and optional sections on sets and number theory. The second course (MATH 50B) addresses probability, statistics, measurement, and geometry. The Department of Mathematics and Statistics uses a common text for the courses. In the 2000-2001 academic year the text was Bassereur (2001a). In the previous three years the adopted text was Long and DeTemple (1997). Local community colleges use Musser and Burger (1999). Of the 10-12 sections on our campus per semester, about half are taught by tenured or probationary faculty, and half are taught by temporary (adjunct) faculty. Most (but not all) instructors choose to make extensive use of collaborative group activities, problem-solving, communication of reasoning and manipulative materials among their instructional strategies and goals.

Approximately 14% of Liberal Studies majors choose mathematics as their Area of Concentration (AOC). The AOC requires 18 semester units of mathematics. In addition to MATH 50A and 50B, two additional three-unit courses are designed specifically for the future teacher. One of these focuses on algebraic thinking and the other on intuitive foundations of geometry. In California a credentialed teacher can be authorized to teach mathematics as a single subject through grade nine by completing 20 semester units of mathematics. Most Liberal Studies majors with mathematics as their AOC take one additional course to obtain their Supplementary Authorization in Introductory Mathematics.

Data for the present study was collected from three sections of MATH 50A at the beginning of the semester, before the numeration unit had been studied. Students had been introduced to the course with some problem solving tasks and informal set operations. One section was taught by an assistant professor, the second was team-taught by a full professor and an elementary teacher-in-residence. The third section was taught by the teacher-in-residence. (The Teacher-in-Residence program is funded in part by a grant from the Exxon Education Foundation.) Data was also collected from five sections of MATH 50B. Two were taught by a full professor, one by an adjunct lecturer with high school teaching experience, and two evening sections were taught by a local high school teacher in our Master's degree program in Mathematics Education. Approximately 90% of these students had completed MATH 50A (or an equivalent transfer course at a community college) with a grade of C- or better. The majority of the remaining 10% were concurrently enrolled in 50A (and had recently completed the numeration unit in MATH 50A).

The numeration unit in MATH 50A typically includes an overview of some historical systems (e.g., Babylonian, Egyptian, Roman, Mayan). Discussion may focus on a comparison of their properties with those of our conventional Hindu-Arabic base-ten system. The numeration unit also includes examination of other bases. The texts all use base five and several others. Students are expected to "count" in another base, and to convert from base ten to base five, for example, and vice-versa. They are also provided opportunities to deepen their conceptual underpinnings of both conventional and alternative and even invented computational algorithms by computing in these less familiar bases of numeration.

Data for the present study was collected from one additional group of 17 undergraduate preservice elementary teachers (PSTs). These were students participating in "MathLinks," a campus program established by a mathematics faculty member, which provides training and field experiences in elementary school classrooms. Students in this program are selected from among those applying. In this sense the tutors are self-selected. Applicants are required to have completed MATH 50A; most have also completed 50B. An interview is used to assess their beliefs about procedural / conceptual learning and teaching of mathematics. Preference is given to those who seem "driven to understand how children make sense of the math" (informal interview with project director, Dr. Eliza Berry, May 31, 2001). They are expected to attend an initial six-hour training session and biweekly two-hour training sessions and spend a minimum of five hours per week as a math aide in an elementary classroom at a partnership school. MathLinks tutors are paid. By October, when the data was collected, several place-value activities had been used in the training sessions (e.g., activities double digit-addition and subtraction on hundreds boards).

For purposes of comparison, data was also collected from 15 teachers, a student teacher, and the principal from one elementary school.

Instrumentation: The Digit-Correspondence Task

To assist in assessing the effects of a teaching experiment with elementary school students, researchers designed a written digit-correspondence task that could be administered to the whole class, rather than in individual interviews (Ross & Sunflower, 1995, Ross, 1999). Based on examination of student work, a descriptive scoring rubric for categorizing the students' responses was developed. The task and rubric used in the present study was adapted from that work.

In this study, each subject received a picture of 25 randomly arranged pennies and two colored pens. Aided by an overhead projector transparency of the picture, the researcher elicited a consensus that the number of pennies in the picture was 25. The researcher then wrote "25" on the transparency and asked students to do the same on their paper.

The researcher then said "We've written twenty-five for the 25 pennies. She then circled the "5" in red and asked the students to do the same. She then asked, "What does this part of your 25 have to do with how many pennies are in the picture? Use the red pen to write what you think, and use the picture to show what you mean by drawing or coloring." The prompt was also printed on the transparency, and revealed after a first reading. The prompt was repeated with the prompt on the screen. After allowing for response time, she circled the "2" with a blue pen, asked students to do the same, and then asked "How about this part? What does THIS part have to do with how many pennies are in the picture?" Use the blue pen to write what you think, and use the picture to show what you mean by drawing or coloring. The second prompt was also printed on the transparency and revealed after the first reading.

Data Collection Procedures

All data from the eight MATH 50A and 50B sections was collected during regular class time, either in the first 10-15 minutes or the last 15 minutes. The regular instructors introduced the researcher. The researcher read to the class the assent script approved by the university's Committee on Research with Human Subjects and then began a warm-up by asking students to estimate the number of pennies flashed on the screen with an overhead projector. When 25 pennies had been estimated and counted, the 8 x 11 paper with a picture of 25 ungrouped circles at the top of the page was distributed to the students, along with red and blue pens. After the digit-correspondence task was completed, instructors collected the student papers. Instructors read the student work, which had names, but to assure anonymity for the research, cut off the names before giving the student papers to the researcher.

For the MathLinks group, data was collected as in the 50AB classes at one of their October bi-monthly evening training meetings.

Data was collected from 17 elementary teachers at the beginning of an after-school faculty meeting, in April. These subjects were asked NOT to identify themselves by name, but were surveyed as to the number of years they had taught various grade levels.

Results

The 85 preservice elementary teachers (PSTs) enrolled in the first semester of a two-semester sequence of required mathematics content courses performed about as well on the 25-pennies task as fifth graders when given the same task, under the same conditions. Only 53% were successful, where success is defined as explanations that convey that "2" represents 20 pennies or two groups of ten pennies, and "5" represents five pennies. One 50A student circled two sets of ten circles in blue, colored each of the circles with them blue, and colored the remaining circles red. She wrote:

> If grouped into two groups of 10s, the number 5 represents the group of 1's that's remaining.
>
> The 2 represents 2 groups of 10s.

The 78 PSTs in the MATH 50B course, most of whom had been provided some instruction in numeration in the previous course, performed somewhat better on the 25-pennies task. 70% of them were successful. This difference favoring the PSTs in MATH 50B was statistically significant at the 0.95 level.

In both groups the most common unsuccessful responses explained that the "5" represented groups of five pennies, or five groups of pennies, or five groups of five pennies (30% in 50A, 27% in 50B). These students usually circled five groups of five pennies in red.

> The 5—there are 5 groups of 5. [Draws a picture of five sets of five circles.]
>
> The 2—2 times 12 + 1 = 25.

Other invented meanings were infrequent. One example is the student who grouped the pennies as described in the writing below.

> There are 5 groups of pennies with 3 in each set.
>
> There are 5 groups of pennies with 2 in each set.

A few students from each group invented meanings not typical of elementary school students' responses. They explained that $5^2 = 25$ and assigned meanings for the digits that map accordingly; "5" is for the base and "2" for the exponent. One 50A student colored nineteen pennies blue and 6 pennies red, and wrote:

> 5 is the square root of 25.
>
> 2 is part of the square root of 25.

Another 50A student circled five groups of five with her red pen, colored the five circles in one group red, colored two circles in another group blue, and wrote:

> The 5 in 25 shows that there are 5 times itself (5 x 5) pennys [sic] in the picture.

The 2 in 25 explains that if you take 5 and square it (5^2) you get 25.

In addition to the six classes who responded to the 25-pennies task, one 50B class was asked the same digit-correspondence questions about 24 pennies (where the "five groups of five" response would not be likely). In the 24-pennies task, 80% of the 31 students were successful. All of the unsuccessful responses explain that the "4" represented four groups, or groups of four, and the "2" represented two groups or groups of two.

Another 50B class of 26 students was asked about 23 pennies (a prime number where "groupings" would be difficult to make). For the 23-penny task, 77% were successful. Half the unsuccessful students made groups of 2 and groups of 3.

In the MathLinks group of 17 students, all were successful. One example of a response from this group was the following, from a one-year veteran of the program. This student had experience conducting individual place-value assessments with primary-grades students. The assessment tasks were adapted by Exxon Educational Foundation teacher-in residence Robert Callahan, from Richardson (1998) and Ross (1986).

1. This part of the twenty-five represents the set of five pennies left over *after* you sort the pennies into groups of ten.
2. The 2 in twenty-five represents the twenty pennies, or two groups of ten.

[This MathLinks tutor circled five pennies in red, and colored those pennies red. There is an arrow from her explanation #1 (above) to that set. Two sets of ten pennies are circled in blue. Each penny within those sets are colored blue. Near the first set she has written, with an arrow, "1 group of ten". After the second set she has a comment, with an arrow, that says "2 groups of ten in all".]

The elementary teachers all successfully made the correspondences between the individual digits and five pennies, and 20 pennies. But like many PSTs, some added extraneous details.

5 is one-fifth of the total number of pennies.

20 is 4/5 of the total amount.

[This sixth grade teacher circled five sets of five pennies. Each penny in one set is colored red and in four sets the circles are colored in blue.]

Discussion

Research Questions

How well do preservice elementary teachers understand place value as assessed by examining written responses to a double-digit correspondence task? Student explanations of the meanings of the individual digits in a double-digit numeral were weak.

Was the students' apparent weakness in providing mathematically adequate explanations due to the nature of the task? The digit-correspondence task used in this investigation was limited in informing us about what PSTs actually know about place value. The digit-correspondence task prompt does not provide any cues to help the responder to access the "place-value" schema in the brain. There is no use of the phrase place-value, or of the word "digit." Nothing in the prompt mentions tens or ones. Providing one or more of these cues, perhaps in a preceding item, might stimulate subjects to respond in ways more like those of their mathematics teachers.

How do the responses of preservice teachers who have completed a curriculum unit about numeration at the university level compare to those of students who have not yet studied numeration at the university level? Although the explanations from PSTs enrolled in the second required mathematics course in their major performed better (70% successful vs. 53% successful), and the difference was statistically significant, the net gains were not heartening for those of us who provide instruction about numeration in the first course.

How dependent are the responses on task differences such as choice of two-digit number? Subjects in MATH 50B whose prompt with the digit correspondence task included 23 or 24 pennies were more successful than the larger sample that had been prompted with 25 pennies. The cognitive salience of making five groups of five out of 25 may account for the differences. The differences, however, were not statistically significant. For 24 pennies $z = 1.08$. For 23 pennies, $z = 0.63$.

Implications for Preservice Mathematics Preparation

How can we provide instruction in undergraduate mathematics content courses that will facilitate deeper understanding of base-ten numeration? One implication of this study is that we might explore the effectiveness of integrating field experiences with the mathematics content courses for elementary teachers. In the MathLinks program undergraduate students assess children's understanding of place-value knowledge.

A second recommendation is to develop and evaluate units of instruction which facilitate undergraduates' deeper cognitive constructions of numeration systems and their properties. Anecdotal evidence from local instructors suggests, for example that the activity "alphabitia" (Bassarear, 2001b, pp. 37-44), in which students invent their own numeration systems, is both challenging and frustrating for MATH 50A students. Similar challenges can be found in the middle school text, *Language of Numbers* (EDC, 1994). Further study of the effects of using activities like these on the PST's understanding is needed.

Implications for Future Research

The written digit-correspondence task can be effective as a gross indicator of understanding of "tens and ones." Preservice elementary teachers performed unexpectedly poorly in this study. Much broader and deeper research, including interviews and a broader range of tasks, is needed to provide what typical PSTs understand, and what they don't understand, about base-ten numeration. Deborah Ball has pointed out that "…relying on what prospective teachers have learned in their *pre* college [italics mine] mathematics classes is unlikely to provide adequate subject matter preparation for teaching mathematics for understanding." (1990, p. 142.) Neither can we rely on what they might have learned in their undergraduate mathematics classes.

The sample of elementary teachers included in this study was extremely small, and all from one school. This group performed well. When and where do elementary teachers learn enough about tens and ones to explain them in a digit-correspondence task? Perhaps half have learned before graduating from high school. Another quarter may develop the "aha" when studying numeration as developed in typical "Mathematics for Elementary Teachers" courses and textbooks as undergraduates.

Place value concepts are central to multi-digit calculating. Liping Ma has helped to direct widespread attention to problems mathematics teacher educators face daily (1999). The study of how teachers develop "profound understanding of fundamental mathematics" needs to include more examination of pre-service and practicing elementary teachers' understandings of our base ten numeration system.

References

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21*, 132-144.

Bassarear, T. (2001a). *Mathematics for elementary school teachers.* Boston: Houghton-Mifflin.

Bassarear, T. (2001b). *Mathematics for elementary school teachers: Explorations.* Boston: Houghton-Mifflin.

Cobb P., & Wheatley, G. (1988). Children's initial understandings of ten. *FOCUS on Learning Problems in Mathematics, 10* (3), 1-28.

Educational Development Center . (1994). *Seeking and thinking mathematically: The language of numbers*. Portsmouth, NH: Author.

Fuson, K. C., & Kwon, Y. (1992, April). Korean children's understanding of multidigit addition and subtraction. *Child Development, 63* (2), 491-506.

Fuson, K., Smith, S., & Cicero, A. (1997). Supporting first graders' ten-structured thinking in urban classrooms. *Journal for Research in Mathematics Education, 28,* 752-758.

Kamii, M. (1982). *Children's graphic representation of numerical concepts: A developmental study.* Unpublished doctoral dissertation, Harvard University.

Kamii, C., & Joseph, L. (1988). Teaching place-value and double-column addition. *Arithmetic Teacher, 35,* 48-52.

Labinowicz, E. (1989). *Tens as numerical building blocks.* Paper presented at the annual meeting of the American Educational Research Association, San Francisco, CA.

Long, C., & DeTemple, D. (1997). *Mathematical reasoning for elementary teachers.* Reading, MA: Addison Wesley.

Ma, L. (1999). *Knowing and teaching elementary mathematics; Teachers understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Erlbaum.

Miura, I. T., Kim, C. C., Chang, C., & Okomoto, Y. (1988). Effects of language characteristics on children's cognitive representation of number: Cross-national comparisons. *Child Development, 59,* 1445- 1450.

Miura, I. T., & Okomoto, Y. (1989). Comparisons of American and Japanese first-graders cognitive representation of number and understanding of place value. *Journal of Educational Psychology, 81,* 109-113.

Musser, G., & Burger, W. (1999). *Mathematics for elementary teachers: A contemporary approach.* New York: Macmillan.

Richardson, K. (1998). *Developing number concepts.* Parsippany, NJ: Dale Seymour.

Ross, S. (1986). *The development of children's place-value numeration concepts in grades two through five.* Paper presented at the annual meeting of the American Educational Research Association, San Francisco, CA. (ERIC Document Reproduction Service No. ED 273 482).

Ross, S. (1989). Parts, wholes, and place value: A developmental view. *Arithmetic Teacher, 36,* 47-51.

Ross, S. (1990). Children's acquisition of place-value numeration concepts: The roles of cognitive development and instruction. *FOCUS on Learning Problems in Mathematics, 12* (1), 1-17.

Ross, S. (1999). Place value: problem-solving and written assessment using digit-correspondence tasks. *FOCUS on Learning Problems in Mathematics, 21* (2), 28-36.

Ross, S., & Sunflower, E. (October, 1995). Place value: Problem-solving and written assessment using digit-correspondence tasks. *Proceedings of the Seventeenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 338-343).

Steffe, L., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

AN ANALYSIS OF THE TEACHING PRACTICES OF A GROUP OF FOURTH GRADE TEACHERS

Roberta Y. Schorr
Rutgers University
Schorr@rci.rutgers.edu

William Firestone
Rutgers University
Wilfires@rci.rutgers.edu

Lora Monfils
Rutgers University
Monfils@rci.rutgers.edu

Abstract: This research presents a preliminary overview of the teaching practices of fourth grade teachers throughout the state of New Jersey. These teachers were all involved in professional development experiences designed to help them revise their approach to the teaching and learning of mathematics. In this aspect of the study, we observed teachers at least twice and then interviewed them about their lesson, their practice, and their professional development experiences. In sum, we conclude that while teachers are adopting new techniques such as using manipulatives or cooperative grouping as part of their instructional practices, they are not changing their basic approach to the teaching and learning of mathematics.

Introduction and Theoretical Framework

Many efforts are underway to reform the teaching and learning of mathematics. Some of these efforts involve "radical change in the mathematics taught in schools, the nature of students' mathematical activity, and teachers' perspectives on mathematics teaching and learning" (Simon & Tzur, 1999, p. 252). The goal of these reforms is to move toward instructional practices that provide students with the opportunity to build concepts and ideas as they are engaged in meaningful mathematical activities (NCTM, 1989, 2000). Reforming the teaching and learning of mathematics is not easy , and sometimes, as a result of reform initiatives, teachers may pick up one or more strategies that they believe are associated with the reform movement. However, these may not result in more thoughtful teaching and learning for students (Schorr & Lesh, in press). Researchers such as Simon and Tzur note that "on the basis of our research to date, we suggest that teachers often interpret the current mathematics education reform as discouraging telling and showing. Although teachers have been able to appropriate particular teaching strategies from the reform movement (e.g., using small groups, manipulatives, and calculators), the movement has not provided them with clear direction for how to help students develop new mathematical ideas" (p. 258). Similarly, Spillane and Zeuli (1999), found that many of the teachers involved in a study which explored patterns of practice in the context of national and state mathematics reforms, made use of the new strategies that Simon and Tzur refer to, however "the conception of mathematical knowledge that dominated the tasks and discourse in these teachers classrooms ...[did not] suggest to students that knowing and doing mathematics involved anything more than memorizing procedures and using them to compute right answers" (p. 16).

Stigler and Hiebert (1999), explain why this may be the case. They state that that teaching is a cultural activity, and cultural activities "evolve over long periods of time in ways that are consistent with the stable web of beliefs and assumptions that are part of the culture... and rest on a relatively small and tacit set of core beliefs about the nature of the subject, about how students learn, and about the role that a teacher should play in the classroom" (p. 87). They go on to state that "trying to improve teaching by changing individual features usually makes little difference, positive or negative. But it can backfire and leave things worse than before"(p. 99).

The entire study, of which this research is a part (cf. Firestone, Monfils, & Camilli, 2001; & Schorr & Firestone, 2001) was designed to obtain a more accurate picture of the teaching practices, perceptions, and professional development experiences of a group of fourth grade teachers across the state of New Jersey. It was felt that through this research, we could learn more about the current state of mathematics teaching among these teachers, as well as their ideas about their current practices and professional development experiences. In addition, as part of this study, we also obtained information regarding how the state mandated tests were impacting teaching practices. This information could be used for many purposes, one of which is to assist in making informed policy decisions regarding professional development for teachers. By better understanding the teaching practices and perceptions of these teachers three years after a new state mathematics test had been adopted—the first in the state for elementary schools , we can provide information that helps to inform teacher education projects which intend to help teachers move toward approaches which encourage the development of deeper mathematical understanding in students. Further, by seeing how teachers in general practice, we may be able to contribute to a teacher education agenda for the future.

Methods and Procedures

This particular aspect of the study consisted of several components that are briefly described below.

Selection of Teachers:

University statisticians selected fifty-eight 4th grade teachers who came from districts throughout the state. The districts were representative of the socioeconomic and demographic characteristics of the state.

Classroom Observations:

Each teacher was observed for two math lessons. The classroom field researcher kept a running record of the events in the classroom, focusing on the activities of the teacher as well as capturing the activities of students. The field notes recorded all of the problem activities and explorations, the materials used, the questions that were posed, the responses that were given—whether by students or teachers, the overall atmosphere of the classroom environment, and any other aspects of the class that they were able to gather.

Interviews with Teachers:

At the conclusion of each lesson, the teachers were asked to respond to a series of open-ended questions. Space limitations restrict the number of questions that can be shared in this preliminary format, however, a more complete list will appear in the full paper. Following are a sample of the questions:

> What were you trying to accomplish for today's lesson? What concept or ideas were you focusing on?
>
> What worked well during today's lesson, and why do you think that it worked well?
>
> What, if anything, would you change about today's lesson, and why?
>
> Why did you do this, or how did you feel about that (referring to a particular instance where for example, students explained mathematical ideas to each other or to the teacher, or with regard to a particular event or activity).

There was also a series of questions relating to the teachers' professional development experiences. The following represents a subset of the questions in that category:

> What personal or professional learning experiences (in the past five years) stick out in your mind as strongly influencing how you think about mathematics (these experiences could be within your school, the district or outside of the district)? How did your teaching practice change as a result?

Also included were a series of questions about the impact of state testing on teaching practices, the math topics that they were teaching (and if they felt that they were teaching more or less of any topics as a result of the new state standards and assessments,) etc.

Developing a Coding Instrument:

During the observations a running seminar was conducted with the field researchers. During these seminars, we conducted detailed analyses of records of classroom observations, seeking to pinpoint a series of important themes or issues that could be explored through the classroom observation data. As the observations drew to a close we adapted several pre-existing coding schemes to be used for coding the classroom data. These were based on the works of Stein, Smith, Henningsen, and Silver (2000); Stigler and Hiebert (1997, 1999); Davis, Wagner, and Shafer (1997); and Davis and Shafer (1998). A preliminary coding scheme was tried out on approximately six observations before being agreed upon. A sheet of code definitions was created and a training session was held for the six coders involved in the activity.

Some of the guiding questions for coding the classroom observations and interview data included the following:

What kinds of activities were the teachers presenting to students?

What kinds of materials were they using, and how were these materials being used?

What kinds of questions were the teachers posing, and how were the students responding?

What was the nature of the discourse that was taking place in the context of the classroom?

Did the lesson provide opportunities for students to make conjectures about mathematical ideas?

Did the lesson foster the development of conceptual understanding?

Coding the Observations:

Two individuals independently coded each observation—at least one of the coders was an experienced mathematics education researcher. After independent coding on all 20 dimensions, inter-rater agreement ranged from a high of 100 percent to a low of 70 percent. Where differences occurred, raters sought to reconcile their differences and were successful in all but 2 of the 108 cases of initial rater discrepancy. In those two cases, another mathematics education researcher discussed differences with the raters and helped them to reach agreement.

Data:

The data includes all 116 (58 teachers, two observations and interviews each) of the transcribed observations and related classroom materials, the coded observations, and transcribed interviews.

Results

The particular aspect of the study that we will focus on for this report pertains to some of the changes that are taking place in classrooms. Our research confirms that teachers are revising their instructional practices to include more group work and hands-on experiences for children. For instance, manipulatives were used in about 60% of all observed lessons. Similarly, students worked in groups for at least a portion of the time, in almost 66% of all observed lessons. We also found that in more than half of all observed lessons, teachers made an effort to connect the ideas that they were teaching to the students' real life experiences.

The adoption of specific strategies was not necessarily accompanied by a change in overall approach to teaching mathematics, however. For example, while manipulatives were used in about two thirds of all cases, they were used in a non-algorithmic manner in less than 18% of all observed lessons. This essentially means that the manipulatives were used in ways that did not necessarily foster the development of conceptual understanding. In fact, in almost two thirds of the lessons where manipulatives were used, they were used in a very procedural manner, where the teacher generally told the students exactly what to do with the materials, and the students did it as best they could. Other times, teachers used manipulatives to demonstrate a particular procedure to the class.

As an example, consider one lesson where the teacher attempted to have students solve a problem using concrete materials—chips, while working in a small group setting (see Schorr & Firestone, 2001 for a more complete analysis and description). The teacher, Ms. J, placed students in small teams (groups) of four to work together to solve the following problem which she had posed: There are eighty-four, fourth graders, and because they've done so well, Ms. J. has decided to take two thirds of them out to dinner with her…so I'm trying to find out what is two thirds of eighty four.

Ms. J. distributed the chips to the students, and told them to work on the problem using the chips—and not using paper and pencil. Many students began to separate the chips into 4 different groups, which was not the strategy that Ms. J. appeared to have had wanted. So, after a very short period of time Ms. J called for the attention of all students as she told them how to divide the chips:

Ms. J: We need three groups, because two thirds means two out of three groups. So if you have a pile of 84, then you need to make three groups.

Ms. J. then demonstrates the procedure for all of the students by placing chips into three piles. She continued by saying the following:

Ms. J: And you keep passing them out into groups until you've used up the 84. Remember the denominator tells you the number of equal groups you need. Just like if you play cards, and each person gets the same number of cards, right.

After checking to be sure that each group had made three groups of chips she continued:

Ms. J: OK, we finished step one. If you want to know what two thirds of 84 is, you have to divide 84 by 3. So how many did you end up with in each group?

Girl: 28

Ms. J: So what do we do now? How do you know what type of equal groups to put them in? What tells you?

Boy: By looking at two thirds?

Ms. J: What tells you? The denominator tells you what number of equal groups to divide by. The divisor or the bottom number of the fraction tells you how many equal groups to make. So does that mean that 28 students can go? Can I get a consensus? (About one half of the students raise their hands). Twenty-eight people cannot go. So what do I need to do now? Two thirds of my 84 students can go. So how many students can go? You're not multiplying; you're using your manipulatives. So what am I doing now? Everyone should have the same amount.

In the above excerpt, the teacher directed the students to make three groups of chips "because two thirds means two out of three groups. So if you have a pile of 84, then you need to make three groups". This was done with little further discussion of, for example, why three groups were needed in this particular problem, or why the three in the two thirds is used to determine the number of groups of chips that the children should have. While the teacher said that the denominator tells you what number of equal groups to divide by, there was no discussion of why that is so, or how that maps into the concrete representation. In fact, there was evidence of confusion throughout the entire lesson.

This excerpt also demonstrates another common thread that emerged in many of the lessons—while many teachers had students physically touch concrete manipulatives, there often was little or no opportunity for the students to develop their own solutions to the problem. As a result, students often did not see the relationship between the problem activity and the concrete (or alternative) representations. To further illustrate, we will continue with the classroom discussion described above, where despite the fact that the students had placed the chips into three groups (as instructed), they were still not able to find a solution to the problem. As a result, Ms. J. allowed a student to demonstrate a written algorithm for division on the board. The student divided 84 by 3 thereby obtaining a quotient of 28. Ms. J. then instructed the students to put all of the chips back into one pile again, and redistribute them so that now they would have only three chips in each group (28 groups of 3).

Ms. J: The whole reason I let you put them into groups of three [referring to the first part of the lesson where the chips were distributed into 3 piles of 28] is that I wanted you to see that when you are dividing, you need to look at the dividend, that is the number I have in all, and the divisor tells me how many in each group... What if I have ten party bags and five people

> coming to my birthday party? How many bags would each person get? Each person would get two bags. The divisor tells you how many in each group, and the quotient tells you how many groups. I told you two from each group can go. So now tell me, how many can go. Take two from each group on a blank area of you desk. Now tell me how many people can I take out to lunch.

Ms. J: (to a student who was proceeding incorrectly) No, you take two from each group, and that's how many can go out. How many students can go?

In the above, Ms. J was using a different model for the solution than she had in the beginning. In either case, few, if any, students appeared to understand how the different representations connected to each other, the algorithm, or to the problem activity. Further, it is not clear that Ms. J. ever realized that. In her post lesson interview, Ms. J acknowledged that there was some confusion, but felt that the lesson went well: "I think the manipulatives and the hands-on experience worked well, and I think the cooperative groups with them working together and learning from each other worked well." Wanting to learn more, the interviewer specifically asked her about the different ways in which she instructed the students to use the materials. He said: "At first you started out with having to break the 84 into three groups, and you let them try that and see what, and you talked about why you didn't think that that worked and you had them go back and put them into groups with only three." Ms. J replied:

> Because I wanted them to see the vision. And they kept saying to me—because they knew the algorithm because I showed it to them on Friday [the class before], but a lot of them kept telling me to make groups of three. And then I went ahead and did the algorithm to show them that it wasn't making groups of three, but indeed putting three in each group. I could have told them that they were wrong, but would they have understood it if I hadn't gone back and showed them what they did wrong? That's what I was saying in my mind. Like after they went ahead and they said "well you're supposed to divide what, three into 84," and they got 28. But, when they looked on their desk, they didn't see 28, they saw these three piles. They didn't see their 28 groups that they were looking for.

This example highlights the difficulty some teachers had in making meaningful use of manipulatives to help students build ideas. It also illustrates the difficulty many teachers had in listening to, or closely observing, the mathematical thinking of their students.

As mentioned above, for many teachers, the fact that students used concrete objects, or worked with each other in a small group setting, was an indicator that they were incorporating reform practices into their instruction. They did not consider that the students were not connecting the different representational systems, or that the concrete objects did not even map into the problem situation or symbolic representations that were used. The fact that they had used concrete materials appeared to be what mattered most, not how they were used, or the level of understanding that was elicited.

As further illustration, consider the following interview with a teacher: "I myself attend a lot of workshops...and I learned that the children need to do the things hands on, because they need to see it, they need to feel it, they need to understand it. And basically everything in my classroom, to the best of my ability, I try to do it hands on." This teacher went on to say that she likes to always have "everything [done as] group work". After the classroom observation, she elaborated on her lesson, and how she felt that she had actually used a hands on and group work approach. She noted that she was "trying to focus on long division, and we have been doing division with two digits in the quotient. And today we took it a step further to do three digits in the quotient, and without remainders". She continued by saying, "I think that [the lesson] went well because instead of doing paper and pencil, and instead of being lectured, and instead of just observing, they actually get to do it, and they use the white boards—those white boards and the markers. They actually are more motivated to do division than they would be any other time. And then having them involved and having them come up to the overhead projector and doing it. And then when they're done, if they're confident, they go around and help the other children."

These and the many other comments made by this teacher help to shed light on what she means by, for example, group-work, hands-on learning, and thoughtful problem-solving activities. Notice that this teacher felt that by teaching in this manner, she was indeed teaching for understanding (and effectively using hands-on and group-work teaching strategies).

Conclusions

This research shows how teachers are adopting specific procedures and techniques and integrating them into their existing paradigm of teaching. Moreover, it suggests that in one state, at least, this adaptation of new techniques to old ways of teaching is rather common. In a more extended paper, we suggest that New Jersey's standards and assessment policies are contributing to the adoption of these new techniques without leading to deeper change (Schorr & Firestone, 2001). This surface change in practice appears to be common to a number of states (Wilson & Floden, 2001).

As a result of the full study, we suggest that teachers lack both the pedagogical and content knowledge to make more radical shifts in practice just because state accountability systems are changing. For the standards movement to lead to deeper changes (NCTM 2000), increases in accountability will have to be accompanied by increases in capacity building in both pre-service teacher preparation programs and continuing professional development programs in schools and districts.

Regarding the specific findings reported in this paper, we suggest that although teachers are incorporating practices they identify with reform, without a deeper understanding of the mathematics they are teaching, no matter how well intentioned, they will continue to approach mathematics instruction as the transmission of an external body of knowledge (i.e., facts and procedures) rather than the creation an inquiry-oriented environment in which students explore and build mathematical ideas. If teachers are to change their practice, they must change their understanding of what it means to know and do mathematics (Schorr & Lesh, in press). Moreover, they will need time and support, in the form of extended and meaningful professional development in the area of mathematics and mathematics pedagogical content knowledge.

References

Davis, J., Wagner, L. R., & Shafer, M. C. (1997). *Classroom observation scale.* Madison, WI: University of Wisconsin.

Firestone, W. A., Monfils, L., & Camilli, G. (2001). *Pressure, support, and instructional change in the context of a state testing program.* Paper presented at the annual meeting of the American Educational Research Association, Seattle, WA.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Schorr, R. Y., & Firestone, W. A. (2001, April). *Changing mathematics teaching in response to a state testing program: A fine-grained analysis.* Paper presented at the Annual meeting of the American Educational Research Association, Seattle, WA.

Schorr, R. Y., & Lesh, R. (in press). A models and modeling perspective on classroom-based Teacher Development. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective on teaching, learning, and problem solving in mathematics education.* Hillsdale, NJ: Erlbaum.

Shafer, M. C. (2001). *Instructional quality in the context of reform.* Paper presented at the Research Presession of the National Council of Teachers of Mathematics, Orlando, FL.

Simon, M. A., & Tzur, R. (1999). Exploring the teacher's perspective from the researchers' perspectives: Generating accounts of mathematics teachers' practice. *Journal for Research in Mathematics Education. 30* (3) 252-264.

Spillane, J. P., & Zeuli, J. S. (1999). Reform and teaching: Exploring patterns of practice in the context of national and state mathematics reforms. *Educational Evaluation and Policy Analysis, 21* (1), 1-27.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction.* New York: Teachers College Press.

Stigler, J. W., & Hiebert, J. (1997). Understanding and improving classroom mathematics instruction. *Phi Delta Kappan*, *79* (1), 14-21.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* New York: The Free Press.

Wilson, S. M.,& Floden, R. E. (2001) Hedging bets: Standards-based reform in classrooms. In Fuhrman, S. (Ed.), *From capitol to classroom: Standards based reform in the states* (pp. 193-216). Chicago: University of Chicago Press.

PERCEIVED EXPERIENCES AND ATTITUDES OF PRESERVICE MATHEMATICS TEACHERS

Julie Sliva
San Jose State University
Sliva@mathcs.sjsu.edu

Cheryl Roddick
San Jose State University
Roddick@mathcs.sjsu.edu

Abstract: This paper explores the usage of a Mathematics Autobiography to investigate the beliefs, attitudes and experiences that preservice teachers have had while learning mathematics in order to inform instruction. The following were found to be important in students' experiences in learning mathematics: the role of the teacher; support and influence of family; challenge; issues of fear, failure and avoidance; learning strategies; and content issues. Implications for instruction are also discussed.

Affective variables related to the learning of mathematics play an important role in the development of preservice teachers. Throughout the process of learning mathematics, preservice teachers collect a wide range of experiences. Both positive and negative, these experiences have led to the development of their beliefs and attitudes about mathematics. Recent research has found that teachers' beliefs about mathematics and the teaching of mathematics are significantly influenced by their mathematical experiences as a student (Brown & Borko, 1992; Brown, Cooney & Jones, 1990; Raymond, 1997). Furthermore, research has found that success in solving mathematics problems is not based solely on one's knowledge of mathematics. It is also based on metacognitive processes related to mathematics strategy usage, the emotions an individual feels when doing a problem and the personal beliefs of one's mathematical abilities (Garafalo & Lester, 1985; Schoenfeld, 1985; McLeod, 1988).

As mathematics educators preparing students to teach mathematics, we have often thought that the beliefs, attitudes and experiences our students have had learning mathematics may have an impact on their learning and performance in mathematics, which ultimately may affect how they teach mathematics. This belief is supported by McLeod's (1992) assertion that affective issues play a central role in mathematics learning. Beliefs that students hold about mathematics and their abilities to perform mathematically are critical in the development of their responses in mathematical situations.

For the purposes of this study beliefs will be defined as the personal assumptions from which individuals make decisions about the actions they will undertake. This notion is consistent with research that indicates that actions are motivated by what individuals perceive are the outcomes of their actions (Kloosterman, 1996). From "I hate math" to "Math is my favorite subject", these statements speak to the range of feelings and beliefs student have about mathematics. Students, in the process of learning mathematics, experience both positive and negative emotions, which influence the development of their attitude towards mathematics as a whole. These beliefs about mathematics, about what they need it for, and how strong they are as mathematics students are related to learning and can significantly affect what students do in a mathematics classroom (Kloosterman, 1996).

Teacher educators are charged with the daunting task of shaping and reshaping the attitudes, beliefs and content knowledge of preservice teachers. Teacher education programs must sometimes help participants to deconstruct, and reconstruct their views on teaching and learning (Brown & Borko, 1992; Wilson & Ball, 1996). It is important for mathematics to be learned in a supportive community of learners (Brown & Campione, 1994). This environment can also provide an arena for discussion of these important affective issues.

The authors' primary purpose of this research was to investigate the beliefs, attitudes and experiences that preservice teachers have had while learning mathematics in order to inform instruction and facilitate the development of a supportive learning environment. This article will describe students' perceived experiences of learning mathematics in grades K-14 and discuss the implications of these experiences for college level instruction.

Methodology

The participants in this study are preparing to teach at either the elementary, middle, or secondary level, and were enrolled in a mathematics or mathematics education course required for students seeking teaching certification in their respective level. Data were gathered in the form of written responses called "Mathematics Autobiographies" during the first week of the semester. We requested that students respond in writing about any experience in mathematics that they felt contributed to their mathematical development. A total of 72 mathematics autobiographies were gathered from three different levels of college mathematics courses and two secondary mathematics education courses. In the second session of each of these courses students were asked to share with the class any experiences they were comfortable sharing, and then as a group we discussed the implications for mathematics instruction.

Analysis of Results

Data were analyzed in two ways: a holistic examination of each student's response, and a comparison across chosen teaching levels: elementary, middle, and secondary. Each individual mathematics autobiography was summarized and coded according to emerging themes. These themes were then compared within and across teaching levels with the help of conceptually ordered displays (Miles & Huberman, 1994).

After analysis, several themes emerged. The following were found to be important in students' experiences in learning mathematics: the role of the teacher, support and influence of family, challenge, issues of fear, failure and avoidance, learning strategies, and content issues. All of these issues mentioned are in some fashion tied to one variable, the teacher. Since the teacher generally guides classroom instruction, many of these issues focus on a teacher's approach to classroom instruction. Each of these experiences is stated from the student's perspective and thus are perceived experiences and beliefs about their learning of mathematics.

The Role of the Teacher

Virtually all students mentioned the importance of the role of the teacher in the development of their understanding of mathematics. This finding is consistent with previous research (Jackson & Leffingwell, 1999). The description of the teachers seemed to fall on a continuum, with one end of the continuum indicating teachers with "enabling" traits, and the other end indicating teachers with "disabling" traits. The "enabling" teacher was often portrayed as one who was patient and understanding who would always fully answer student questions and explain a concept until all students understood. Further, this type of teacher would not embarrass a student, and in fact, would often single out a student independently to provide additional assistance and encourage them to continue developing their mathematical skills. Teachers classified as "enablers" were typically remembered for the ability to make learning mathematics fun. In contrast, the "disabling" teacher was depicted as one who would often ridicule students if they did not know an answer, intimidate students, not fully explain concepts or homework problems when asked, and who was often inconsiderate of students' feelings. This type of teacher repeatedly embarrassed students for their lack of understanding.

Generally, an experience with an enabling teacher left students, even those who were not as successful in the class as they would have liked, feeling as if they had been successful in the class, and could be successful in future mathematics classes if they worked hard. Commenting about such an experience Corrine states, " My high school mathematics teacher had a positive attitude toward me which gave me the confidence to do well in math and not feel stupid any longer." Sean recalls his trigonometry teacher in this manner " He gave me confidence and I still have bad math anxiety but it goes away when I think of that experience (in Trig Class) and how it made me believe in myself and how I could learn mathematics."

Students clearly articulated their feelings about learning in a class with a disabling teacher. Rich recalls "I was always nervous about going to my Calculus class because the teacher made me feel dumb. If I would raise my hand and ask a question, he would give me the answer but would give it in a way that made me feel I was wasting his time by asking a question…I felt so dumb. I ended up barely passing the class. I didn't want to ever have to take anymore math." Jessica wrote about her fifth grade teacher "who wrote continuously on the board, not once turning around to see if we were understanding the material. He would go over it once and if we didn't understand it we would have to depend on our friends to help us. By this time I lost all interest in math and began to loose my self-esteem, I had no desire to continue with math. The only thing standing in my way was that I had to complete three years of math to go to a good college." Bernadette also had a difficult experience learning her multiplication tables and stated "I had a teacher who would make us recite them (multiplication tables) in front of the class and often made me feel very stupid and embarrassed when I made a mistake. I was a very shy child who didn't have a lot of friends and I was very insecure about myself, which made this experience especially painful for me. For the years that followed I decided that I was not good in math." A bad experience with a teacher who made a student feel uncomfortable with their mathematical abilities tended to deter students from wanting to pursue further mathematics study. Many students felt that they were just "not math students."

The teacher was a pivotal experience for each of the students. A good experience with an "enabling" teacher promoted the students' positive outlook on their mathematical abilities and the subject. In contrast, an experience with a "disabling" teacher discouraged positive student beliefs of mathematical ability and also dampened outlook on future mathematical experiences.

Support and Influence of Family

The family was mentioned as a significant contributor to many of these students' beliefs about their abilities in mathematics and their ability to learn to do mathematics. Many students spoke of family members, such as grandparents and siblings, who supported their learning by providing additional help outside of class. Brian spoke of how proud he was when he finally mastered his multiplication tables, "My parents worked with me each night creating flash cards which helped me memorize my multiplication tables." Glenn, who was caught cheating on his timed tests to keep up with his peers, used this embarrassing event to "light a fire" under him to do well in mathematics. Nadara also said that "I made flash cards with mom and it helped me to learn math which made me feel good about myself." Kim also spoke of asking her father for help. She recalled "bitter classroom memories of learning multiplication facts.... I owe it all to my Dad, he helped me learn them." Other students spoke of turning to family members to help them learn how to complete simple division problems, multiplication and addition/subtraction concepts. However, in some instances, the persons with the greatest ability to influence—the parents—were the ones who had the greatest and most memorable negative influence. A scathing or careless remark made by a mom or dad in the formative years remained a sizable part of the preservice teachers' mathematical identity. Mina, who received an 'F' in 9th grade algebra, internalized her parents' scolding that she was stupid, and did not use the brains she was given. Throughout her career, and to this day, Mina has doubts about her ability to solve mathematics problems. Celeste's mom, who referred to her sarcastically as "my daughter, the math genius," led Celeste to feel incapable of doing math as a child.

Although she appears confident and has had much success in college mathematics classes, she admits that sometimes she doubts her abilities and falls prey to her old assumption that she doesn't "have a mathematical mind." Monica also shared "My mom's immediate response (about my math grades) to my father was, 'Oh! She takes after me and her grandmother; we were never good in math.' I heard comments like this throughout my life and believed that it was o.k. to do badly in math. My father always made comments like 'Boys are better in math than girls.'"

Challenge

Challenge for the purposes of this paper is described as any situation in which students are asked to compete with themselves or their peers to demonstrate their understanding of mathematics topics. It may also include being rewarded for completing a task successfully. A typical challenge described by our preservice teachers was an activity where they were asked to compete with each other to see who could recite multiplication facts the quickest. Other situations were mentioned, such as being required to orally recite mathematics facts using a hand-held timer, and taking timed tests. In addition, award ceremonies both in the classroom and school-wide were mentioned as significant experiences in their mathematics history. Ironically, the use of challenge was both positive and negative for these students. For some students, the experience of having to compete in class with their peers was devastating and created detrimental effects on their feelings toward mathematics and their perceived ability to learn the subject. For others, this negative experience was an impetus to learn the material and thus feel successful in their abilities to learn mathematics. Many students who used this negative experience to learn mathematics turned to family members to help them with their learning.

The research of Meyer, Turner, & Spencer (1997) supports our findings with "challenge seekers" and "challenge avoiders." Challenge seekers appear to have a self-reported tolerance for failure, a learning goal orientation, and a higher than average self-efficacy in mathematics. Several of our preservice teachers recall challenge warmly in their mathematics classes. Jody fondly remembers a competition in her second grade class. "I still have the little trophy (from the competition), because I was so proud. I believe that is the reason why I loved math." She also recalled another incident when she had to orally compete in front of the class. "I happened to finish 3rd. Again, another rewarding experience to make me love math." For Jody and others, challenge was a driving force in shaping their positive attitudes toward mathematics.

However, this was not true for all. For the "challenge avoiders" in our courses, this was not a memorable occasion. These challenge avoiders tended to have a self-reported higher negative affect after failure, a more performance-focused goal orientation, and a lower self-efficacy in math (Meyer, Turner, & Spencer, 1997). Jake remembered "having to get up in front of the entire class to do a timed problem on fractions that I did not really understand. It was embarrassing not knowing how to do the problem, but it made me want to learn even more so that I would be able to do the problem in front of everybody."

Fear, Failure and Avoidance

A general trend that emerged from the data is fear, failure and then avoidance. First a student experiences fear of mathematics, often followed by failure which can then lead to avoidance of the subject. Avoidance of mathematics is often overcome only after a positive experience. This cycle may repeat several times throughout a student's education. Students may also experience fear, failure and avoidance in exclusion of each other. Our findings are supported by previous research, which has found traditional mathematics instruction to cause fear, anxiety, and avoidance of mathematics by students (Tobias, 1981).

Bruce's experiences in geometry class were representative of this cycle. "As a result of my shortcomings in geometry I began to fear math for the first time in my life. So I spent the next few years shying away from the subject for fear that I may fail at something that I was once so in love with. This continued until I took my first physics class.... Fortunately physics intrigued me so much that I simply knew I had to stand up to my fears of math and overcome them in order to proceed in that area of study."

A majority of students at all levels spoke of fear of mathematics. These are some of their feelings,

- "By this time, I internalized and truly believed that I hated math, was stupid and couldn't do it";
- "From the age of 15 until the age of 41, I avoided math like the plague.";
- " I had begun to hate math. I did not take a class during my senior year; we only had to go to Algebra II to get into college so that was enough for me. But now, I have to take this class.";
- " Midway through my Junior year I dropped his class...I wasn't about to take that teacher again so instead I just gave up on math for a while."

These comments highlight the important component that fear played in students' learning of mathematics. Students whose last memory of learning mathematics resulted in fear or failure will likely have quite different perceptions about their learning of mathematics than students who recall a more positive resolution to a fearful situation in a mathematics class.

Learning Strategies and Content

Two topics closely related that were consistently mentioned were learning strategies used to learn mathematics and several specific mathematics topics, which included number sense (basic counting), multiplication, and geometry. One of the major approaches to the successful learning of mathematics is the use of learning strategies. The National Council of Teachers of Mathematics (NCTM, 2000) states that

> to meet new challenges in the work place students will have to be able to adapt and extend whatever mathematics they know.

This requires problem solving which is dependent upon students' knowledge of strategies. Learning strategies mentioned consistently throughout the mathematics autobiographies were using concrete manipulatives, learning aides, and asking for outside help. A combination of strategies was used in most situations.

Counting strategies come naturally to young children and in an environment that supports children's development of personal strategies these will come naturally. (Carpenter, Fennema, Franke, Levi, & Empson, 1999) However, many of our preservice teachers had difficulty with this topic at a young age. Tina had difficulties learning how to count; she commented on this and the problems she had learning division. "My Grandmother taught me how to count, she gave me chopsticks to help me." Bernadette's mother helped her learn math by "counting socks and lima beans." Simon also used manipulatives to learn to count, "My parents knew I was having trouble they helped me learn to count with pinto beans." Nicole's third grade teacher brought in candies and nuts to help the class with their mathematics lessons. Use of manipulatives has been found to be an effective tool to facilitate the learning of mathematics (Lambert, 1996).

Several students like Michelle used flash cards to help them learn. Michelle says her sister "helped me create flash cards and helped me learn my multiplication facts. I remember. I was so proud of myself. I know I need visual aids, that's when I learn best." As many students mentioned previously, help from family members was often used as a strategy to learn mathematics. In addition, many students mentioned help from other friends, roommates and tutors.

Mathematics content was a frequently discussed topic, mostly the difficulty of learning the specific topic. Counting was frequently mentioned as an area where students elicited help from family members. Multiplication, which is typically taught in the third grade, was an area where these students had a mixture of experiences. Though the majority of these students had difficulty learning multiplication facts, others expressed they did not. Those who had difficulty learning their facts were very upset by teaching strategies their teachers used to facilitate their learning. These techniques usually included some form of timed assessment to test a student's knowledge, and/or creating a competitive environment to assess learning. These assessment strategies again elicited opposing feelings from students. Those students who understood their multiplication facts were excited and felt proud of their achievements in these assessments. Whereas, students who did not perform well on these types of assessments were embarrassed and left feeling that they "could not do mathematics" at a very early age.

If there is one course that is most often singled out for its anxiety and frustration-provoking memories, it is geometry. Glenn attributes his [illegible] the subject to his lack of mathematical maturity and his inability to get along with his teacher. Bruce excelled effortlessly in all mathematics classes until he reached geometry. He states "as a result, I did poorly in a math class for the first time in my life." Marc recalls that his math courses in high school went rather smoothly until he reached geometry. He "hated making columns and memorizing rules for proofs." He related, like others, that he was just happy to get out of the class.

Discussion

We previously mentioned the commonalties among students choosing to teach at the elementary, middle school and high school levels. However, we also found marked differences. The following paragraphs are composite sketches of each of these groups of teachers.

Typically preservice elementary teachers had more negative feelings, attitudes and less confidence toward learning mathematics. They felt like mathematics "was just not their strong suit", "was a horrible subject" and that they were "just plain stupid at math" and thus "hated math." These students also expressed their fear of their current math class and had math anxiety. In addition, they experienced increased failure in their entire mathematical experience from kindergarten through college.

Middle school preservice teachers felt some of the same anxiety as their elementary teacher counterparts. However, this anxiety was mixed with enough successful experiences in mathematics to encourage a greater commitment to the teaching and learning of mathematics. For the most part, their experiences in mathematics class were positive, although the love/hate relationship with mathematics was evident in their self reports: "As for my relationship with math today, I hesitate to call it dysfunctional, but it is definitely wrought with anxiety and at times exhilaration."

High school preservice teachers' memories of their mathematics classes were the least tainted by frustration and failure as depicted by this student, "I like to think I've always been good at math and never had to struggle." They all spoke of influential teachers who helped them develop a love of math, teachers who also often singled them out and instilled a belief that they were talented in doing mathematics. They had a desire to emulate the characteristics of this beloved teacher. Many spoke of great admiration for their parents' understanding of mathematics and the help they received growing up which helped them learn mathematics.

The major difference we found in each of these groups was in confidence level. It seems obvious that students who experience less failure would have more confidence in their abilities. The preservice elementary students generally reported more anxiety over learning mathematics. They also spoke, as a group, of more failure and self-doubt toward their abilities to learn and do mathematics well than secondary preservice teachers.

Many of our students spoke of perceived good experiences in learning mathematics, even if they had not achieved high grades, when they had a teacher that gave them the impression that they could succeed in mathematics. Others spoke of bad experiences when they had the opposite, a teacher who did not express an interest or a belief that the student could learn mathematics. Of concern to the authors is the question of whether or not a student would continue to pursue mathematics if they never had a good experience learning mathematics. Another concern is that all of our students are working toward a teaching license with the goal of teaching as a career. This is a concern because our belief is that if they did not have good experiences learning mathematics, how would they then know how to create good mathematical learning experiences for their students. Will they express a love of the subject or exude a dislike of the subject, which their students could then also adopt? Specifically, could the negative fears and experiences these preservice teachers have transfer to their students, thus continuing the cycle of negative feelings and beliefs about mathematics? This is a major concern for all students, especially those students who may have already had a negative learning experience in mathematics.

Implications for Teacher Educators

Mathematics autobiographies can be used as a tool for teacher educators because they can provide insight into the mathematical learning experiences of their students. However, this assignment may also be altered to be used in any content area, not just mathematics, to gain important insights into influential learning experiences. Specifically, this information can be useful for instructional planning and curriculum development as it enables an instructor to link a student's past learning experience to instructional techniques which may facilitate the student understanding of the topic in a deeper manner. However, it is essential that a discussion occur that highlights the differences and similarities between past learning experiences and present instructional strategies used to learn the same content.

Research has supported our findings that some students believe they cannot do mathematics (National Research Council, 1989). Kloosterman, Raymond, and Emenaker (1996) found that students pick up teachers' unintended as well as intended messages about what it means to know and do mathematics. Thus, it is essential that appropriate modeling be used. This is especially important in preservice courses since the teaching for which teacher educators are preparing their students is different from most of what students recall from their own learning experiences (Wilson & Ball, 1997). For example, traditionally mathematics students have learned mathematics from a "sage on the stage." Now the approach to teaching mathematics has shifted to a student- centered constructivist approach to learning where students are often given problems that require them to work in considerable depth and often, struggle with confusion rather than be told the "one method" of solving the problem. In addition, there are shifts in teacher interactions with students and the expectation of high achievement for all students. This deconstruction of past views on instruction and teaching is often a difficult task for teacher educators, but a necessary one (Wilson &

Ball, 1997). It is imperative that preservice teachers experience current approaches to teaching and learning advocated by current reforms in an explicit manner.

Mathematics autobiographies can also be used to facilitate a positive classroom environment, one in which all students feel that their experiences and background are valued. Sharing autobiographies in a classroom discussion can help students identify with others' similar experiences as well as enlighten students on the diversity of student backgrounds. For example, as our students shared their experiences, a common discussion that developed focused on difficulties learning mathematics. The discussion progressed into exploring a variety of different learning strategies used to learn mathematics. We have found that an important component of using mathematics autobiographies is having our students verbally share and discuss their experiences. Shared experiences begin to develop commonalties such that they begin to connect to one another and often realize they are not alone with their feelings, beliefs, experiences, and approaches toward learning mathematics. In addition, such discussion facilitate the development of a supportive mathematical community, one in which students can construct their own ideas, find their own representations, and connect mathematical ideas in their own ways while doing mathematics together. Students can share their work in a comfortable atmosphere where discourse and collaboration are valued (Brown & Campione, 1994; Bruner, 1996).

Utilization of mathematics autobiographies may also be appropriate in a K-12 environment. As students get older, teachers become less confident in remediation, and since low achievers think that their teachers believe they cannot be helped, they translate this feeling into a belief that they are not capable of learning. Thus, elementary school teachers need to make a consistent, long-term effort to get low achievers to believe that the effort to learn is essential (Eccles et al., 1993). Further, Eccles et al., (1993) and Stipek (1993) state that rethinking about classroom environments is especially important to improve motivation and performance of low achieving students. Using autobiographies as a window into the experiences of children may provide insight into their learning experiences and used to keep students on a positive track toward success in mathematics.

Finally, since prior experiences appear to have shaped these preservice teachers' attitudes and self-efficacy about math and math learning, it is important to consider how their experiences may affect their future teaching of math. Though our students did not explicitly mention the impact their learning experiences might have on their future teaching, we believe there is an impact. There seems to be a range in possibilities of how this might impact teaching. There is the possibility that these students may teach just as they were taught. Conversely, there is the possibility that preservice teachers, when taught using appropriate methodologies, may embrace these methods and utilize them in their own teaching. The impact that a positive or negative experience learning mathematics has on future teaching is an area, which needs to be explored further. This information may also provide insights into choice of grade level to teach. Without the opportunity to experiment with new methods of teaching and learning, to examine their own beliefs and knowledge (Kloosterman, Raymond, & Emenaker, 1996), teacher educators are likely to fall into past patterns of teaching and learning (Wilson & Ball, 1997).

References

Brown, C. A., & Borko, H. (1992). Becoming a mathematics teacher. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 209-239). New York: Macmillan.

Brown, S. I., Cooney, T. J., & Jones, D. (1990). Mathematics teacher education. In W. R. Houston (Ed.), *Handbook of research on teacher education* (pp. 639-656). New York: Macmillan.

Brown, A. L., & Campione, J. C. (1994). Guided discovery in a community of learners. In K. McGilly (Ed.), *Cognitive science and educational practice.* Cambridge, MA: MIT Press.

Bruner, J. (1996). *The culture of education.* Cambridge, MA : Harvard University Press.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction.* NH: Heinemann.

Eccles, J. S., Wingfield, A., Midgley, C., Reuman, D., MacIver, D., & Feldlaufer, H. (1993). Negative effects of traditional middle schools on students' motivation. *Elementary School Journal, 93*, 552-574.

Garofalo, J., & Lester, F. K. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for Research in Mathematics Education, 16*, 163-66.

Jackson, C. D., & Leffingwell, R. J. (1999). The role of instructors in creating math anxiety in students from kindergarten through college. *The Mathematics Teacher, 92*, 583-586.

Kloosterman, P. (1996). Students' beliefs about knowing and learning mathematics: Implications for motivation. In M. Carr (Ed.), *Motivation in mathematics* (pp. 131-156). Cresskill, NJ: Hampton.

Kloosterman, P., Raymond, A., & Emenaker, C. (1996). Students' beliefs about mathematics: A three year study. *The Elementary School Journal, 97*, 40-56.

Lambert, M. A. (1996). Mathematics textbooks, materials, and manipulatives. *LD Forum, 21* (2), 41-45.

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan.

Meyer, D. K., Turner, J. C, & Spencer, C. A. (1997). Challenge in a mathematics classroom: Students' motivation and strategies in project-based learning. *The Elementary School Journal, 97*, 501-521.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Research Council (1989). *Everybody counts: A report to the nation on the future of mathematics education.* Washington, DC: National Academy Press.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis.* Thousand Oaks, CA: Sage.

Raymond, A. M. (1997). Inconsistency between a beginning elementary school teacher's mathematics beliefs and teaching practice. *Journal for Research in Mathematics Instruction*, *28*, 550-576.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Sipek, D. J. (1993). *Motivation to learn: From theory to practice.* Boston: Allyn & Bacon.

Tobias, S. (1981). Stress in the math classroom. *Learning, 9*, 34-38.

Wilson, S. M., & Ball, D. L. (1996). Helping teachers meet the standards: New challenges for teacher educators. *The Elementary School Journal, 97*, 121-137.

ELEMENTARY TEACHERS' ABILITY TO ANALYZE STUDENT ERRORS MADE WHEN SOLVING PART-WHOLE DIVISION PROBLEMS

Walter E. Stone, Jr.
University of Massachusetts , Lowell
Walter_Stone@uml.edu

Abstract: The purpose of this paper is to report results from a piece of a larger research study (Stone, 1999) that investigated the factors that contribute to prospective teachers' abilities to solve part-whole division problems. In this study, elementary teachers' abilities to analyze students' errors to part-whole division problems was investigated. It was found that many teachers were able to provide written explanations of important relationships given in the problem. These teachers learned about a pedagogically useful explanation in the context of analysis of errors that was not uncovered when the teachers solved the problems on their own.

Introduction and Rationale

Although elementary teachers have difficulties with rational number concepts and skills, they are responsible for recognizing and remediating students' errors and misconceptions in this content area. Teachers must have the knowledge of and about rational number that allows them to identify and analyze students' errors in order to plan appropriate instructional interventions (Ashlock, 1998; Cox, 1975). Since teachers must understand and teach rational number concepts and skills, more must be learned about their knowledge of these concepts and skills and their abilities to apply them in order to enhance their teaching (Sowder et al., 1998; Sowder et al., 1993).

There has been little research on elementary teachers' abilities to analyze errors and on the effect of error analysis on prospective teachers' content knowledge of mathematics. Borasi (1996) suggests that mathematics teachers and educators have "so far overlooked the potential of many errors to stimulate reflection and inquiry about both specific mathematical content and mathematics as a discipline" (p. 10). Inquiry into errors permits the refinement of hypotheses and can lead to the reorganization of teachers' subject-matter content knowledge (Kremer-Hayon, 1993; Russell & Munby, 1991). Communication about errors allows for students and teachers to reflect on their knowledge of mathematics, thus creating more numerous and stronger knowledge networks and connections among those networks (Hiebert & Carpenter, 1992). Lerman (1996) suggests that students and teachers learn by participation in and reflection on social situations that occur in the classroom. Brown, Collins, and Duguid (1989) endorse the idea that learning is most effective when situated in context. Analysis of errors is context-based; it is situated in teaching and is part of teachers' learning about students' knowledge of mathematical topics.

Brown and Burton (1978) asked prospective teachers to analyze systematic subtraction errors, called "bugs," and investigated the effect of error analysis on their knowledge of the whole number subtraction algorithm. Results showed that following "bug" analysis, prospective elementary teachers' knowledge of the subtraction algorithm improved. Furthermore, these teachers developed ways to talk about errors that they felt would help them in future instruction of the subtraction algorithm.

More information is needed on elementary teachers' knowledge of rational number and how it can be improved. Thus, the focus of this study was to investigate elementary teachers' abilities to analyze students' errors when solving part-whole division problems. Information about these abilities may shed light on timely and economically feasible approaches for helping elementary teachers extend their knowledge about rational number.

Theoretical Framework

Part-whole division problems were selected for this study because they are one type of multiplicative situation with rational numbers, or fractions (Greer, 1992; 1994). The use of fractions in multiplication and division problems reflects the complexity of rational number concepts and ideas in which teachers, as well as students have difficulty (Post, Harel, Behr, & Lesh, 1991). Van de Walle (1998) recommends that teachers present part-whole situations to children in order to develop their understanding of fractional parts and the meanings of numerator and denominator of a fraction.

In part-whole tasks, given any two quantities, the whole, the part, or the fraction, the third quantity can be determined. Part-whole multiplication problems are problems in which the fraction and the whole are given, and the part is determined. An example of a part-whole multiplication problem is:

> There are 36 children in a class, of whom 2/3 are girls. How many girls are there in the class? (Greer, 1992, p. 278)

Part-whole division problems are problems in which the fraction and the part are given and the whole is determined. Part-whole division problems are the inverse of part-whole multiplication problems. Where the task in a part-whole multiplication problem is to find y given that y is a/b of x, the task in part-whole division problems is to find x when y is a/b of x. Behr and Post (1988) call this task a "construct the unit problem." Consider the following part-whole division problem.

> There are 24 girls in a class. This is 2/3 of the children in the class. How many children are in the class?

Since 24 is 2/3 of the children, 12 is 1/3 of the children. Thus, the total number of children in the class, 3/3, can be found by multiplying 12 by 3. There are 36 children in the class.

To correctly solve part-whole division problems, the problem solver must have an understanding of the quantities given in the problem. In this case, the problem solver must understand that the fraction given in the problem and the whole number problem both relate information about part of the whole. For example, in the part-whole division problem above, the fraction, 2/3, relates information about the fractional part of the whole number of children in the class, where 24 relates this same information in terms of a corresponding whole number amount.

This study investigates elementary teachers' abilities to present this information about the relationship between the part and whole of a quantity given in a part-whole division problem. In other words, are elementary teachers able to reflect on the relationships between the fractional part and the whole in a part whole division problem and relay that knowledge to students when engaged in analyzing students' errors to part-whole division problems?

Methodology

The sample for this study consisted of 66 students enrolled in a graduate elementary mathematics methods course at a private urban college in the Boston metropolitan area. During November of 1998, data were gathered for the study using a written instrument called "The Students' Solutions Instrument," an instrument consisting of six part-whole division problems. Each problem in the instrument is followed by the correct answer to the problem and a transcript of a short interview conducted by a teacher with a middle school student who had just incorrectly solved the problem. For example, the following is an item from the Students' Solutions Instrument.

Yesterday, Ariel spent $\frac{4}{11}$ of her homework time, or 36 minutes, doing her math homework. How much time did she need to do all of her homework assignment?

The correct answer is 99 minutes, or 1 hour and 39 minutes.

Sydney solved the problem in the following way:

Sydney: "She spent $\frac{4}{11}$ of her homework time and 36 minutes doing her math."

Teacher: "What would be your answer?"

Sydney: "I would add $\frac{4}{11}$ and 36 to get $36\frac{4}{11}$ minutes."

After reading each problem statement and interview, participants in the study were asked to describe the error that the student made in solving the problem and why the student may have made that error. For example, one participant wrote that "Sydney added the 36 to the 4/11 rather than dividing the 36 to find out how many minutes in each set of 11." When asked why the student made that error the participant responded, "the student does not understand that 4/11 and 36 are different ways of stating the amount of time Ariel spent on her math homework."

Data Analysis

Participants written explanations were analyzed based on the theoretical framework of this study. That is, the written explanations are analyzed on the basis of whether or not these elementary teachers were able to explain the part-whole relationship given by the data in the problem. Written explanations were also analyzed in terms of participants' ability to emphasize this relationship in their discussion of student errors and how the student may remediate these errors.

Results

Table 1 shows the number of participants that provided a written explanation that emphasized the relationship relating the fraction and the quantity as referring to the same part of the whole. Nearly all of the participants were able to state the relationship between the fraction and the whole number given in the problem, but success ranged depending on the problem. The lowest percentage of participants that were able to state the relationship occurred with problem 5, the temperature problem. The largest percentage of participants that were able to state the relationship occurred with problem 3.

Conclusions

The results of this study indicate that the elementary teachers have the ability to state the relationship between the part and the whole, particularly when this relationship is made explicit, as in problem 3. In a previously reported study based on elementary teachers' abilities to solve part whole problems (Stone, 2000), it was learned that the same group of teachers use a combination of proportional reasoning, concept of unit, and fraction as operator strategies when solving part-whole division problems. Many of the teachers did not explicitly express the relationship examined in this paper but were able to do so when asked to analyze a students' solution error.

It is interesting to note that when asked to solve problems in the context of teaching, such as analyzing a student's error, the teachers in this study often used pedagogically useful written explanations. Use of error analysis in mathematics methods courses in the discussion of rational number knowledge may be useful to help highlight the thinking that is involved in solving problems and what is useful in helping elementary students gain this knowledge.

Table 1. The Number of Participants (Percentage) Who Successfully Analyzed Students Errors based on the Part-Whole Relationship Stated in Each Problem

Problem	Part-Whole Relationship	Number of Participants (%)
1. Gerald used one-fourth, or 12 postage stamps to mail postcards. How many postage stamps did he start with?	One-fourth is 12	34 (52%)
2. Suzanne noticed that two-fifths of the calculators in the classroom set were broken. Ten calculators were broken. How many calculators were there in the whole set?	Two-fifths is 10	45 (68%)
3. Yesterday, Ariel spent 4/11 of her time, or 36 minutes, doing her math homework. How much time did she need to do all of her homework assignments?	4/11 is 36	50 (76%)
4. One-twelfth of the distance from Cambria to Netley is ten miles. What is the distance from Cambria to Netley?	One-twelfth is 10	45 (68%)
5. In Phoenix last week, one-half of the daytime temperature was the nighttime temperature. The nighttime temperature was 25 degrees. What was the daytime temperature?	One-half is 25	27 (41%)
6. Two-thirds of the students in the chorus are girls. There are 14 girls in the chorus. How many students are in the chorus?	Two-thirds is 14	31 (47%)

References

Ashlock, R. B. (1998). *Error patterns in computation* (7th ed.). Upper Saddle River, NJ: Merrill.

Behr, M., & Post, T. (1988). Teaching rational number and decimal concepts. In T. R. Post (Ed.), *Teaching mathematics in grades K-8* (pp. 190-230). Boston: Allyn & Bacon.

Borasi, R. (1996). *Reconceiving mathematics instruction: A focus on errors*. Norwood, NJ: Ablex.

Brown, J. S., & Burton, R. R. (1978). Diagnostic models for procedural bugs in basic mathematical skills. *Cognitive Science, 2*, 155-192.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18*, 32-42.

Cox, L. S. (1975, February). Diagnosing and remediating systematic errors in addition and subtraction computations. *The Arithmetic Teacher*, 22.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 276-295). New York: Macmillan.

Greer, B. (1994). Extending the meaning of multiplication and division. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 61-88). Albany, NY: SUNY Press.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 276-295). New York: Macmillan.

Kremer-Hayon, L. (1993). *Teacher self-evaluation: Teachers in their own mirror*. Boston: Kluwer.

Lerman, S. (1996). Intersubjectivity in mathematics learning: A challenge to the radical constructivist paradigm? *Journal for Research in Mathematics Education, 27*, 133-150.

Post, T. R., Harel, G., Behr, M. J., & Lesh, R. (1991). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 177-198). Albany, NY: SUNY Press.

Russell, T., & Munby, H. (1991). Reframing: The role of experience in developing teachers' professional knowledge. In D. Schon (Ed.), *The reflective turn: Case studies in and on professional practice* (pp. 164-188). New York: Teachers College Press.

Sowder, J. T., Bezuk, N., & Sowder, L. K. (1993). Using cognitive principles from cognitive psychology to guide rational number instruction for prospective teachers. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 239-259). Hillsdale, NJ: Erlbaum.

Sowder, J. T., Philipp, R. A., Armstrong, B. E., & Schapelle, B. P. (1998). *Middle-grades teachers' mathematical knowledge and its relationship to instruction: A research monograph*. Albany, NY: SUNY Press.

Stone, W. E. (1999). *Factors that contribute to teachers' abilities to solve part-whole division problems*. Unpublished doctoral dissertation, Boston University, Boston, MA.

Stone, W. E. (2000). Prospective teachers' part-whole division problem solution strategies. *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Tucson, AZ. Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Van de Walle, J. A. (1998). *Elementary and middle school mathematics: Teaching developmentally* (3rd ed.). White Plains, NY: Longman.

CLASSROOM REFORM: THEORY AND PRACTICE IN REFLECTIVE ACTION

Sharon B. Walen
Boise State University
walen@math.boisestate.edu

Kathy M. Ivey
Western Carolina University
kivey@wpoff.wcu.edu

Abstract: This paper is a critical evaluation of the impact of long-term research from a single theoretical framework on the practice of teaching by two university teachers. The changes in instruction were not accomplished through a step-wise process but rather through an injection model (Williams, Smith, Mumme, & Seago, 1998). Perhaps the most important aspect of this paper is the exploration of the conflict between internal and external perceptions of the change process. Although the changes we have made in our instruction appear to our colleagues to be complete and our classrooms are viewed by our peers as examples of reform, neither of us view our progress as complete. This paper documents the dynamic tension we have felt and continue to feel between research about a theory of mathematics education and placing that theory into practice. In particular, we discuss the differences in the two challenges of teaching mathematics and teaching mathematics teachers to teach.

Theoretical Underpinnings

The analysis presented here proceeds from a framework that assumes teachers and students alike bring many beliefs and attitudes into the classroom. (For a more complete exposition of this framework, see Williams, 1993.) We agree with many researchers that the perceived relationships between teacher beliefs and teaching practice are related in part to the goals of the researchers in any specific project. However, in our dual roles of researchers and teachers our research goals become intertwined with our teaching goals. Because we approach classroom research within a Heideggerian framework, we assume that both students' and teachers' actions reflect the Heideggerian notion of being-in-the-world. From this perspective, classroom participants may judge a situation and act within that situation at a non-reflective level, and we, too, participate as teachers in exactly this way. However, because of our long-term use of a Heideggerian framework in our work as researchers, the conflicts that we as instructors have felt and continue to feel as we implement reform practices in our daily classroom instruction force an examination of the basis of our own actions. In typical Heideggerian terms, we have deliberately broken our hammer, and we have continued without the comfort of operating within our familiar unreflective practice since we've begun our move to reform. As we move through the Heideggerian circle, we become aware of new possibilities and see a new horizon. Our movement from being-in-the-world of our teaching to reflecting on our teaching practices presents new opportunities. However, the continuous struggle resulting from students who are "not expecting to be taught this way" and from within ourselves often leaves us exhausted. Examination of this very struggle reveals the tensions and interactions between our researcher roles and our teacher roles.

Modes of Inquiry and Data Sources

Prior to this analysis of our attempts to change our university classrooms, we have participated in multiple studies gathering data with respect to changing P-12 teaching practice and have employed a variety of qualitative methodologies including participant observation, videotape analysis, audiotape analysis, student interviews, student questionnaires, examples of student work, student journals, teacher interviews, and classroom artifacts. The studies include multiyear, multisite projects in middle grades (Ivey, 1996; Ivey, 1994; Ivey & Williams, 1993; Williams & Ivey, 1995), high school mathematics classes (Walen, 1996; Walen, 1994; Walen 1993), and college classes (Ivey, 1997). These studies have served as an ongoing series of events and have informed and influenced our own university teaching. Some of the phenomena that we have previously described in our research assist us as we attempt to explain our current frustrations. However, the individual perspectives brought to light by the authors are unique to this paper and to our common predicament–changing university classrooms to reflect the teaching and learning goals described in the National Council of Teachers of Mathematics documents: *Curriculum and Evaluation Standards for School Mathematics* (1989), *Assessment Standards for School Mathematics* (1995), *The Professional Standards for Teaching Mathematics* (1991), and *Principles and Standards for School Mathematics (2000)*. For the purposes of this report, two specific examples will be discussed in limited detail. In addition, we will reference non-university examples and other complications without presenting a complete analysis.

Discussion and Implications

Lampert (1985) suggests that she and other teachers manage a dilemma "by putting the problems that lead to it further into the background and by bringing other parts of my job further to the foreground" (p.186). This view of teaching as managing dilemmas, rather than making didactic choices, presents decisions in a light of "coping rather than solving" (p. 189) and allows us to frame our discussion. We, with Lampert, see "the person she wanted to be" (p. 188) and this vision assists us as we, too, manage dilemmas. We begin our discussion of change with the observation that our university students come to our classrooms with certain expectations–expectations that have been well established through years of traditional classroom instruction. Our students know what it is like to learn in a classroom and they assume from observation that they also know how it is to teach in a classroom setting. Some of our students have even practiced teaching within their chosen role in the classroom. This phenomenon of the "little teacher" has been described elsewhere (Ivey & Walen, 1999). However, our classrooms are not like those with which they have prior experience, and this is a source of both joy and frustration for our students. We ask our students to explore problems together in small and large groups, to create sensible ways of doing mathematics, to write about their work, and to explain their understandings to others. Those students who find joy in this kind of non-traditional classroom setting may not have experienced success in a traditional mathematics classroom. Such students become enthusiastic, excited about learning and understanding real mathematics, and serve to motivate our continued efforts. However, they do not constitute the whole of our students. Students who come to our classrooms skeptical of our methods and frustrated with our expectations are also common. To alleviate their frustrations, our students often try to recreate a familiar traditional classroom environment within their small groups. In this attempt to reproduce traditional roles and expectations, they subdivide the work of their group and assume more traditional roles within the group itself. When this occurs in our classrooms, the members of the group do not participate as partners in a joint problem solving setting but, rather, as students in a traditional classroom setting. Their allocation of responsibilities within the group most often results in one student, the little teacher, assuming the mathematical power and responsibility for group learning. Other students in the setting accept little or no responsibility for their own learning and, for the most part, resort to an inactive consumption and regurgitation of what the little teacher has decided is required for success within the group and classroom as a whole. This restructuring of classroom equity and responsibility is contrary to our philosophy of teaching and learning. Trying to change the structure of the small groups who have encapsulated themselves in a traditional bubble within our reform classroom setting can be difficult for us and traumatic for the group members. We have no choice but to take away the comfort zone in which they have cocooned themselves. We do not have a good way to teach students to adopt a more productive group structure. It is our philosophy that telling them what to do when working in a group is no more appropriate than lecturing to the whole class about mathematics. Assigning roles for each group member, for example, appears to be counterproductive to a classroom environment that expects to empower students as genuine learners of mathematics. We have written about teachers' successful uses of cases or vignettes in P–12 classrooms as they assisted their students to reflect upon group processes (Walen & Ayers, 2000, and Walen & Hirstein, 1995). In addition, using cases in small group discussions have provide practicing teachers in the midst of reforming their own classrooms opportunities to reflect, identify, articulate, and address common concerns. (For a complete discussion of these issues, see Walen & Williams, 2000.) However, the use of cases to remedy this particular college classroom situation has been less successful than we had hoped. This may be a reflection of the nature and strength of our students' need to cultivate and maintain a more traditional classroom environment. This particular conundrum has become a source of discomfort for us as we continue to teach and research using our theoretical framework–Hermeneutics. As researchers, we are able to identify and describe but not solve this particularly difficult situation in our classrooms, while, as teachers, we look to the research for assistance in solving our dilemma. As a result, our phenomenological perspective leaves us with an identified problem and a solution that does not fit within our theoretical framework.

It is also true that we are not the only ones looking for solutions to teaching dilemmas. Many times we are approached, as the mathematics education experts, by mathematics colleagues with requests for answers to their own dilemmas. (How should I group my students in a calculus class with only three females? Do I keep the women together, or do I separate them?) Unfortunately the answer often is, it depends. Without more knowledge of the instructor's and students' views of the classroom, it is difficult to suggest answers. It is particularly difficult to provide the type of answer that the colleague wants—a single, correct, definitive answer that can be used over and over much like a proven theorem. Our theoretical framework for research does not start or proceed in a disembodied fashion, but rather sets out to make sense of a certain aspect of human existence. Similarly, our teaching does not start out with a curriculum to be lectured but with the goal to provide a classroom experience through which all students may find their own voice and speak with mathematical power.

Most college instructors were successful in learning mathematics in traditional classroom settings. To expect them to reproduce other than what they have experienced as successful seems unreasonable. Furthermore, we do not wish to impose on them our view of what is important to instruction. Each person's horizon is necessarily distinct, and the process of change is continual. What we view as a work forever in progress, our colleagues may view as a completed whole. Their road to reforming their own classrooms must be one that they design, not a mirror image of ours. So, in a serious sense, there really is not anything specific to tell our colleagues about how to mechanically reform their classrooms. From our theoretical perspective, we cannot simply tell them what to do. However, from their experience, this is precisely what they expect from us.

Overall, our experiences have been positive in teaching mathematics from a nontraditional approach. Students may resist, and we may have to help them learn different ways of working in a mathematics class, but generally we have been successful. Students see that they are learning mathematics or that their peers are, and they become more willing to try a different approach to learning mathematics. Perhaps the immediacy of experiencing success or watching others experience success with mathematics problems helps to make students willing to try new ideas and to participate in new ways in the classroom. However, we have not been as successful in teaching students about how to teach using reform methods. Students who have chosen to become secondary mathematics teachers seem particularly reluctant to release the stranglehold they have on their traditional expectations for teachers and students in a mathematics classroom. We typically come in contact with these students as they enroll in Teaching Secondary Mathematics–one of their last courses prior to student teaching. At this point in their college education, they have completed all but one or two of their required mathematics classes. We believe that this course should be the most exciting class that we teach. We have a vision of the course as a collegial setting where we (students and instructors) all work to "become the teachers we want to be." Many of the students have exactly this experience, but not everyone.

It is in this course that we meet those students who are the most resistant to reform mathematics teaching methods and ideas. These students also have a picture of the "teacher they want to be." Typically, the person they wish to emulate is someone who "turned them on to mathematics." The problem is that their role model is often someone who taught using a very traditional method: lecture with repeated individual practice. And these students have evidence that their role model's methods worked–they themselves serve as proof. They were well served by this wonderful teacher. They are successful. Their desire is to complete their college education and become just like the person that they admire, a high school teacher who "covers material by lecturing clearly and efficiently." For example, one student, Jane, recently argued that reform methods were "just too much trouble," and that she was going to go back to her home town after graduation and "take over for Mr. Johnson. He's retiring and he asked me if I would be willing to take over for him." Students like Jane with this type of role model argue that reform methods take too much classroom time, require too much preparation, and are not an efficient way to cover the required course content.

What students with these views want from the instructor in their methods course is to "tell us exactly what to do. Teach us how many questions to work on the board, how many problems to assign for homework, when to give a quiz, when to give a test, etc." It appears that they wish to become teachers who will be able to reproduce a classroom environment that replicates their own high school mathematical experience. They wish for a methods class that would tell them exactly how to become their idols. These students wish to become the teachers of yesterday who had a great influence on their own success. For these students, no amount of discussion, reading of the Standards, or classroom experience with alternative materials seems to make a difference in their vision of "the teacher they want to be" and in their attitudes toward reform methods. Their frustration with the class begins early and lasts. Jane described her frustration on the course evaluation, "I learned to use different things and different ways to teach but I feel as though I didn't become a better teacher or presenter of lessons. I scored poor on every lesson and I didn't seem to improve. Mr. Johnson said that what you had us doing in class was not going to work in his real classroom." Recall that Mr. Johnson is Jane's idol and is the teacher that Jane intends to replace when he retires. Mr. Johnson continues to support traditional methods and continues to oppose changes in instructional methods. His position as idol and as practicing teacher holds the ultimate power to influence Jane's views. During this course, Jane and others with similar role models are not moved from their original views of teaching. Thus, our efforts to encourage students to look at teaching in a new way, in effect to "break their hammers," were not consistently productive.

Interestingly, getting students to look at teaching in new ways is often more effective with those who are practicing teachers returning to school to complete a Master's Degree. These students are often more willing to explore new ideas because they have seen that the traditional methods do not work with all students. Take for example Olivia's comment, "every week I leave with a new idea of something I want to look at in my own teaching; which is great, but exhausting." Notice that Olivia is less concerned about the specific technique or problem that has been given in class and is more interested in the possibility it presents in her own work as a teacher. Olivia is ready to examine the hammer that she has been using to see if it is the correct tool. It appears that actual classroom teaching experience may influence one's willingness to examine non-traditional teaching techniques. From our theoretical perspective, our own methodology consists of a basic desire to find out what a certain phenomenon means and how it is experienced. In this sense we, like Olivia, teach in our classrooms sensitized to the subtleties of our everyday experiences.

Mathematical proficiency and mathematical confidence also influence students' willingness to examine alternative methods for instruction and assessment. Students who are reluctant to attempt to present a lesson in any form other than a lecture format express fear that they will not know the answers to questions when asked. The NCTM's Principles and Standards for School Mathematics (2000) emphasizes a teacher's mathematical competence. Mathematically weak pre-service teachers intuitively suspect, perhaps correctly, that teaching in a format where they have to think mathematically on their feet will be their downfall. Jane's comment summarizes this view, "Besides even if Mr. Johnson did think you were doing things that would work in real classrooms, I don't think that I would do it. I don't want anyone thinking I don't know the math. I'm going to be a good teacher even if you don't think I will. If I don't know an answer, I'll look in the teacher's book." This view of "looking in the teacher's book" is a view that most likely would provide solutions for lecture-type lessons but not for non-traditional types of lessons. Mathematical confidence strongly influences this student's willingness to take risks in classroom instruction.

Perhaps we are less successful with teaching students how to use reform methods in their own teaching because of the lack of immediate application. In our methods classes, we use observations, peer teaching, and videotape analysis as means of mimicking practice with students in real classrooms. One unavoidable problem with these practice sessions is the pretend nature of all these activities. Our methods students do not have real experiences with using alternative methods. The students that they teach in these practice activities generally already know the content, so there is no chance to observe real learning made possible by alternative methods.

Lampert's view of the "contradicted 'self' as a tool of her trade" (p. 190) also allows us to describe how we feel as our colleagues come to us for education solutions in their classrooms. It is frightening that we, who are still in the throes of change and uncertainty, are called upon to lead the way in improving classroom instruction. What we offer in the way of advice today may not be what we would suggest tomorrow and, most certainly, will not be what we would say a year from today. However, according to our peers and according to many of our students, we are models for educational solutions. This paper helps to document the difficulty we have met and continue to meet in making changes to our college classrooms. It also provides a warning and a reminder to all who would be instruments of change: unless we verbalize our concerns, our outward appearance is all our students see. The nature of the inevitable conflict between who we are and who we want to become is, at its foundation, the essence of the Heideggerian framework that is our lens on educational practice. Our work as researchers must influence our work in the classroom and vice versa. However, as a field, we have not carefully examined the apparent separation of our work as researchers and our work as teachers. We proceed in a generally unreflective mode of being-in-the-world. Yet how change proceeds depends on our ability to critically examine the interactions between research and teaching, i.e. to break the hammer that separates our roles of researcher and teacher.

References

Ivey, K. M. C. (1997). Warning: Asking questions in small groups may lower your mathematical status. In J. A. Dossey (Ed.), *Proceedings of the Nineteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 619-625).

Ivey, K. M. C. (1996). Mechanistic and organic: Dual cultures in a mathematics classroom. *Journal of Research and Development in Education*, 29 (3) 141-151.

Ivey, K. M. C. (1994). World views in the mathematics classroom: Students translating beliefs into actions (Doctoral dissertation, Washington State University, 1994*). Dissertation Abstracts International*, *55*, 3773A.

Ivey, K. M., & Walen, S. B. (1999). The little teacher: The development of a student role in mathematics small groups. In F. Hitt & M. Santos (Eds.), *Proceedings of the Twenty-First Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 624-630). Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Ivey, K. M. C., & Williams, S. R. (1993). Investment of self as a component of mathematical engagement. [Summary] In J.R. Becker & B.J. Pence (Eds.), *Proceedings of the Fifteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol.

2, p. 257). Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Lampert, M. (1985). How do teachers manage to teach? Perspective on problems in practice. *Harvard Educational Review, 55* (2), 1978-194.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (1995). *Assessment standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Walen, S. B. (1994). *Identification of student-question teacher-question student-response pattern: Students' interpretation of empowerment.* Paper presented at the Annual Meeting of the National Council of Teachers of Mathematics. Indianapolis, IN.

Walen, S. B. (1993). An analysis of students' knowledge of the mathematics classroom (Doctoral dissertation, Washington State University). *Dissertation Abstracts International, Vol. 54-08A,* 2935.

Walen, S. B. (1996, April). *An examination of small groups in secondary mathematics classrooms: Views from inside, outside, and somewhere in the middle.* Paper presented at the Research Presession of the Annual Meeting of the National Council of Teachers of Mathematics, San Diego, CA.

Walen, S. B., & Ayers, K. L. (2000). Using vignettes in a sophomore math classroom. In C. R. Duke & R. Sanchez (Eds.), *Assessing writing across the curriculum* (pp. 89-107). Durham, NC: Carolina Academic Press.

Walen, S. B., & Hirstein, J. (1995). Classroom vignette: An alternative-assessment tool. *Teaching Children Mathematics, 1* (6), 362-365.

Walen, S. B., & Williams, S. R. (2000). Validating classroom issues: Case method in support of teacher change. *Journal of Mathematics Teacher Education, 3* (1), 3-26.

Williams, S. R. (1993). Mathematics and being in the world: Toward an interpretive framework. *For the Learning of Mathematics, 13* (2) 2-7.

Williams, S. R., & Ivey, K. M. C. (1994). Exploring the social in social construction. In D. Kirshner (Ed.), *Proceedings of the Sixteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 105-111).

Williams, S. R., Smith, S. Z., Mumme, J., & Seago, N. (1998). Ports of entry into curricular change. In L. Leutzinger (Ed.), *Mathematics in the middle* (pp. 75-82). Reston, VA: National Council of Teachers of Mathematics.

DR. ROBERT B. DAVIS, THE TEACHER

Carol W. Bellisio
Rutgers University
Carol@bellisio.com

The purpose of this paper is to document the teaching techniques of a master teacher who demonstrated the philosophy of the Madison Project. It will be shown that this teaching style resulted in thoughtful, successful work by the students. The Madison Project was begun in 1957 in order to change children's school mathematical experiences. Educators wanted to provide students with experiences in which they were successful, and to make the classroom atmosphere more conducive to learning. Students in this project were given a larger role in learning mathematics, and teachers were encouraged to use informal language and choose problems that related to the children's culture and interests (Davis, 1972). Teachers respected the children and had high expectations for the quality and sophistication of their work. Teachers and students jointly explored the mathematics, and student solutions and techniques were often named after the children who invented them.

This research was motivated by a study of the data collected during a three and a half year, longitudinal research project conducted by mathematics educators at Rutgers University (funded in part by NSF grant # MDR-9053597). The students described in this paper were members of a fourth grade class of twenty-four students from a suburban, middle-class community and Dr. Davis was the teacher. Eight of the same students were interviewed by Dr. Davis the following year. Data for this report came from transcripts of videotapes taken of the classroom session and the interview, student work, and researcher notes. Dr. Davis gave students activities that involved the use of negative numbers, the rule for substituting numbers in an equation, and the discovery of numbers that could be substituted into a quadratic equation. The dialogue between Dr. Davis and the students will be described in this talk.

Dr. Davis set a tone of encouragement and respect as he began the class by stating. "I know that you are very good at math " It was interesting to note how Dr. Davis treated a wrong answer. After students had found two numbers that worked for the quadratic equation, a student thought she had found a third solution. Dr. Davis' response was, "You found another number that works? Perhaps we left something out. Let's go back to this one." All during the class Dr. Davis acted as a partner in discovery rather than as a teacher. When he reminded the students of the Pebbles-in-a-Bag activity they had previously done he asked, "What are we going to have to do to make this work, do you remember?" He acted as the liaison between student groups with comments such as, "They claim they have a number that works," and "John says he has a number that works and he would like to show you." He let students decide the method of presentation and he assured the students they would all get a chance to talk.

Dr. Davis interviewed eight of these students one year later. His philosophy of teaching was again apparent. He asked the students to continue working on a problem they had previously begun. He discussed with them the patterns they had discovered, "Suzanne's method," "Brendan's law," and "Nadia's extension." Throughout the interview many of his teaching technique were evident as he encouraged the students and gave them time to work out a written notation for their verbal rules.

Many aspects of the Madison Project were demonstrated during the class period and group interview. Over forty years ago, Dr. Davis helped to develop a philosophy of teaching that encouraged students to discover mathematical ideas and to talk about their ideas and solutions. The role of the teacher was very important but in a different way than had previously been considered. They were no longer the authoritarian figure in the front of the class, but served as mentors and coaches. Student-teacher language and interaction reveal Dr. Davis' style of teaching that embodied this philosophy. Success of this teaching style will be shown in the students' enthusiasm and involvement with the activities and in their discoveries and solutions.

References

Bellisio, C. (1999). *A study of elementary students' ability to work with algebraic notation and variables.* Unpublished doctoral dissertation, Rutgers University, New Brunswick, NJ.

Davis, R. B. (1962). *The Madison Project: A brief introduction to materials and activities*. U.S. Department of Health, Education and Welfare. (ERIC Document Reproduction Services No. ED028948).

Davis, R. B. (1972) Observing children's mathematical behavior as a foundation for curriculum planning. *Journal of Children's Mathematical Behavior, 1* (1), 7-59.

PRESERVICE ELEMENTARY TEACHERS' PERFORMANCE ON NONROUTINE WORD PROBLEMS INVOLVING ADDITION AND SUBTRACTION OF WHOLE NUMBERS

José N. Contreras
The University of Southern Mississippi
Jose.Contreras@usm.edu

Word problems play a prominent role in school mathematics. One reason for the inclusion of word problems in the mathematics curriculum is that they provide practice with real life problem situations where students apply what they learn in school. Another is that word problems help students to make connections between real-world situations and mathematical concepts. Some studies (e.g., Verschaffel, De Corte, & Vierstraete, 1999), however, have documented children's tendency to ignore contextual realities embedded in realistic problematic word problems. In this paper, a problematic or nonroutine word problem is defined as a problem in which the solution provided by the mathematical model does not necessarily represents the solution to the problem, at least if one seriously takes into consideration the reality of the situational context. This study extends previous research to preservice elementary teachers' performance on problematic story problems involving addition and subtraction of whole numbers.

A paper-and-pencil test was administered to a sample of 115 prospective elementary teachers enrolled in their second mathematics content course in a southestern state university in the USA. The test contained 9 experimental items and 7 buffer items. The experimental items were adapted from Verchaffel, De Corte, and Vierstraete's (1999) study. Three of the experimental items were straightforward and six were problematic. An example of the former type of item is "In January, 1985 a youth orchestra was set up in our city. In what year will the orchestra have its twenty fifth anniversary?" An example of the later kind of problem is "For a long time the city held a fireworks display every year on the last day of the October festival. In October, 1982 we had our last fireworks, and thereafter there was no fireworks display. In October 1999 they restarted the tradition of the annual fireworks display. How many years did we miss the fireworks?" As expected, based on the fact that straightforward addition or subtraction yields the correct answer to the nonproblematic items, prospective elementary teachers performed well on the three nonproblematic items. Only 5% of the responses were incorrect. On the other hand, prospective elementary teachers' performance on the problematic items was poor. The percentage of correct responses for each problematic problem varied from 2.6% to 10.4%. Overall, only 5.5% of the solutions to the problematic items were correct, 87% contained ±1 errors, 6% contained other types of errors but could lead to ±1 errors if an appropriate model were executed correctly, and 1.5% contained errors due to other factors such as adjusting both numbers, using the inverse operation (subtraction instead of addition), etc.

While it was expected, based on previous research, that some prospective elementary teachers would not provide the correct solution to the problematic word problems, it was alarming to find out that such a high percentage of the total responses to the problematic items contained unrealistic solutions. One explanation might be that students are conditioned to approach problems in a superficial or mindless way because the problems posed in the traditional instructional environment can be solved by the straightforward application of arithmetical operations. In this sense, students probably expected that all the test problems were of that kind. Another explanation may be students' insufficient understanding of the meaning of subtraction and addition involving ordinal numbers or to an insufficient repertoire of useful heuristic strategies (such as thinking of an easier analogous problem or making a diagram). A question worth investigating is: what are the effects of minimal intervention, such as giving students a hint that some of the given problems are problematic or "tricky", on students' modeling strategies to solve nonroutine word problems? Research using clinical interviews would provide additional insights into the nature of students' thinking and reasoning when solving nonstandard addition and subtraction problems involving whole numbers. In any event, the low percentage of correct solutions to the problematic problems is alarming. It is very likely that prospective elementary teachers need some type of instructional intervention to learn how to model whole number addition and subtraction problems in which the solution is 1 more or 1 less than the sum or difference of the two given numbers.

Reference

Verschaffel, L., De Corte, E., & Vierstraete, H. (1999). Upper elementary school pupils' difficulties in modeling and solving nonstandard additive word problems involving ordinal numbers. *Journal for Research in Mathematics Education, 30* (3), 265-285.

SPATIAL SENSE VS. COUNTING ON FRANK: PRESERVICE TEACHERS' COMPARATIVE ANALYSIS OF MATHEMATICAL SOFTWARE

Sandra Crespo
Michigan State University

Technology requirements have become increasingly common in teacher preparation programs. Prospective teachers around the country are now expected to be proficient users of technology, not only to increase personal productivity, but to also use technology as an educational resource in their classrooms. Technology has also been widely recognized as an important tool in the learning and teaching of mathematics with the power to "influence the mathematics that is taught and enhancing students' learning" (NCTM, 2000, p. 24). Yet, opportunities and models for using technology as an educational tool are in short supply in most teacher preparation programs.

This study set out to investigate the ideas that prospective teachers might have about using mathematical software in their classrooms. Specifically, this study asked: (a) How do preservice teachers go about deciding what makes a good mathematical software?, For instance, are they swayed by surface features of the software or do they attend to substantive issues such as content and pedagogical considerations. (b) What criteria/rationale/justification do they use in order to cast their decisions? And (c) What factors influence and affect their decision making process?

Participants: The participants in this study are elementary prospective teachers attending a mathematics "methods" course taught by the researcher in this study. *Experiment*: The "evaluation of educational software" is one of the four "technology requirements" adopted by my institution's teacher preparation program. To complete this requirement prospective were presented with the scenario of having to review two pieces of mathematical software in order to write a recommendation to their school Principal for purchasing one of them. *Data*: The data collected consisted of preservice teachers' write up explaining their reasoning for why they would recommend their chosen mathematical software. In addition, the teacher-researcher observations made during the activity and later recorded were also collected.

About the Software: The two mathematical software chosen for this experiment were a commercially produced game called "Counting on Frank" (by EAKids) and an educationally produced software called "Spatial Sense" which include "The Factory", "Building Perspective" and "Super Factory" (by Sunburst). These two particular products were chosen for a variety of reasons. They are not drill-and-practice. They have a game-like interface. They focus on different mathematical content, one focuses on number and operations while the other focuses on geometry. Yet they both emphasize problem solving.

The design of the experiment and analysis of the data are based on the works of critical theorists such as Apple (1988) and Ben-Peretz (1990) who advocate that teachers be freed from the "tyranny of texts" and engage in critical analysis of curriculum materials. In particular, Ben-Peretz' construct of "curriculum potential" is used to describe what prospective teachers attend to, chose to ignore, or remain unaware of, when analyzing the educational value and potential of mathematical software. The analysis of the data is further informed by the research project E-GEMS which for the past few years have been studying the educational potential and design of electronic games in mathematics and science classrooms. In particular, Saxton and Upitis' (1995) categories of students' orientations towards playing computer games as "players, resistors, and creators" are used to describe patterns in preservice teachers' engagement with the software. The relationship between prospective teachers' engagement and uncovering of the software's curriculum potential is also explored.

References

Apple, M. (1988). *Teachers and texts: A political economy of class and gender relations in education*. Albany: SUNY.

Ben-Peretz, M. (1990). *The teacher curriculum encounter: Freeing teachers from they tyranny of texts*. Albany: SUNY.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Saxton, S., & Upitis, R. (1995). *Players, resistors, and creators: Adolescents and computer games.* (Available online at http://taz.cs.ubc.ca/egems/home.html)

MATHEMATICAL DISCUSSION DURING STUDENT TEACHING: WHO HELPS, WHO HINDERS, AND WHY

Vari Nelson Durrant
Brigham Young University

This study considers the mathematical discussions that take place or do not take place between student teachers and cooperating teachers and why such conversations occur or do not occur. Deborah Ball's 1991 study of prospective teachers prior to the student teaching experience found that: (1) traditional school mathematics can be difficult, (2) precollege education does not sufficiently provide subject matter knowledge, and (3) majoring in mathematics does not guarantee adequate subject matter knowledge. Since the translation of subject matter knowledge into pedagogical content knowledge appears to be a critical point for many new teachers, this study attempts to gain insight into this process by examining the student teaching stage of teacher preparation. This study looks more closely at the motivations and discouragements that affect mathematical discussion between cooperating teachers and student teachers.

Questionnaires were given to cooperating teacher/student teacher pairs in the 1997-1998 school year. From this data, research directions were developed and a more focused questionnaire was given to 25 such pairs expected to interact during Winter Semester 1999. From these 25, 8 pairs were selected for the study according to their responses to questions concerning expected interaction frequency and content.

Data for the study was collected in several forms. The main sources for this data consisted of two interviews for each participant, self-reports of contacts collected randomly throughout the semester, audiotaped recordings of selected conversations, and classroom observations.

Results can be summarized in terms of the research questions.

1. Do student teacher/cooperating teacher pairs discuss mathematics at a conceptual or procedural level? Six of the eight pairs observed discussed mathematics almost exclusively at a procedural level if at all. The cultural misconception that the mathematics taught in the secondary schools is easy and needs little discussion that was discovered by Ball in 1991during pre-student teaching continues to be present during the student teaching experience. However, it may also be the case that many student teachers and even cooperating teachers had a shallow, insufficient understanding of the mathematics they were teaching. Pedagogical content knowledge was widely recognized as one of the needs of the new student teachers, but discussions and interactions that foster such knowledge were often lacking.

2. What beliefs seem to motivate or discourage mathematical discussion and who possesses these beliefs? The attitudes observed in this study seem to stem from both cooperating and student teachers, though cooperating teachers' beliefs were more apparent in most cases. However, the culprit underlying the attitudes seems to be the culture of teaching. Student teachers are usually bound by the role of the cooperating teacher as an evaluator; cooperating teachers are intimidated by the mathematical knowledge student teachers supposedly obtain during their preparation for teaching. It also seems to be a commonly held belief, as mentioned before, that once a person gains adequate personal knowledge of the material, it requires little effort to achieve understanding of how to teach the same material. On the surface the cooperating teacher and student teacher seem to be the source of the attitudes expressed, but it is not difficult to see that these attitudes are a product of the culture in which they have been taught and are living
3. What are the general characteristics of cooperating teacher and student teacher interaction that facilitate or hinder mathematical discussion? As this study shows, few (2 of 8) student teachers are receiving the subject matter understanding needed to teach under current NCTM Standards during their student teaching experience. The reasons for this are many and varied. Most are a result of societal misconceptions carried as beliefs both by current and prospective teachers. Mathematical discussion is discouraged by cultural norms. It is encouraging, however, to see that a few cooperating teachers and student teachers were able to recognize and move past those constraints and progress toward a better and more conceptual mathematical understanding.

I believe that the first step that must be taken to improve mathematical discussion between cooperating teachers and student teachers is simply to make them aware of the beliefs of the teaching culture that prevent mathematical discussion. Simply recognizing that these obstacles exist enables both student teachers and cooperating teachers to move past them and discuss the mathematics in a way most beneficial to them both. This might be done through pre-student teaching training for both student teacher and cooperating teacher.

Reference

Ball, D. (1991). Research on teaching mathematics: Making subject-matter knowledge part of the equation. In J. Brophy (Ed.), *Advances in research on teaching* (Vol. 2, pp. 1-48). Greenwich, CT: JAI.

MATH JOURNALS: TOOLS FOR THE GROWTH OF MATHEMATICAL UNDERSTANDING

Hari P. Koirala
Eastern Connecticut State University
Koiralah@easternct.edu

The National Council of Teachers of Mathematics (NCTM, 2000) and the American Mathematical Association of Two Year Colleges (AMATYC, 1995) emphasize that students should be able to communicate mathematically, both in written and oral forms, using mathematical vocabulary and notations. Following these recommendations, many teachers have used journals as a regular feature of their school and college mathematics classrooms. Some of these teachers report that the use of journals in mathematics classrooms has helped them understand students' mathematical thinking (Chapman, 1996). The purpose of this presentation is to analyze the use of journals in mathematics classrooms intended for prospective elementary school teachers and discuss how these journals help develop students' growth of mathematical understanding.

The data for this study were collected from a "Number Systems" course over a period of five years, taught by this researcher. This course is primarily intended for future elementary school teachers. In the past five years more than 200 students have participated in journal writing, totaling approximately 1800 journal entries. The course is taught using a problem-solving approach in which students are required to monitor their thinking processes using journals as a tool. Some journals are fully open and students are free to choose any topics as long as they relate to course materials and purpose. Other journals require students to provide specific responses to a mathematical task or scenario. Regardless of the types of journals, students are encouraged to demonstrate their mathematical understanding.

Each journal entry was read by the researcher/instructor and responded to students during teaching. The entries were also analyzed to assess student growth of mathematical understanding. In response to open topics, more than 80 percent of the students simply chose to express their beliefs about mathematics. Although they were asked to demonstrate their mathematical understanding in their journals no such understanding was noted. They made statements like "I liked the last week's class because it was fun" or "I did not understand what you were trying to do in the last class." Moreover, the journals were rather brief and did not demonstrate any in-depth reflection from students. As the course progressed, the students developed sufficient confidence to express their concerns, difficulties, frustrations, and excitements in their journals. They also began to identify mathematics concepts. While the information obtained from students' general entries were helpful to address their problems and concerns, it was difficult to explore their mathematical thinking. Students' mathematical thinking was more evident on specific journal entries, which required specific responses.

From the students' point of view, journals were powerful tools for them to express their concerns. As one student commented, "I like the journals because they give me time to think and reflect about what I learned and ask questions as they come up, and not have to wait until class-time." Other students made similar comments in their journals.

The use of journals in this course helped the instructor improve his teaching of mathematics and develop students' mathematical understanding. The instructor was personally close to each student in the class and was able to respond to their concerns. The non-specific journal entries provided information about students' beliefs and attitudes, which were helpful for the course. The specific entries solicited student's specific mathematical thinking, which helped in assessing students' growth of understanding. Overall the journals created a positive atmosphere in the class both in terms of classroom climate and students' understanding of mathematics.

References

American Mathematical Association of Two-Year Colleges. (1995). *Crossroads in mathematics: Standards for introductory college mathematics before calculus.* Memphis, TN: Author.

Chapman, K. P. (1996). Journals: Pathways to thinking in second-year algebra. *The Mathematics Teacher*, *89* (7), 588–590.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

DEVELOPMENTAL PROCESS IN THE TRANSITIONAL PERIOD: PRESERVICE MATHEMATICS TEACHERS BECOMING CLASSROOM MATHEMATICS TEACHERS

Hea-Jin Lee
The Ohio State University at Lima

This study was designed to understand M.Ed. students' conceptual changes in teaching, their teacher education program, and self-identification as they progressed through their 5th year teacher preparation program. This study was conducted from 2000 through 2001, during which time two one-month field experiences were included in the teacher preparation program. Data was collected throughout three different time lines: at the beginning of the academic year; after the first field experience; and after student teaching. Fifty-two preservice teachers in an elementary teacher education department participated in this investigation. The main data sources in this study were observations, interviews, and written documents such as concept maps, journal entries, and questionnaires.

Research on both preservice and inservice teacher change indicates that changes in ideas and attitudes, and action and behaviors, occurs in a mutually interactive process. That is, change in attitudes and behaviors is interactive; well-conceived professional learning experiences address both, knowing that change in one brings about and then reinforces change in the other. As noted by Borko and Putman (1996), "[t]he knowledge and beliefs that prospective and experienced teachers hold serve as filters through which their learning takes place. It is through these existing conceptions that teachers come to understand recommended new practices" (p.675). In addition, the statement by Pajares (1992) "Beliefs are instrumental in defining tasks and selecting the cognitive tools with which to interpret, plan, and make decisions regarding such tasks: hence they play a critical role in defining behavior and organizing knowledge and information" (p. 325) supports the idea of that teachers' beliefs shape their practice, and that we must understand and take into account prospective teachers' beliefs in developing and assessing our teacher-education programs.

The changes observed throughout this study were in emphasis from transmission of knowledge to experimental learning; from reliance on existing research findings to examining one's own teaching practice; from individual-focused to collaborative learning; and from mimicking best practice to problem-focused learning. Individuals who change their beliefs and practice over time went through stages in how they feel about the change and how knowledgeable and sophisticated they are in using what they have learned in their teacher preparation program. The questions that they asked evolve from early questions that are more self-oriented to questions that are more task-oriented. In this study concept maps were analyzed to look for changes in the following: total number of items on the maps, number of item streams, hierarchical organization, increased similarity to one another, use of technical vocabulary introduced in the program, and content analysis of frequently used terms. Many students listed terms such as lesson preparation, attention, enthusiastic teaching, and teaching aids. Other frequent item streams included humor, reinforcement, and classroom management. There were general lack of technical vocabulary evidenced on the maps, as well as a lack of detail and hierarchical organization.

It appears that successful programs encourage students to discuss the beliefs that guide their thinking and actions, pinpoint the differences between those beliefs and the perspectives that their professors want them to consider, and analyze the advantages and limitations of thinking with and acting on their current beliefs. Prospective teachers can be encouraged to "try on" ways of interacting or teaching that conflict with their beliefs, as long as the outcomes are likely to be positive and not simply confirming of the beliefs. Often change in behavior proceeds change in beliefs (Guskey, 1989). Feiman-Nemser and Buchmann (1989) describe a student teacher who "combined past experiences with ideas she encountered in formal preparation in a way that reinforced earlier beliefs and reversed the intended message of her assigned readings…" (p. 371). The tension between challenge and support – between assimilation and accommodation, between program elements that are consistent with students' current understanding of teaching and elements that question those conceptions-is a tension that must be tolerated and cultivated.

References

Borko, H., & Putnam, R. (1996). Learning to teach. In D. Berliner & R. Calfee (Eds.), *Handbook of educational psychology* (pp. 673-708). New York: Macmillan.

Feiman-Nemser, S., & Buchmann, M. (1989). Describing teacher education: A framework and illustrative findings from a longitudinal study of six students. *The Elementary School Journal, 89*, 365-377.

Guskey, T. (1989). Attitude and perceptual change in teachers. *International Journal of Educational Research, 13*, 439-453.

Parjares, F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*, 307-332.

INSIGHTS FROM A MATHEMATICS-SPECIFIC FIELD EXPERIENCE FOR ELEMENTARY EDUCATION MAJORS

Denise S. Mewborn
University of Georgia
dmewborn@coe.uga.edu

Elementary education majors at the University of Georgia participate in a mathematics-specific field experience in conjunction with their first mathematics methods course during their junior year. The purpose of the field experience is to provide the preservice teachers (interns) with opportunities to listen to and make sense of the mathematics that children generate and to plan appropriate instruction based on a child's mathematical thinking. Thus, each intern works with one child once a week for 45 minutes for eight consecutive weeks. (For a thorough description of the field experience, see Mewborn, 2000.) The field experience provides mathematics methods instructors with a venue for coaching each intern in the teaching of mathematics, an opportunity not afforded in traditional field experiences that span multiple content areas in a whole class setting. The corpus of data that formed the basis of this manuscript included weekly lesson plans and reflections prepared by interns, weekly observation notes made by faculty and graduate assistants, and final reflective portfolios prepared by interns over a five year period.

Our analysis of the data shows evidence of several positive outcomes of the field experience. First, the interns become convinced that children have powerful, sensible ways of thinking about mathematics that are very different from the ways that the interns think. They learn to suspend their preconceived ideas about problem solutions and really listen to the children, and they become more adept at asking questions that elicit children's mathematical thinking (because most of these children are not accustomed to explaining their thinking). The interns also become aware of the shortcomings and inadequacies of procedural explanations, and, in some cases, of manipulatives when they try to use these as "band-aids" for children's confusion or questions.

The way in which the interns interpret the goal of the field experience has a significant impact on their teaching practice during the field experience. Despite the fact that the sessions are not framed as "tutoring" sessions, some interns see their goal as getting a student to be procedurally proficient with a particular piece of mathematical content. These interns tend to take a less exploratory attitude toward planning and teaching and are concerned about both themselves and the children "getting it right" (Lovin, 2000). Interns who see their task as investigating a child's mathematical thinking and planning activities that are appropriate for the child tend to take a more problem solving approach to their planning and their instruction.

An unanticipated consequence of the field experience has been that interns become convinced of the value of listening to children's mathematical thinking and try to enact this type of discourse in a whole class setting in a subsequent field experience. They generally do not have the skills to orchestrate such a discussion, and they default to the idea that individualized instruction is the only way to achieve the desired type of discourse.

References

Lovin, L. A. H. (2000). In the eyes of the beholder: Elementary preservice teachers' experiences in a reform-based mathematics methods course. *Dissertation Abstracts International*, *61* (09), 352A.

Mewborn, D. S. (2000). Meaningful integration of mathematics methods instruction and field experience. *Action in Teacher Education, 21* (4), 50-59.

CONFRONTING CONSTRUCTIVIST PEDAGOGY: INTERPRETING TWO FIRST-GRADE TEACHERS' CONCEPTIONS OF INNOVATIVE PEDAGOGY FOR ARITHMETIC

Jo C. Olson
Illinois State University

Jeffrey Barrett
Illinois State University

This case study examines how two first grade teachers in an urban school system negotiated their deep beliefs on teaching and learning mathematics during the first year of a four-year professional development project. It further examines the role of staff development to support change by promoting reflection and by modeling instruction centered on rich tasks. Using the Activity-Reflective Cycle (Simon, 2000) as a model for staff development, we analyzed the teaching practice of Ellen (sixteen year veteran) and Becky (beginning her second year) using accounts of practice (Simon & Tzur, 1999) to hypothesize conceptual advances that would help them shift from focusing on memorized answers to student thinking. Ellen, the more experienced teacher is characterized.

Ellen's perception that students could not create mathematical strategies by themselves or verbalize numerical patterns influenced how she utilized the curriculum (Ball & Cohen, 1996). At the end of the intervention, Ellen expressed concerns about feeling behind in their lessons and that students lost ground in knowing their facts because they were slower. She struggled negotiating her deep belief that understanding evolved from practice, speed, and accuracy with her new experiences in developing understanding. Ellen noticed, "Perhaps they are slower because they are thinking. Figuring out the answer instead of pulling and answer from memory takes longer." She began to see the purpose and methods for eliciting student strategies. We believe Ellen was unable to make the conceptual changes we expected because she herself lacked multiple solution strategies to combine numbers using related number facts. During discussions she ignored alternate strategies suggested by students, continuing to ask leading questions until her strategy was described. Without experience in flexible thinking, she was unable to extend children's mathematical thinking beyond the strategy she knew. Furthermore, she was unable to view the development of mathematical thinking as an interconnected process with overlapping components described in the district plans and felt considerable time constraints to develop understanding to meet mandated quarterly plans.

This study illustrates the importance of assessing not only a teacher's knowledge of pedagogy but also their content knowledge. In this case-study, the limited strategies of combining numbers prevented the teachers from helping children make the conceptual advances expected by the investigators. Further studies are needed to examine whether teachers can gain "pedagogical courage" to examine children's developing strategies for regrouping and place value thorough knowledge of these concepts directly, or whether teachers can learn such courage by gradual changes in their own classroom questioning and explorations.

References

Ball, D. L., & Cohen, D. K. (1996). Reform by the book: What is ? or might be? The role of curriculum materials in teacher learning and instructional reform? *Educational Researcher*, *25* (9), 6-8.

Simon, M. (2000). Research on the development of mathematics teachers: The teacher development experiment. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design mathematics and science education* (pp. 335-360). Mahway, NJ: Erlbaum.

Simon, M., & Tzur, R. (1999). Exploring teacher's perspective from the researcher's perspective. *Journal for Research in Mathematics Education*, *30*, 252-264.

REPRESENTATIONS AS A RESOURCE FOR TEACHER LEARNING

Miriam Sherin
Northwestern University
msherin@northwestern.edu

Andrew Izsák
Northwestern University
izsak@northwestern.edu

This study draws together research on teacher learning and research on external representations (e.g., graphs, symbols, drawings). We suggest that teachers' pedagogical content knowledge with respect to representations is an important facet of teacher cognition that should be studied in greater depth. We examine this issue by analyzing how teachers learn from students' interpretations of representations that are contained in reform-based instructional materials. Such materials often include novel conceptually-based representations intended to support students' learning of mathematics. We provide an initial framework for research in this area.

Recent research has documented the need for teacher learning in the context of reform. Yet, such work has not explored how teachers might develop pedagogical content knowledge through teaching with novel representations. Furthermore, existing empirical research on representations has focused on students', not teachers', understandings (Leinhardt, Zaslavsky, & Stein, 1990). We claim that a close examination of cases in which teachers deepen their understandings of external representations, and of content, from students' interpretations of such representations is critical in light of recent research that finds access to student thinking is key to effective implementation of reform (Franke, Fennema, & Carpenter, 1997).

Data for this study came from one high-school and one elementary-school study that shared four features. (1) The researchers developed instructional units intended to help students develop deep conceptual understanding of core mathematics topics. The high-school materials focused on linear functions and their graphs. The elementary-school materials focused on connections among two-digit multiplication, areas of rectangles, and the distributive property. (2) The instructional materials introduced rich representations as supports for students' thinking. (3) The teachers had taught the same topics before, but in a "traditional" manner. (4) The researchers worked with the teachers implementing the new units and videotaped all of the lessons. We analyzed the videotaped data using fine-grained analyses. We focused on the features of representations to which teachers drew students' attention (e.g., through gesture, verbal reference, and underlining), ways that teachers interpreted students' understanding of these features, and ways that teachers responded to students' understandings by modifying their pedagogical approach.

We outline an initial framework that highlights aspects of elementary- and high-school teacher learning that occurred as teachers taught with new representations. (1) *New representations prompted teachers to use new pedagogical techniques*. Many studies of reform have discussed teachers' tendency to transform new materials into more traditional and familiar lessons (e.g., Cohen, 1990). In contrast, we found that teachers were less likely to transform lessons that used novel representations, which seemed to emphasize for teachers that the goals of the new lesson were different from those of more traditional lessons. Thus new representations helped teachers shift their practice from direct procedural instruction to classroom discussions in which students explained their thinking. (2) *Representations provided teachers access to student thinking*. Discussions in which students offered their interpretations of representations helped teachers become aware of students' alternative approaches to the mathematics and of particular difficulties that students were having with the subject matter. (3) *Representations fostered connections for teachers between more traditional and more reform-oriented approaches to mathematical content*. As teachers responded to students, they connected their familiar approaches to the topics with the approaches presented in the novel curriculum. Making such connections deepened teachers' understandings of the mathematics involved and allowed teachers to construct pedagogical content knowledge for using the representations contained in the reform-based materials.

References

Cohen, D. K. (1990). A revolution in one classroom: The case of Mrs. Oublier. *Educational Evaluation and Policy Analysis, 12,* 311-329.

Franke, M. L., Fennema, E., & Carpenter, T. (1997). Teachers creating change: Examining evolving beliefs and classroom practices. In E. Fennema & B. S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 255-282). Mahwah, NJ: Erlbaum.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, *60* (1), 1-64.

LESSON STUDY AND COLLABORATION IN COLLEGE ALGEBRA

Emily Dixon Simpson
Brigham Young University

The objective of this study was to examine the dialogue and characteristics that occur in a mathematics teacher collaboration before and after the implementation of Lesson Study. Stigler and Hiebert (1999) state that, "the often described isolation of U.S. teachers has greatly hindered our discussions about teaching and hence our ability to improve it" (p.123). Research has shown that collaborating with other teachers is an important step in developing professionally as a teacher. But once together in collaboration, what do the teachers discuss? Stigler and Hiebert (1999) express that the best way to improve teaching is to improve the classroom lesson, yet teachers in the United States "who do collaborate, generally do everything except work on the improvement of classroom lessons" (p. 157). In Japan, teachers participate in a professional development model entitled, Lesson Study. Collaboratively, they determine a goal or objective and write out a lesson plan, carefully studying and planning each concept, example, and student response. In this study we focus on the teacher's perception of the usefulness of the collaboration, and the depth of mathematical discussion before and after the implementation of Lesson Study.

Five College Algebra teachers were assigned to mathematics teacher collaborations according to his/her teaching experience. Three teachers made up the more experience group, and two in the less experienced group. The teachers met in their groups for forty-five minutes each week for a total of eight weeks. The first four weeks, the teachers were instructed to collaborate and to discuss anything regarding their class. After four weeks, an inservice on Lesson Study was given, and teachers were encouraged to implement the idea into their collaborative group and create a lesson plan for a specific topic.

The primary means of gathering data was by audio taping the weekly conversations in each teacher collaboration, a midterm and exit interview administered to each teacher, and the video taping the final lesson being taught to the students.

The results of this study can be given by answering the research questions posed in this study.

1. *How does the introduction of Lesson Study affect the teacher's perception of the usefulness of the collaboration?*

   From the evidence found in the results and analysis, we see that after the introduction of Lesson Study, the teachers' perceptions of collaboration included that the Lesson Study model was a quality model for professional development. Reasons for the perception change were a result of Lesson Study: having a focus or goal, focusing on student perspectives, reflection on teaching practice, and better ideas and lessons. These reasons helped make the Lesson Study experience a more enjoyable and worthwhile professional development model. Teachers felt they were accomplishing something. They were improving their teaching and lesson plans, while focusing on the most important recipient of their practice, the student.

2. *Does Lesson Study increase the depth of mathematical discussion in a mathematics teacher collaboration?*

Before lesson study, the discussion within the mathematics teacher collaborative groups had different focuses than after the implementation of Lesson Study. Teachers would discuss course administration issues such as quizzes, homework, and discipline problems frequently. Relating to the mathematics content and lessons they were teaching, the teachers discussed basic ideas about how they taught certain concepts. These concepts were discussed at surface level, mostly just communicating what they had taught the previous day or where they were at in the textbook. The dialogue didn't delve deeper than that.

After the introduction of Lesson Study, teachers discussed the mathematics content of the lesson at a much deeper level. Teachers were themselves finding connections, understanding formulas better, and ultimately obtaining a conceptual understanding of the lessons they were teaching. These teachers were acquiring knowledge while pursuing a great lesson that would give students the opportunity to discover, create, and build a conceptual understanding of mathematics. The participants in the collaborative groups would grapple for tens of minutes over such things as which numbers to use in a particular problem to prevent any possible student misconceptions and what would be the best problem to pose at the beginning of the class period.

Reference

Stigler, J. W. & Hiebert, J. (1999). *The teaching gap.* New York: Summit.

PRESERVICE TEACHERS' ATTITUDES, BELIEFS AND CONCERNS ABOUT THE TEACHING AND LEARNING OF MATHEMATICS

Cynthia Smith
Lisa Hitchcock
SUNY College at Fredonia

Frances Fuller's research (1974) has challenged teacher educators to address the possibility that they may be *answering* questions that preservice teachers are simply not *asking*. Since preservice teachers are most likely to seek and learn information about what concerns them (Fuller, 1974), recognizing their beliefs and concerns will have significant implications for teacher education. In this study, we attempt to answer the question: *What do the questions that preservice elementary teachers pose reveal about their evolving beliefs and concerns about the use of resources (concrete and technological) in the teaching and learning of mathematics?*

Fuller and his colleagues have done a considerable amount of research on the topic of teachers' concerns. The researchers originally classified concerns into two types: *self-concerns* and *student concerns* (Fuller, 1974). Later, Fuller and Brown introduced a third model of concerns: *task concerns*. Research suggests that teachers progress through relatively predictable stages of concerns during their professional development. Preservice teachers' concerns, for example, are mainly self-concerns. Typical examples of self-concerns among preservice teachers include the desire for children to like them and the need for acceptance among staff members. In this category of self-concerns preservice teachers generally focus on "*Can I do it*?" and "*How do I do it*?" (Fuller, 1974). Beginning teachers, on the other hand, generally have concerns about content knowledge (task concerns) as well as teaching performance (self-concerns). More experienced teachers are concerned with students' understanding and students' learning processes (student concerns). In this study, Fuller's model of self-concerns will be used to analyze the data.

The participants in this study were both male (n=25) and female (n=114) senior-level preservice elementary school teachers at the SUNY College at Fredonia. During the 1998-1999 academic year these participants were enrolled in a required mathematics teaching methodology course taught within the School of Education. The data used in this study are textbook-related student-generated questions. In the first stage of data analysis each question was independently and repeatedly read looking for keywords and prevailing themes. A significant number of the preservice teachers' questions on the use of resources did not fit either the "Can I do it?" or " How do I do it?" categories. As a result of this study's analysis, a third self-concern category was developed- the "Should I do it?" category. With the addition of this third category, data were further coded within each of the three categories of self-concern. Only four questions fell within the "Can I do it?' category. This is not surprising given that the mathematics methods course and the texts fostered the understanding that one *can* use resources. Within the "How do I do it?" category, data (n=65) were further coded to reveal the following concerns: (1) How do I know **when to stop** using resources? (2) How do I use **calculators** in the classroom? (3) How do I know **when and how** to use resources? (4) How do I know the **age appropriateness** of resources? (5) How do I know how much **time** to devote to the use of resources daily? (6) How do I **assess** students' comprehension? (7) How do I approach the use of resources if the **school** is unfamiliar with them or unsupportive of their use? (8) How do I **teach a specific task**?

The remaining data (n=53) were classified within the "should I do it?" category. These data indicate that preservice teachers are concerned and often unsure about what they should and should not do in the classroom. *Should I* concerns focused primarily on (1) **technology** and (2) **age appropriateness**. With respect to technology, study participants questioned whether they should use technology given that, in the opinion of many preservice teachers, technological resources are likely to become *crutches* for students. With respect to age appropriateness, research indicates that many teachers believe that it is inappropriate for students beyond fourth grade to use manipulatives (Tooke, 1992). This belief was echoed in the participants' questions. Once again the participants asked if they should use resources with older students given that these resources are likely to become *crutches*.

Reference

Fuller, F. F., Parsons, J. S., & Watkins, J. E. (1974). *Concerns of teachers: Research and reconceptualization* Paper presented at the Annual Meeting of the American Educational Research Association, Chicago, IL. (ERIC Document Reproduction Service No. ED 091 439)

FOSTERING TEACHER EDUCATION THROUGH RESEARCH ON CHILDREN'S PROBLEM SOLVING

Emily Dann
Rutgers University
dann@rci.rutgers.edu

Objectives

To share the use of video research data in pre- and in-service teacher education, and to introduce the six one-hour videotapes produced by Harvard-Smithsonian Center for Astrophysics in collaboration with the Robert B. Davis Institute for Learning, Rutgers University.

Perspective

Consistently and uniformly, feedback from practicing teachers, pre-service teachers, and graduate students viewing videotapes of children's work in mathematics, and in teachers' examining their own practice suggests the effectiveness of incorporating videos in teacher education. Materials and activities are presented in a set of six 60-minute videotapes that provides a sequence of episodes showing children and/or teachers engaged in doing and talking about mathematics. Each videotape contains episodes from a 12-year research study carried out in a K-12 public school district in NJ within a partnership with the Robert B. Davis Institute for Learning at Rutgers University that began in 1984. The practice of viewing these videotapes has resulted in greater attention to children's thinking on the part of teachers and mathematics educators, making them more aware and more attentive to what children can accomplish. In addition, the remarkable performance of students observed in these episodes has prompted teachers to want to try some of the problems in their own classrooms. All of this is evidenced in teacher feedback. Such opportunity leads to possibilities for discussing and extending teacher's ideas and beliefs about student problem solving, and helps to point out aspects of mathematics often overlooked by schools. In addition, it focuses teachers on techniques that they might use for eliciting problem solving growth in their own students.

Modes of Inquiry

Three graduate-level classes comprising students who are also in-service teachers have been held at Rutgers using the newly produced videotape series, referred to as the Private Universe Project in Mathematics. The professional development workshop series for K-12 teachers can be found online at the web site: www.learner.org/channel/workshops/pupmath. The workshops are being held throughout the country. Preliminary evidence of teacher growth has influenced presenters to want to share the possibility of extending this experience to other teacher education settings.

Results

Teachers document their growth in understanding through a combination of solving problems together, observing children developing similar ideas, and making changes in their own practice.

Goals

The goal of this work is to foster both the disposition in teachers to pay attention to their own mathematical development and that of students.

PRE-SERVICE ELEMENTARY TEACHERS' UNDERSTANDING OF PLACE VALUE

Bradley E. Garner
Boise State University
garner@math.boisestate.edu

Margaret Kinzel
Boise State University
kinzel@math.boisestate.edu

Sharon B. Walen
Boise State University
walen@math.boisestate.edu

Understanding the place-value structure of the base-ten number system is an important goal for elementary school students (NCTM, 2000). It follows that prospective elementary school teachers need to gain a deep understanding of place value. Often counting and arithmetic operations using bases other than base ten are included in the curriculum for pre-service teachers in an effort to enhance their understanding of place value. Yet the study of bases other than base ten regularly poses great difficulties for pre-service teachers and inclusion of various numeration systems in the pre-service curriculum has long been questioned (see for example Rappaport, 1977). This study investigates two questions. First, what level of understanding of place value are pre-service teachers actually gaining? Second, does the study of various bases help pre-service teachers gain the understanding they need?

Like Zazkis and Khoury (1993), we used "non-decimals" to explore and identify pre-service teachers' misunderstandings of computational algorithms. Non-decimals are representations of mixed numbers in bases other than ten, such as 2.314_{five}. Students did not encounter these representations until the interview. We gave seventy-five pre-service teachers a survey about their understanding of place value concepts before they encountered other bases in their course. We administered a second assessment after the non-base-ten material (i.e., conversions and computations) had been studied in class. Subsequently, we selected seven of these students for in-depth videotaped interviews about their understanding of place value and computation in base ten and other bases.

Based on our study of the data to date, we hypothesize that there are two main elements in a complete understanding of place value. We call these elements "power structure" and "dynamic grouping." The essence of power structure is that the various places for digits in a numeral are labeled by powers of the base. For example, in the numeral 234, the "units" or "ten to the zero power" place is occupied by the digit 4, the "ten to the first power" place is occupied by the digit 3, and the "ten to the second power" or "ten squared" place is occupied by the digit 2. This structure can be extended to include as many digits as required by the task, to include digits to the right of a decimal point. This framework becomes the basis for arithmetic computations for students who view numbers in this way. The essence of the "dynamic grouping" element is that a multi-digit number is understood to be one number comprised of different-sized collections. For example, the numeral 234 is the sum of 4 units, 30 (which was obtained by collecting 3 tens), and 200 (which was obtained by collecting ten sets of ten 2 times). The motion of replacing ten objects in one place with one object in the next higher place becomes the basis for understanding computations for students holding this view. This dynamic grouping can also be used to move from a higher place to a lower place, and can be extended to the right of a decimal point.

We present evidence from interviews with three students: one exhibiting primarily a "dynamic grouping" view of numbers and computation, the second exhibiting primarily a "power structure" view, and the third exhibiting a combination of both views. The third student showed the most flexibility and facility in computations, conversions between bases, and explanations of place value. We argue that the study of bases other than base ten can force students to examine their own understanding of place value and help them obtain a connected understanding comprised of both essential elements.

References

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Rappaport, D. (1977). Numeration systems—a white elephant. *School Science and Mathematics, 77* (1), 27-30.

Zazkis, R. & Khoury, H. (1993). Place value and rational numbers representations: Problem solving in the unfamiliar domain of non-decimals. *Focus on Learning Problems in Mathematics, 15* (1), 38-51.

THE CHANGES TEACHERS ENCOUNTER IN APPLYING STATE STANDARDS TO LESSON PLANNING

Catherine Pomar Herzog
Florida State University
cherzog@mail.pc.fsu.edu

Kenneth L. Shaw
Florida State University
kshaw@mail.pc.fsu.edu

Teachers in the state of Florida have used the Sunshine State Standards (SSS) for six years. Understanding of the SSS varies among teachers, as does the approach to the use of the SSS. Many mathematics teachers will sit down with their lesson plans at the end of a week and add whatever standards they feel were covered that week to their lesson plans. However, the Beacon Learning Center (BLC) is trying to change this approach. Created by teachers in 1997 to provide curriculum resources and professional development, the BLC teaches curriculum design that is based on the state standards. In July 2000, the BLC expanded upon receiving a Technology Innovation Challenge Grant from the United States Department of Education. This five-year, ten-million dollar grant has enabled the Beacon Learning Center to offer professional development throughout Florida and to develop free lesson and unit plans that are available from their website. The Florida State University was contracted to evaluate the BLC for this grant. The evaluators became curious about the affect of BLC professional development on teachers' approach to the SSS.

In the BLC lesson and unit planning workshops, teachers are encouraged to think about their lessons with the end in mind. First the standards to be taught are chosen, second the assessment to be used is determined and then the instruction is designed. Upon completion of a curriculum workshop, teachers have an aligned lesson plan that has been validated and can be published to the BLC website. The validation process ensures all lesson plans include appropriate standards, an assessment that assesses the chosen standards, and an easy-to-follow instructional plan.

A positive affect has been realized through qualitative inquiry. Interviews were conducted with current and past classroom teachers. These interviews provided evidence that teachers who attend workshops begin their planning with the SSS and then focus on assessment and instruction. For each teacher interviewed this was a new approach. Teachers became more analytical of their textbooks and chose classroom activities that led students to understanding as prescribed by the standards. One mathematics teacher explained how her school year wrapped up with less stress due to this new approach. She had a "big picture" view of where the class needed to go and was able to get there with fewer hours of grading and planning.

The professional development offered by the BLC has strengthened the understanding and use of the SSS. Teachers now have accountability and confidence in the use of the standards rather than utilizing them as afterthoughts to their classroom instruction.

THE USE OF VIDEO TO FACILITATE TEACHER REFLECTION

Debbie Junk
University of Texas at Austin
junkdeb@mail.utexas.edu

Erin Turner
University of Texas at Austin
eturner@mail.utexas.edu

Susan Empson
University of Texas at Austin
empson@mail.utexas.edu

Brief Description

This study explores the reflective processes of teachers as they view themselves interacting with small groups of students on video. We sought to identify the ways video technology, which easily captures teaching moments, can help teachers to reflect upon their interactions with students and use what they learn by viewing and discussing these interactions as resources for their own teaching and learning. We will report on the criteria we used to choose clips from video of teachers' own classrooms, and how well those clips functioned in helping teachers reflect.

The ten third grade teachers involved in this study were in their first year of using the *Investigations in Number Data and Space curriculum*. The teachers were videotaped as they each taught their class the same lesson from the third grade unit *Fair Shares*, and then individually interviewed shortly after the lesson by a researcher. Researchers selected video clips of teacher / student interactions that included interesting, surprising, and sometimes perplexing student thinking to show to the teachers during the interviews. The teachers' reflective comments during the interview provided insights into what teachers actually notice and make sense of during interactions with their students.

In addition, after analyzing the videotaped lessons, we identified aspects of teachers' interactions with students, such as non-routine student strategies or unexpected student questions, that have the potential to serve as resources for teaching (Empson, 2000). Researchers acknowledge that the repertoire of resources a teacher has available and employs contributes to more effective teaching. When teachers are able to recognize and make use of the resources that emerge during these teacher to student interactions, student learning is facilitated. What resources teachers choose to make public, bypass or perhaps misinterpret greatly influences their students' opportunities to learn with understanding.

Results indicate that video clips did function to help teachers reflect upon their teaching. However, the reasoning behind researchers' selection of clips, and the potential resources that they identified within those clips, did not always coincide with the teacher's perceptions of the interaction. Based on several interviews, we learned that researchers may need to help teachers focus on particular aspects of the interaction, such as the questions they posed to students, and even replay those interactions to provide teachers with multiple opportunities to reflect. To make full use of this powerful video technology, we must further investigate the context in which these video clips are viewed and discussed.

A COMMON AGENDA FOR CONTENT AND METHODS INSTRUCTION

Gavrielle Levine
Elliott Bird
C.W. Post Campus, Long Island University

This interdepartmental project addresses the needs of preservice elementary school teachers by reducing the artificial gap between mathematics content and methods courses. In most situations, undergraduate elementary school education majors enroll in both mathematics content courses offered by the mathematics department and mathematics methods courses offered by the school of education as part of their teacher preparation program. This dichotomy creates a sharp schism between such courses, which can lead to an artificial distinction in the student's mind between content issues and instructional issues.

Content and the way it is taught are irrevocably interconnected (NCTM, 2000). It is not possible to teach content without an underlying instructional strategy, and, similarly, it is not possible to introduce mathematics instructional strategies devoid of mathematics content. Nevertheless, without thoughtful attention to both dimensions, content and teaching strategies are presented separately. Collaboration between faculty teaching mathematics content and mathematics instructional strategies may improve this situation.

Current research suggests that understanding does not happen in an instant (Glasersfeld, 1995). The learner individually and alone constructs knowledge. When learning occurs in an environment that includes a teacher, the teacher can both monitor and facilitate knowledge construction. Self-reflection becomes a key aspect of learning and teaching and is therefore an appropriate ingredient in both content and methods courses.

In this project, we consider elementary education majors' difficulty with multiplication and division of fractions. Most students solve problems involving rational numbers in a rote computational manner with little attention to understanding the meaning of the relationship or the operations. As a familiar example, when calculating one-third of one-half, students automatically arrive at an answer of one-sixth by multiplying numerators and denominators. However, given a model of one-third, and asked to find half of it, students are often unable to solve the problem and usually make no connection between this experience and the rote computation they have just completed.

To address preservice teacher's limitations in this area, we developed activities and teaching strategies that emphasize meaning rather than mechanics. The content class emphasizes conceptual relationships while modeling appropriate teaching practice. For students who have previously been taught mathematics in a rote manner, this provides an important alternative instructional model. In order to teach in a constructivist mode, it is helpful to have been taught using constructivist strategies. The methods class emphasizes instructional strategies for teaching operations with fractions while revisiting the content. By interconnecting content and methods during these two courses, students have an opportunity to appreciate the role that the solution process plays in computation and to become self-reflective in both their own problem solving and their teaching. These curricular innovations will be described as well as changes in students' understanding of the mathematics and the pedagogy.

For preservice elementary school teachers, the interrelated experiences of learning content and instructional methods in a coordinated fashion can contribute to their ability to teach mathematics effectively.

References

Glasersfeld, E. von (1995). *Radical constructivism: A way of knowing and learning.* London/Washington, DC: Falmer Press.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

IMPLEMENTING STANDARDS-BASED INSTRUCTION IN A MATHEMATICS SEQUENCE FOR PRESERVICE ELEMENTARY TEACHERS

Cheryl Roddick
San Jose State University
roddick@mathcs.sjsu.edu

Julie Sliva
San Jose State University
sliva@mathcs.sjsu.edu

Research about student attitudes toward learning mathematics, revisions in *Principles and Standards for School Mathematics* (NCTM, 2000), Ma's research about "profound understanding of fundamental mathematics" (PUFM) (Ma, 1999), and other research about common teacher understandings and misunderstandings (Ball, 1990; Behr et al., 1992) were all impetus for a revision of a mathematics course sequence for preservice elementary teachers. This study was a curriculum evaluation and revision designed to address the above research and national guidelines for successful teaching and learning of mathematics. This study was piloted on 106 preservice elementary teachers in one of the courses in the sequence. The final subjects in this study include all preservice elementary teachers enrolled in the mathematics course sequence.

Several components were included in this comprehensive evaluation and revision. These include changes in curricula in addition to affective and cognitive assessments. Curricular changes include the following: 1) inclusion of activities designed to encourage deeper understanding of key concepts; 2) implementation of a comprehensive technology plan; and 3) incorporation of journal entries related to key understandings of each course. Affective issues were assessed through mathematical autobiographies and an attitudes and belief questionnaire. Content assessments were designed to address the fundamental concepts in each course and were implemented in several different ways: open-ended questions given at the beginning and end of each course, common exam questions, journal entries, concept maps, and interviews.

This poster presentation will include materials developed for the course sequence as well as preliminary results of assessments. A discussion of this research project will address the following: Can attitudes and beliefs toward learning mathematics be changed over an elementary course sequence?; Does technology have an impact on the learning of mathematics? Does writing impact teachers' understanding of mathematics? and, What are preservice elementary teachers understandings of basic operations and their relationship to each other?

References

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

TRACING A PRESERVICE TEACHER'S GROWTH OF UNDERSTANDING OF RATIO, PROPORTION AND RATE OF CHANGE

Katrina N. Staley
North Carolina State University
kstaley@unity.ncsu.edu

Clarence E. Davis
North Carolina State University
ce_davis@ncsu.edu

The objective of this study was to examine the ways in which undergraduates who are preparing to teach high school mathematics grow in their understanding of the high school content. The conceptual framework was the Pirie and Kieren's model of mathematical understanding. "[T]he growth of mathematical understanding is a whole, dynamic, leveled but non-linear, transcendently recursive process. This theory attempts to elaborate in detail the constructive definition process of organizing one's knowledge structures" (Kieren & Pirie, 1994, p. 16). Descriptions that are relevant from this model include primitive knowledge, image making, image having and folding back. We interviewed a pre-service teacher who was asked to plan a lesson to introduce the concept of rate of change to an Algebra I class and relate that concept to ratio and proportion. The lesson plan was viewed as acting in context while content competencies were needed to complete the task. The sources of data were a transcriptions of two videotaped interviews, student notes of the planned lesson, and descriptive artifacts. Berenson, Cavey, Clark, and Staley (in press) stated that teacher preparation needs to include instructional teaching strategies that are associated with the content of high school mathematics. We also made the conjecture that after the lesson-planning task, Karen appeared to have folded back to make new images of ratio, proportion and rate of change. For example, in the pre-lesson interview, Karen discussed her images of ratio and proportion in a procedural manner.

> K: *When I think of a ratio I think of two to three and 2:3 and sometimes you can write a fraction 2/3*
>
> K: *I think a lot of things deal with proportion. You know ... things being equal to each other.*

After the lesson-planning task, Karen's images of ratio and proportion have changed .

> K: *So, it's* [ratio] *a comparison of numbers or measures.*
>
> K: *I might start* [my lesson] *with fractions since all fractions are ratios.*
>
> K: *I would probably say that proportion is a statement of equality between two ratios. ... When I think of word problems I always think of proportions. Like if you have 4 pieces of candy for $1, how much are 12 pieces? Then*

> *I would go into cross multiplication ... this would be something they can figure out*

The lesson-planning task appeared to have contributed to the pre-service teacher's growth of mathematical understanding of ratio and proportion encouraging her to fold back to revise her images. These results from the lesson planning task support Tower's (2001) conjecture that students' understanding is partly determined by teacher interventions.

References

Berenson, S., Cavey, L., Clark, M., & Staley, K. (In press). Adapting Pirire and Kieren's model of mathematical understanding to teacher preparation. *Proceedings of the Twenty-Fifth Annual Meeting of the International Group for the Psychology of Mathematics Education.* Utrecht, NL: Fruedenthal Institute.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics*, *26* (2-3), 165-190.

Towers, J. (2001, April). *Teachers' interventions and the growth of students' mathematical understanding*. Paper presented at the meeting of the American Association of Education Research. Seattle, WA.

EFFICACY OF PRESERVICE ELEMENTARY MATHEMATICS TEACHERS

Robert G. Wenta
DePauw University
rwenta@depauw.edu

The purpose of this study was to investigate the effects of an elementary mathematics methods course (E343) and its associated field experience (M201) on the Teacher Efficacy (TE) of preservice teachers (PSTs). Teacher efficacy is defined as the belief a teacher has that he/she can enhance a student's learning. Resarch has shown TE to be composed of two independent components, personal teaching efficacy (PTE), defined as a teacher's beliefs in his/her ability to enhance a student's learning, and general teaching efficacy (GTE), a teacher's beliefs that teachers in general are able to enhance a student's learning despite factors beyond his/her control. The TE score was the sum of the PTE and GTE scores.

Interest in teacher efficacy has been on the rise in recent decades, at least in part because traits of teachers found to be high in TE correspond to traits of teachers recommended by the NCTM Standards. In-depth study of TE is seen as a means of improving the achievement level of our students, as well as improving the sense of professional fulfillment of classroom instructors.

This study is based on previous research in teacher efficacy, teacher change in mathematics and related reform literature, and levels of reform strategy use in the mathematics classroom. I used a collective case study methodology, using a TE Survey to provide a quantitative approach to determining consistencies and differences between PSTs, and using interviews and observations to provide more details on these issues.

All 93 students of four E343 classes were asked to complete a TE Survey, from which PTE, GTE, and combined TE scores were calculated. Due to schedule limitations, selection of participants in more in-depth investigation was restricted to three of these classes. The students of these three classes were asked to write a personal reflection on their prior educational experiences with mathematics. PSTs who scored in the lower half of those taking the TE Survey on the PTE, GTE or TE aspects of the survey, and who also admitted in their reflection paper as having had an efficacy inhibiting experience in their past were considered as candidates for the study. While twenty students from the three E343 classes met the criteria, eleven agreed to participate in the study.

As part of the M201 field experience, PSTs taught four mathematics lessons to a group of 2-6 area elementary students during the students' math class. The eleven PSTs who agreed to participate further were observed by the researcher during the teaching sessions to determine the extent that PSTs were using "reform" strategies in their lessons. The eleven subjects also took part in interviews with the researcher before, during and after the set of teaching experiences to determine the extent the PSTs saw themselves using reform strategies, and the extent that efficacy inhibitions noted in their reflections were being addressed. The students of all four E343 classes were asked to retake the TE Survey after the four teaching sessions were completed. Eighty-seven PSTs took both the pre-test and post-test TE Survey.

Results indicate that PST participation in the math methods course and field experience corresponded to a significant increase in PTE and TE. PSTs who claimed and were observed to use reform strategies routinely in their lessons showed significant increase in PTE, GTE and TE. PSTs who claimed, but were not observed to use reform strategies in their lessons saw an increase in PTE, but no change or a significant decrease in GTE. PSTs who were not negatively influenced during their teaching lessons by their past math-related fears showed an increase in PTE. Those who were not able to get past expressed inhibitions to learning math (acquired early in their education) showed a decrease in PTE and TE.

Results suggest that TE enhancement is associated with PSTs who are applying themselves toward NCTM-supported efforts, and are moving away from some negative influences of their previous math teaching. PST self-perception of reform strategy use positively influenced their PTE, or personal self-confidence, but actual observable implementation of the reform strategies corresponded to enhanced GTE, or PST belief in teachers' abilities as a whole. The poster session includes further discussion on this topic.

PSTs should be made aware of research that demonstrates TE as a legitimate issue and could be associated with the achievement level of their students, as well as with the level of professional fulfillment in their upcoming profession. Math educators should provide opportunity for PSTs to express their memories that have influenced their perspectives of math. In recognizing the source of any fears, PSTs saw the cause as inconsistent with reform-based approaches now being used, and moved away from being influenced by the negative memory. Other findings regarding specific fears consistently expressed by the PSTs, and benefits realized by sharing with their peers and hearing of successes of their peers in using the reform-based teaching will also be included in the poster session.

This study of preservice teacher efficacy is consistent with the goals of PME-NA as it has required investigations into the disciplines of math education and psychology and contributes toward furthering understanding of the psychological aspects of teaching and learning mathematics.

MEETING THE CHALLENGE OF NEW MATHEMATICS STANDARDS: EXPLORING TEACHERS' UNDERSTANDING OF GEOMETRY, STATISTICS, AND PROBABILITY

Joseph T. Zilliox
University of Hawai'i
zilliox@hawaii.edu

Neil A. Pateman
University of Hawai'i
pateman@hawaii.edu

There is a fundamental issue that remains unresolved in our minds, and that is whether teachers can be effective in helping their students acquire knowledge of mathematical content when that same content is at best unfamiliar to teachers themselves. Some evidence exists to suggest that students learn more from teachers who are well prepared in the mathematics content that they are teaching (Swafford et al., 1997). We believe that this issue is relevant at all levels of mathematics teaching in schools.

Tirosh (2000) defines two essential knowledge types for prospective elementary teachers: *subject-matter knowledge* (that is, the teacher's own knowledge of mathematics), and, following Shulman (1986), *pedagogical content knowledge* (that is, the teachers' knowledge of how their students learn mathematics). We find this a useful framework and we think that it in fact could be extended and be applied to middle level and high school teachers of mathematics.

In an earlier report (Zilliox & Pateman, 2000) we presented the results of an analysis of one aspect of a workshop series with elementary teachers and a small group of middle school teachers. The workshop series was funded by an Eisenhower grant and was designed in collaboration with state mathematics resource staff from the Hawai'i Department of Education. The collaboration arose as part of a state-wide move to implement a new set of standards in mathematics. As reported in the earlier paper, teachers were expected to develop pedagogical content knowledge through discussions with other teachers of student work and teacher analysis of that work. The results of this aspect of the workshop series indicated that teachers are able to develop potentially useful pedagogical knowledge about their students through careful collaborative analysis of student work. (See the fuller version of Zilliox & Pateman (2000) for more details). However we have little information about whether teachers are able to effectively make subsequent use this knowledge.

A second aspect of the workshop series was to develop content knowledge about geometry, statistics and probability. It is this second aspect that is of concern for this report. We ask two questions related to the development of subject matter knowledge: (1) *How do teachers, especially at the elementary and middle levels, react to new state-adopted mathematics standards that require those teachers to teach unfamiliar* *content, in particular, geometry, statistics and probability?* and (2) *How do teachers respond to activities designed to introduce them to this content?*

We will answer these questions by describing the activities used and by presenting the results of an analysis of the written responses from the 177 participating teachers. A guiding element for our analysis will be Tirosh's (2000) outline of three dimensions of subject matter knowledge, "algorithmic, intuitive, and formal" (p.6). We expect to use these dimensions to categorize and further describe the responses of the teachers. We will conclude with a summary of suggestions for future work with teachers in relation to their developing stronger content knowledge of the mathematics that they are expected to teach.

References

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15 (2)*, 4-14.

Swafford, J. O., Jones, G. A., & Thornton, C. A. (1997). Increased knowledge in geometry and instructional practice. *Journal for Research in Mathematics Education, 28* (4), 467-483.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31* (1), 5-25.

Zilliox, J. T., & Pateman, N. A. (2000). Implementing state mathematics standards in Hawai'i: A professional development model. In M. L. Fernandez, (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, p. 628). [A fuller version of this Oral Report is available from pateman@hawaii.edu.] Columbus, OH: ERIC Clearinghouse for Science, Mathematics & Environmental Education.

Technology

STUDENTS MATHEMATICAL EXPLORATIONS VIA THE USE OF TECHNOLOGY

Luis Moreno-Armella & Manuel Santos-Trigo
Matematica Educativa, Cinvestav, Mexico

Abstract: The impact of algebraic calculator in the tasks and questions that students examine during their solution processes at school is studied. We try to answer questions such as: How can we design learning activities in which the use of technology help or enhance the study of mathematics? What is the role of teachers/instructors in an enhanced technology class? What features of mathematical proof are privileged via the use of technology? We report examples of learning activities that were used in a problem-solving course with grade 11-12 students during one semester. Specifically, we are interested in characterizing the type of tasks used during the course and documenting what aspects of mathematical thinking emerged from implementing them during the development of the course.

Introduction

Recent mathematics curriculum reforms have pointed out the relevance of using technology in the learning of mathematics. Indeed, the use of graphic calculator or particular software such as dynamic geometry has produced changes not only in the type of tasks and questions that students examine during their solution processes; but also in the role played by both teachers and students throughout the development of the class. The National Council of Teachers of Mathematics (2000) identifies the use of technology as one of the key organizer principles of Pre-K-12 curriculum. How can we design learning activities in which the use of technology help or enhance the study of mathematics? What is the role of teachers/instructors in an enhanced technology class? To what extent mathematical arguments or ways to approach problems vary from traditional approaches (paper and pencil). These are fundamental questions that are used to frame and discuss four different approaches to the use of technology in the learning of mathematics. In particular, we document the use of the TI-92 Plus calculator and Dynamic Geometry software as a means to work on non-routine problems, to make and explore conjectures, to determine general patterns of functions. It is shown that the use of this type of calculator helps to visualize problems from various representations that include the use of graphs and algebraic forms. These representations become important for students to identify and examine diverse mathematical qualities attached to the solution process.

Gaining Access to Basic Mathematical Resources

Several non-routine problems that require powerful mathematical resources for their solution with traditional approaches can become accessible to a variety of students. We have used the algebraic calculator TI-92 as a means to examine particular or simpler cases associated with various problems. This process leads students to gain access to basic resources that help them to approach the problems under study.

It is recognized that the use of calculators and computers is important to promote a mathematical way of thinking that is consistent with the practice of the discipline. These tools, for instance, furnish visual images of mathematics ideas and facilitate organizing and analyzing data. One also learns that when "technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving" (NCTM, 2000, p.24).

When students are encouraged to make use of technology in their learning experiences, it becomes important to document what type of methods and strategies appear as fundamental in their problem solving approaches. In particular, some questions that need to be discussed in terms of what students demonstrate in their learning experiences include:

1. What type of mathematical thinking can be enhanced or constructed via the use of technology while learning the discipline?
2. To what extent is students' thinking compatible or consistent with approaches based on paper and pencil?
3. What features of mathematical proof are privileged via the use of technology?

In this paper, we report examples of learning activities that were used in a problem-solving course with grade 11-12 students during one semester. Specifically, we are interested in characterizing the type of tasks used during the course and documenting what aspects of mathematical thinking emerged from implementing them during the development of the course.

We designed and organized a series of tasks around mathematical activities that involved

1. Generalization and formalization of patterns,
2. Representation and examination of mathematical situations through the use of algebraic symbols and graphs.
3. Formal reasoning within a computational environment.

To illustrate these ideas we first select a problem suggested by Polya (1945) and later used by Schoenfeld (1985) in his problem-solving course. Although the essence of the solution process rests on the use of a particular strategy (modifying the original conditions of the problem and examining particular cases), it is clear that the use of technology offers advantages to visualize and evaluate cases that eventually lead to the solution.

> *Inscribe a square in a given triangle. Two vertices of the square should be on the base of the triangle, the other vertices of the square on the two other sides of the triangle; one on each.*

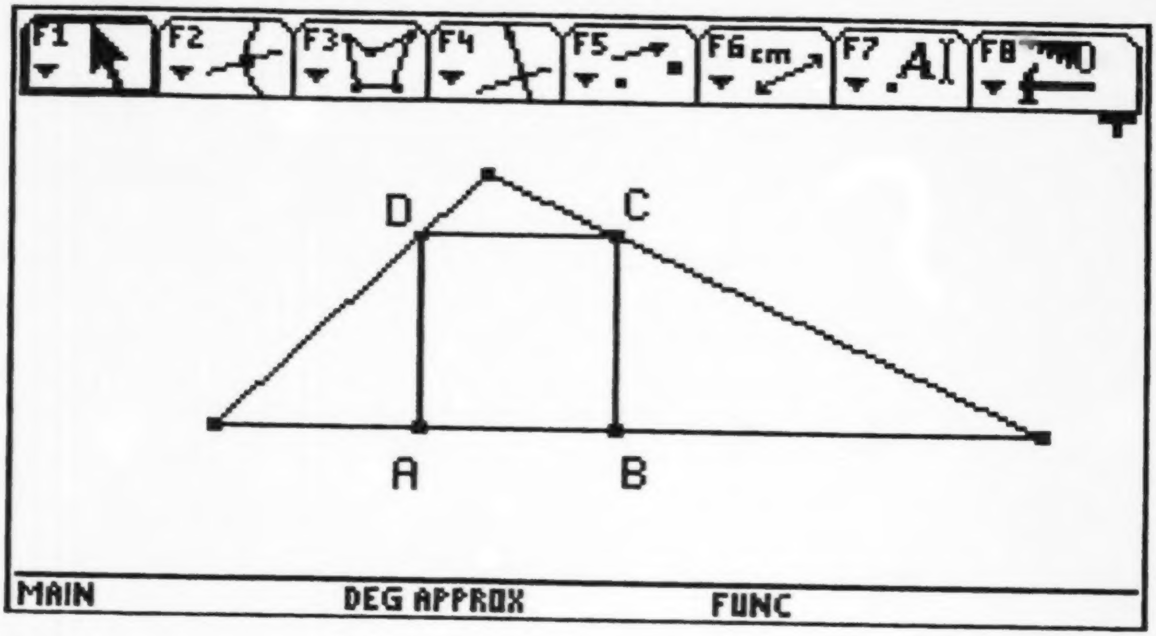


Figure 1.

The students' initial approach was framed through an open discussion of the following questions: What is the given information? What does it mean to have a triangle? Does it mean that it is possible to know its vertices, or the length of its sides and the measure of its angles? What is the question or task? What are the conditions? Is it possible to quantify properties (areas, perimeters, slopes) of the shown figures and observe a particular relationship?

Eventually, students suggested that an important problem solving strategy that can be useful to consider when the problem involves various conditions is to reduce them and explore their behavior through the analysis of particular cases (Polya, 1945). For instance, in this problem a key condition is that the four vertices of the square should be on the perimeter of the triangle. Hence, we can think of drawing a set of squares with only three vertices on the perimeter as in Figure 2.

The construction is done in such a way that when we move the point A on the side of the triangle, the other vertices also move drawing a new square (See Figure 3.).

After a while, students begin to observe that the vertex C moves in what seems to be a straight line. Using the trace capability of Cabri, they are able to produce this figure (obtained by dragging A to its right) (See Figure 4.). This visual evidence suggests that the point of intersection of the moving vertex C with the opposite side of the triangle gives the solution we are looking for.

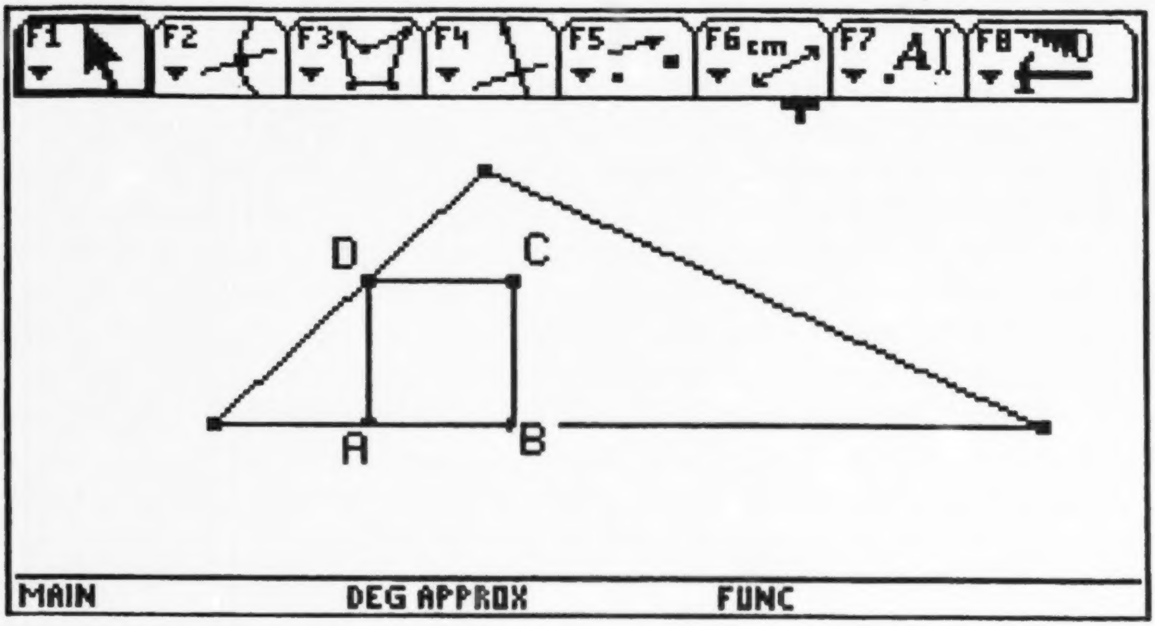

F1
F2
F3
F4
F5
F6 cm
F7 A
F8
D
C
A
B
MAIN
DEG APPROX
FUNC

Figure 2.

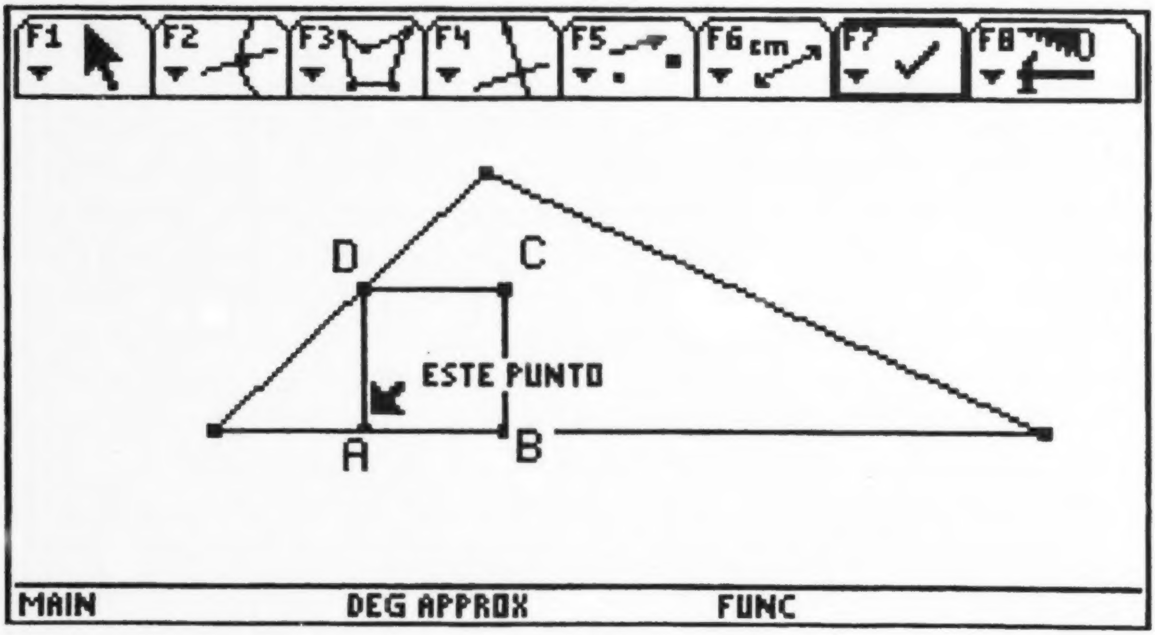

F1
F2
F3
F4
F5
F6 cm
F7 ✓
F8
D
C
ESTE PUNTO
A
B
MAIN
DEG APPROX
FUNC

Figure 3.

Basic Mathematical Resources to Explore Non-Routine Problems

Problems that require powerful mathematical resources for their solution with traditional approaches can become accessible to a variety of students while using of the TI-92 as a means to examine particular or simpler cases associated with these problems.

To illustrate this idea we have chosen (from the tasks posed to the students) a problem related to harmonic sums. Although the essence of the solution process rests on the use of a particular strategy (examining particular cases, for instance), it is clear that the use of technology offers advantages to visualize and evaluate cases that eventually lead to the solution.

Compare the harmonic sums Σ(1/k, k,1,n) with the function log (n+1).

This task was used with advanced students and also with high school teachers. The students' (both groups: teachers and advanced students) initial approach was framed through an open discussion of the following questions:

1. What is the given information?
2. What is the question or task?
3. What are the conditions?
4. *Is it possible to quantify properties in a graphical representation of the problem and observe a particular relationship?*

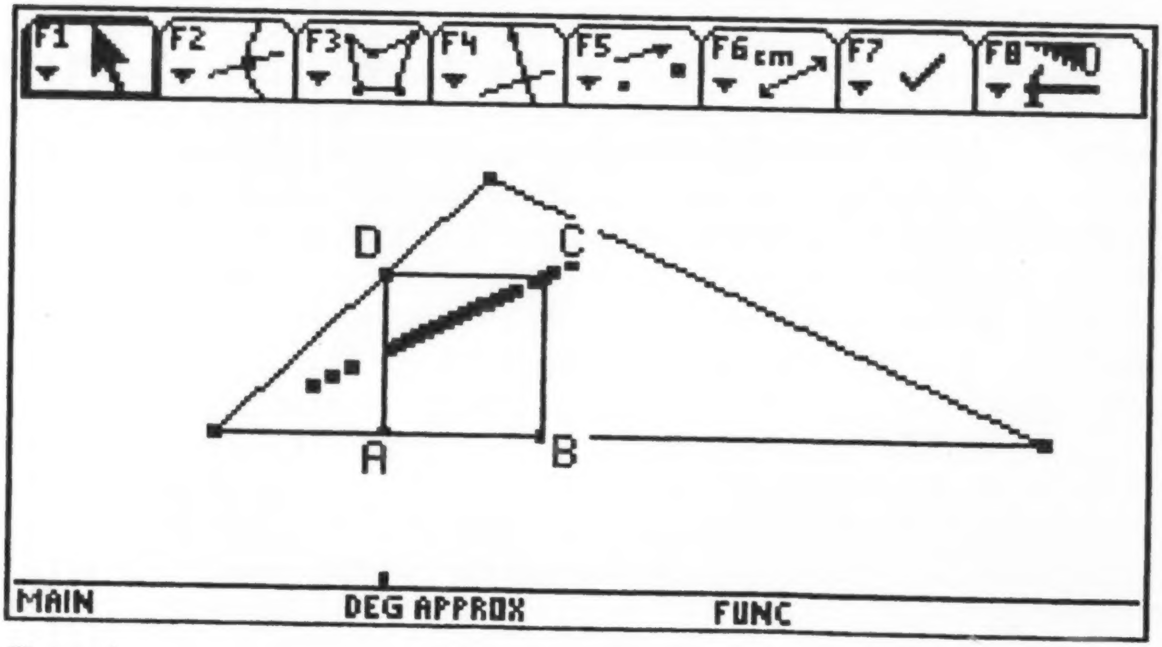


Figure 4.

The instructor's intention while posing this problem was that the students used the graphical representations of sequences. This is a technique with which they were familiar from former tasks.

After some preliminary explorations, they produced the following graph (See Figure 5.):

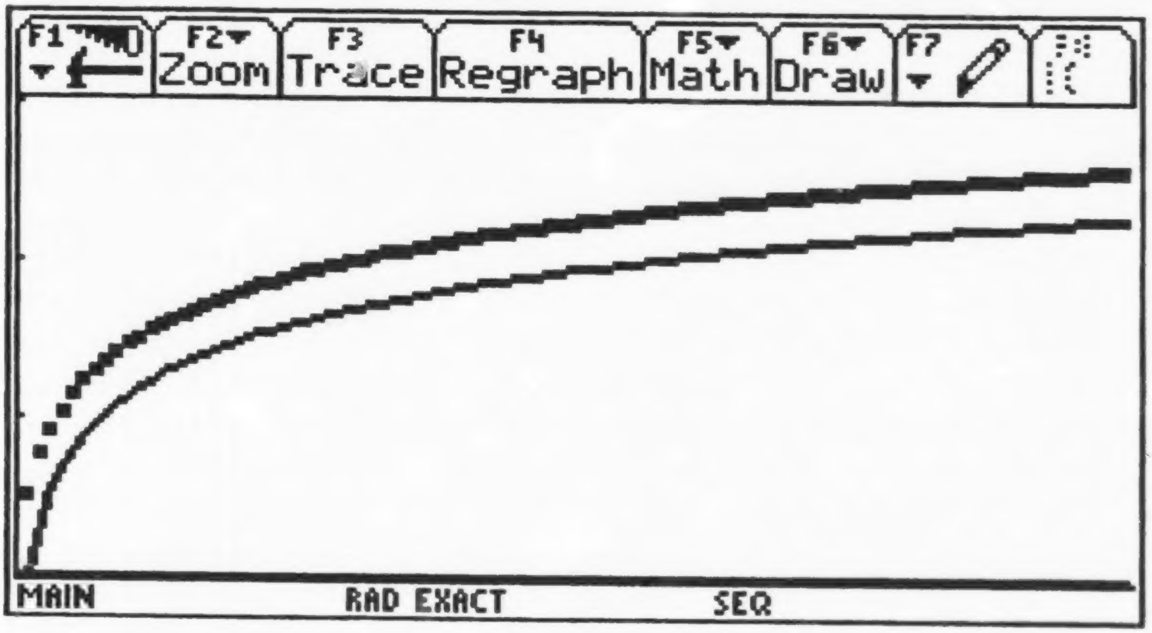


Figure 5.

The thick (lower) one is the logarithm function; the other represents the accumulated harmonic sums. Guided by former questions students were able to arrive at the following conjecture: **The graphs are parallel.**

Before going further on, let us mention that this kind of statement can be called a *theorem in action* after Vergnaud. A theorem in action is a piece of knowledge a person can use with a particular goal in mind but she/he cannot produce a formalized version of the said knowledge. For instance, what does it mean for these graphs to be parallel? Students could not produce a formal statement that captured their idea about parallel graphs, but they were able to translate that condition into an arithmetical one. We must say this was not a simple task, but the effort to make it happen is worth the result obtained. These are precious moments when the teaching make it possible for the students to work from their own version of a piece of knowledge. This situation opens a window of opportunity to conceptual learning. Without conceptual understanding sooner or later skills will be forgotten.

Lets make some notes about the development of this activity. To say that the graphs were parallel was translated into the condition: Σ (1/k, k,1,n) – log(n+1) is more or less constant. This can be easily verified using the calculator. From this point on, the instructor used the result to give an estimate of the number of terms one needs to add (in the harmonic sum) to reach a pre-established natural number.

Formal Reasoning Within a Computational Environment

The use of technology offers great potential for students to search for invariants and to propose corresponding conjectures. We have illustrated how students build dynamic environments to represent problems that eventually lead them to propose conjectures and *prove* them using the tools of the environment. Thus, the software becomes a tool for students to look for and document the behavior of objects and relationships and explore their structural nature. Our last example has to see with *features of mathematical proof* privileged via the use of technology. The mathematical discussion involving this example was much more subtle and students (teachers included) had difficulties when trying to understand it. But it was to introduce mathematical proofs within a computational environment. In a sense we can interpret this part of our work as a *teaching* experiment. At first we discussed the notion of macro construction within the Cabri dynamic environment. After a while, it became clear for the students that a geometric object built usinga macro was a genuine geometric object living in the Cabri universe. This way we could answer the question: To what extent mathematical arguments or ways to approach problems within a Cabriworld vary from the traditional approaches with paper and pencil?

We all know how controversial can be to discuss the place of computers and calculators in the field of mathematical proofs. And how important it is for students to understand what a proof is.

Can we prove a geometry theorem using Cabri? We want to illustrate how this can be made feasible. Let us study *Napoleon theorem*. This theorem says:

> *Given an arbitrary triangle, construct on each side the corresponding outer equilateral triangle. Then the triangle that results by joining the centroids of these three triangles is always an equilateral triangle.*

In Figure 6, the thick triangle is the Napoleon triangle corresponding to triangle ABC. Let us recall how we proceed with this construction. We built two macros:

1. Given two vertices, the first macro determines the third one so that we have an equilateral triangle.
2. The other macro produces the centroid of a given triangle.
   Then, after playing with the construction trying to "destroy" the equilateral triangle (Napoleon's triangle) one "has to accept" the validity of the proposition. This is a natural approach if we are exploring a (possible) theorem within a

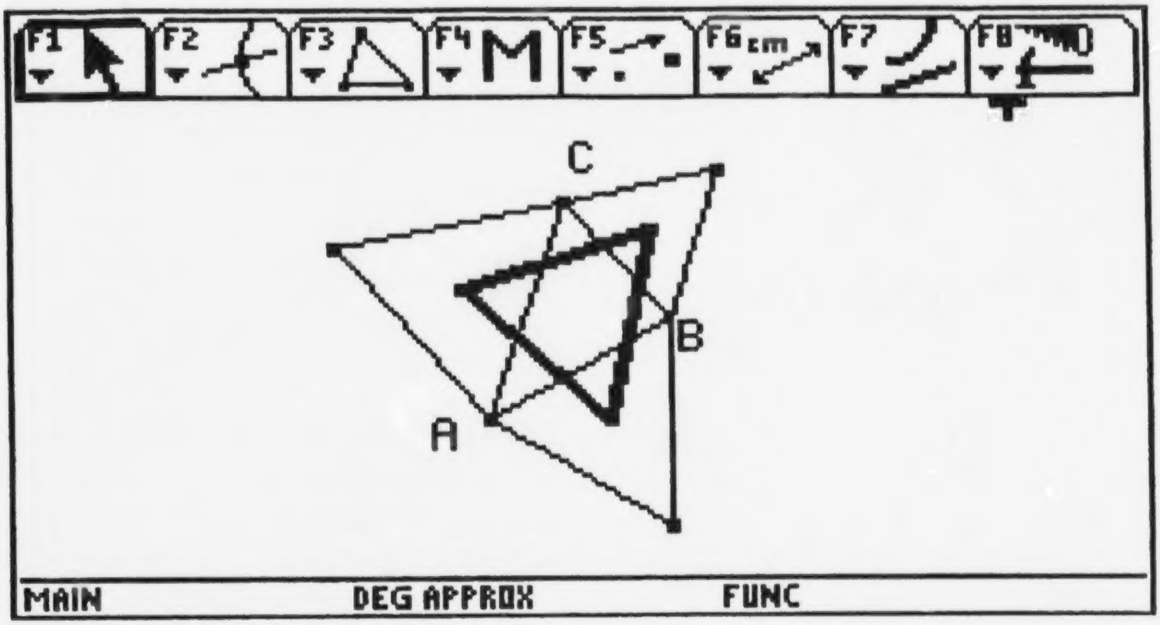


Figure 6.

dynamic environment. That is, we try to make sure that the claim made is an invariant with respect to dragging. But we can go farther than that: *we can give a proof within the Cabri world.*

In fact, we design a macro that enables us to construct equilateral triangles and each time we use it, the result is a genuine equilateral triangle. Likewise, as we have a macro that determines the centroid of a triangle, each time we use it, the result is the genuine centroid of the given triangle. Taking this into account, we realize that when we point out a vertex of the Napoleon triangle we can read the question: "what object?" We can answer "Napoleon" or "centroid" and that means that the vertices of the Napoleon always coincide with the centroids. We know then, that the Napoleon triangle is always equilateral. It is important to remark that this kind of reasoning takes us beyond the perceptual level: this is precisely the case when we intend, for instance, with paper and pencil, to prove a geometrical assertion. Working within a computational environment forces us to adopt a different strategy: we have to resort to the nature of the mediating tools we have at our disposal. Of course, we cannot loose sight of the internal mathematical universe residing in the innards of the calculator.

Final Remarks

Explorations within a computational environment eventually allow students to generate and articulate relationships that are general in the environment in which they are working. Those relationships which encapsulate general statements have been called *situated abstractions*, precisely because they are bound into the medium in which they are expressed (Noss & Hoyles, 1996). What we have introduced in the last section is a kind of proof we could call *situated proof.* In a sense every proof is situated but emphasizing the situatedness while working within a computational environment pays an extra bonus. In our study, whose goal was to explore how students "proved" a mathematical proposition within a computationsl environment, we worked with 17-18 years olds, trained in dynamic geometry–as it comes in the calculator TI-92. For the development of the activities, teams of two or three students were formed. In this, as in other related cases, students became aware of invariants and they could express the relevant ideas *but only within the expressive medium* made feasible by the calculator.

Acknowledgement. The authors want to thank Texas Instruments for all their support during the writing of this paper.

References

National Council of Teachers of Mathematics, (2000). *Principles and standards for school mathematics*. Reston VA: NCTM.

Nesher, P., & Kilpatrik, J. (Eds). 1990). *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education*. Cambridge: Cambridge University Press.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings*. Dordrecht: Kluwer.

Polya, G. (1945). *How to solve it.* Princeton: Princeton University Press.

Schoenfeld, A. (1985). *Mathematical problem solving*. New York: Academic Press.

THE EFFECTS OF MULTIPLE LINKED REPRESENTATIONS ON STUDENT LEARNING IN MATHEMATICS

S. Asli Özgün-Koca
The Ohio State University

The utilization of technology in multiple representations has become one of the significant topics in mathematics education in the last decade. Here, multiple representations are defined as external mathematical embodiments of ideas and concepts to provide the same information in more than one form.

One example of this type of environment is educational software with linked multiple representations. Linked multiple representations are a group of representations in which, upon altering a given representation, every other representation is automatically updated to reflect the same change (Rich, 1995/1996). We define semi-linked representations as those for which the corresponding update of changes within the representations are available only upon request but are not automatic. It is the premise of this study that semi-linked representations are as effective as linked representations and that there is a role for each in different situations, at different levels, and with different mathematical concepts. The focus of this study is comparing three groups of students: one group using linked representation software, the second group using similar software but with semi-linked representations, and the control group. Briefly, the research questions of this study were:

1. What are the effects on students' understanding of linear relationships using linked representation software compared to using semi-linked representation software?
2. What are students' attitudes towards and preferences for mathematical representations—equations, tables, or graphs?

Theoretical Framework

Although there are a number of theories emphasizing multiple representations in the history of mathematics education, with Dienes' "multiple embodiment principle" this issue gained a significant prominence. The multi-embodiment principle suggests that conceptual learning of students is enhanced when students are exposed to a concept through a variety of embodiments (Dienes, 1960).

Constructivism suggests that students construct their knowledge by themselves actively in *their experiential world.* Through communication and interaction with other people, learners test how like (consistent) their constructs are with others' (Confrey, 1990; Goldin, 1990). Because of differences in experience, we cannot expect that everyone will understand a concept the same way from one representation or that one representation will be equally meaningful for everyone.

Integrating theoretical components from a number of mathematics educators, the theory of understanding relative to multiple representations is as follows:

- Students should be able to identify a given mathematical idea across different representations;
- Students should be able to manipulate the idea within a variety of representations;
- Students should be able to translate the idea from one representation to another;
- Students should be able to construct connections between internal representations;
- Students should be able to decide the appropriate representation to use in a mathematics problem;
- Students should be able to identify the strengths, weaknesses, differences, and similarities of various representations of a concept. (Dufour-Janvier, Bednarz, & Belanger, 1987; Hiebert & Carpenter, 1992; Lesh, Post, and Behr, 1987; Schwarz and Dreyfus, 1993).

The question is how understanding across multiple representations can be improved with educational technology. Kaput (1992) advocates the use of linked representations as follows:

> All aspects of a complex idea cannot be adequately represented within a single notation system, and hence require multiple systems for their full expression, meaning that multiple, linked representations will grow in importance as an application of the new, dynamic, interactive media (p.530).

According to Piaget's theory, cognitive development is driven by a series of equilibrium-disequilibrium states. If everything is in equilibrium, we do not need to change anything in our cognitive structures. Linked representational software gives students immediate feedback on the consequences of their actions with machine accuracy, but it may not engender the disequilibrium necessary for learning. Semi-linked software, by not showing the corresponding changes in other representations, by giving time to reflect or asking questions about what kind of changes will result from a change in any representation, forces students to resolve the dissonance in their cognitive structures. If their organization of knowledge is well established, they can deal with the question. However, if not, then they will need accommodations in their cognitive structures. Thus, semi linked representational environment puts students in a more active role as learners.

Data Collection Methods

Subjects of this study were ninth-grade Algebra I students. The class was divided into three groups of students—two experimental groups and a control group. Two experimental groups used the same software but with different linking properties—linked and semi-linked. VideoPoint is a software package that allows one to collect position and time data from QuickTime movies of two cars driving in the same direction with different constant speeds or two fish swimming towards each other. These data can be combined to form calculations such as distances between points and can be presented using different representations such as tables, graphs, and equations. Although VideoPoint was designed as linked representational software, the linkage for the table representation was not two-way. So, the software developer made changes, at the request of the investigator, to create the fully linked and semi-linked versions of the VideoPoint for this study.

The differences between the linkages in the linked and the semi-linked representations are summarized in Figure 1. As one can observe, the graph, table, and movie representations are linked two-way in the linked version. This means that when the user clicks on a point in those representations, the corresponding data points in all other two representations are highlighted. Moreover, when the linked version user clicks to see the algebraic form (the equation of best fit) of the phenomena, the line of best fit is also graphed in the graph window automatically. On the other hand, the user of the semi-linked version is not able to see any updates when s/he clicks on one representation. The only linkage that is available in the semi-linked version is between the graph and equation form. When the user estimates the coefficients in the algebraic form, s/he has an option to see the graph of the predicted equation.

Data collection methods included mathematics pre-and posttests, follow-up interviews with all students after the mathematics posttest, clinical interviews at the end of the treatment with 5 students from each experimental group, and classroom and com-

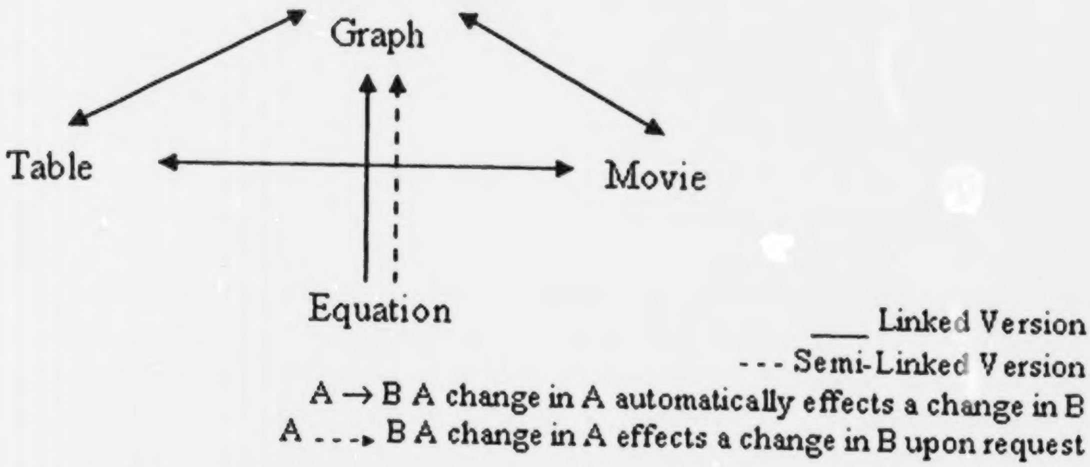

Graph
Table
Movie
Equation
___ Linked Version
- - - Semi-Linked Version
A → B A change in A automatically effects a change in B
A - - -▸ B A change in A effects a change in B upon request

Figure 1. The linkages among linked and semi-linked representations in VideoPoint.

puter lab observations. A survey was conducted at the end of the study in order to see students' opinions about mathematics, representations in general, and the computer environment. Table 1 summarizes the research design with data collection and data analysis methods.

Results

Instead of studying each question separately, questions in all written tests used in this study were clustered into categories, and those categories were compared across the three groups—linked, semi-linked, and control. The categories were: Word Problems, Interpreting/Constructing and Reading Graphs, Solving and Constructing Equations, Reading and Constructing Tables, and Misconceptions (Height/Slope, Point/Interval, Graph as Picture). These categories were compared using a nonparametric test—Kruskal-Wallis (a test for several independent samples)—to identify differences between the linked, semi-linked, and control groups. The results of this test showed that there were no differences in achievement between the groups in any category of problems on either the pretest or posttest at either the .05 or .1 confidence level. A nonparametric test—the Wilcoxon Test (a test for dependent samples)—was used to identify the improvement or decline from pretest to posttest within groups in each category (see Table 2). Some of the improvements were significant at the .05 level, such as experimental groups in the categories of interpreting graphs and constructing equations, the semi-linked group for the height/slope misconception category, and the linked group for the graph as picture category. Other improvements were significant at the .1 level, such as the linked group for the height/slope category.

In order to study the mathematical learning within the computerized environment, clinical interviews were conducted. It was found that in the linked software environment, when a question was asked, students either used the linkage directly to answer the question or they assimilated this new information and drew upon their previous knowledge to answer the question. When they used the linkage, their explanation for their answer was based more on the software; especially the movie. However, their answers were more based on the mathematical aspects of the question, when they did not use the linkage. When students provided an inappropriate answer to a question and they saw that they were wrong according to the linkage or computer feedback, disequilibrium occurred. Then they needed to go back and interpret this new information with their existing knowledge. If they could not interpret the new information, they needed to accommodate their preexisting knowledge in order to reach equilibrium. Sometimes students did not have the enough background to interpret this new information with their existing knowledge. Some students did not use the linkage at all, when they trusted their knowledge and answers..

When a question was proposed in the semi-linked environment, students relied mainly on their own existing knowledge with the help of the software. They assimilated new information and drew upon their existing knowledge to answer the ques-

Table 1. Data Sources.

Research Questions	Data Collection Methods	Description of the Data Collection Methods	Criteria or Indicators for Analysis
1. What are the effects on students' understanding of linear relationships using linked representation software compared to using semi-linked representation software?	Clinical and Follow-up Interviews	Five students from each experimental group were interviewed while using the computer software. Follow-up interviews after the pre- and posttests provided information about their reasoning in answering the questions.	Codes, patterns and themes were searched throughout the data.
	Mathematical Pre- and Posttest	Students' paper and pencil performance were analyzed.	Descriptive analysis Nonparametric tests for group differences and for achievement differences between pre- and posttest
	Teacher Interviews	To see the teacher's views about students' growth mathematically and their preferences	Codes, patterns and themes were searched throughout the data.
	Observations	Everyday classroom and computer lab sessions observations	
2. What are students' attitudes towards and preferences for mathematical representations: equations, tables, or graphs	Survey	Students' attitudes towards mathematics, mathematical representations and their rationales for their preferences towards representations were studied with Likert scale and open-ended questions.	Descriptive analysis Nonparametric tests for group differences Qualitative analysis for open-ended questions

Table 2. *Improvement Significance Scores*

		Improvement Significance Scores		
		Control	**Linked**	**Semi-Linked**
Word Problems (Verbal)		0.083*	0.058*	0.014**
Graphs	Interpreting/Constructing Graphs	0.059*	0.026**	0.007**
	Reading Graphs	0.785	0.038**	0.729
Equations	Solving Equations	0.157	0.317	0.180
	Constructing Equations	0.066*	0.042**	0.042**
Tables	Reading Tables	0.317	0.02**	0.102
	Constructing Tables	1	1	0.317
Height/Slope Misconception		0.684	0.061*	0.024**
Point/Interval Misconception		0.102	0.317	0.317
Graph as Picture Misconception		0.317	0.014**	0.157

*.1 significant **.05 significant

tion. Although the semi-linked environment did not provide such rich feedback as in the linked environment, a ready-made graph or table presented powerful visual information/feedback for students to use while answering the questions. Lack of linkage forced more mathematically-based explanations instead of movie-based explanations and empowered students to trust their answers and convince themselves and construct the linkages between representations by themselves. Some students needed the linkage in some situations in order to construct more empowering mathematical concepts.

The researcher tried to follow up the teacher's regular class sessions in the computer labs with the aim of giving opportunities to students to apply knowledge learned in class. Moreover, it was hoped that students would also carry their learning from the computer labs to the regular class sessions, which mainly consisted of paper and pencil tasks. There were a couple of incidents that showed students were carrying ideas back and forth from the class to the computer lab and vice versa.

Finally, the results from the survey revealed students' attitudes towards mathematics and their preferences for particular representations in paper and pencil and computer environments. All students exhibited somewhat positive attitudes towards mathematics. Students in each group all had similar attitudes towards the use of representations in mathematics. Most students agreed that mathematics problems can be solved in various ways by using different representations. Although they reported that they liked using more than one representation in solving mathematics problems, they also agreed that they found it easier to focus on one representation. They agreed that using different representations does not lead to totally different answers. Students reported that they preferred tables and equations to graphs. They indicated that they usually start solving mathematics problems with tables or equations. Previous experience/knowledge with a representation and knowing how to manage it was a common reason for students to choose a particular representation. There were specific reasons for choosing a particular representation, such as being able to find an exact answer with an equation, the visual advantages of graphs, or the organized information provided by tables.

Most of the students indicated that they found VideoPoint helpful in learning mathematics. Easy access to all representations at once was a common theme mentioned by students as a reason for finding VideoPoint helpful. Students reported that tables and graphs were the types of representation they liked the most while using VideoPoint. Graphs came to be the preferred representations due to the easy access to them with VideoPoint. Some students also mentioned how VideoPoint helped in constructing relationships among representations. They reported that they liked being able to see different kinds of representations all at once since it gave them a choice to work with one that they were more comfortable with or showed them there were various forms available. Several students also pointed out that VideoPoint was helpful in comparing different representations or checking their answers.

References

Confrey, J. (1990). What constructivism implies for teaching. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 107-122). Reston, VA: National Council of Teachers of Mathematics.

Dienes, Z. P. (1960). *Building up mathematics*. Great Britain: Anchor Press, Hutchinson Educational.

Dufour-Janvier, B., Bednarz, N., & Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 109-122). Hillsdale, NJ: Erlbaum.

Goldin, G. A. (1990). Epistemology, constructivism, and discovery learning mathematics. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 31-47). Reston, VA: National Council of Teachers of Mathematics.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan.

Kaput, J. J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515-556). New York: Macmillan.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33-40). Hillsdale, NJ: Erlbaum.

Rich, K. A. (1996). *The effect of dynamic linked multiple representations on students' conceptions of and communication of functions and derivatives*. (Doctoral dissertation, University of New York at Buffalo, 1995). Dissertation Abstracts International, 57 (1), Z142.

Schwarz, B., & Dreyfus, T. (1993). Measuring integration of information in multirepresentational software. *Interactive Learning Environments, 3* (3), 177-198.

VideoPoint 2.1.2 (Modified by the developer for this study) [Computer Software]. (2000). Lenox, MA: Lenox Softworks.

Von Glaserfeld, E. (1996). Aspects of radical constructivism and its educational recommendations. In L. Steffe, et. al., (Eds.), *Theories of mathematical learning* (pp. 307-314). Hillsdale, NJ: Erlbaum.

VISUALIZING GEOMETRY WITH A CALCULATOR

Marco Antonio Santillán Vázquez
CCH-UNAM, México
msantill@mail.cinvestav.mx

Luis Moreno-Armella
Cinvestav-IPN, México
lmorenoa@data.net.mx

Abstract: This article is part of a study in process about the use of the (GC) in mathematics learning. This study aims to research how this instrument induces cognition; it seeks for clarification of the function a tool develops in a mental activity. Some results of the application of the calculator in elementary plane geometry topics with high school students are shown here.

Introduction

Illustrating an idea with drawings or diagrams is a heuristic that has been used in mathematics for many centuries. In geometry prior to Euclid's The Elements, visual representations developed a very important function. For example, in the Plato's Dialogue known as Menon, we have a beautiful example of how a teacher uses the intuitive power of drawings to awaken understanding in the learner. In this passage, a visual representation is used to make a theorem evident. On this issue, Szabó (1960) points out: "it's known that the former Pythagoras' followers considered geometry as "a science inseparable from vision", and that in the early Greek geometry demonstrations were not probably more than simple visualizations". Showing when an affirmation was correct or not lay in the drawing. Later, greek geometry started a fundamental transformation process when it began to prove the theorems apart from concrete considerations. In this way, the importance of the visual aids was declined.

Due to its concrete character, visual representations are a link to access abstract ideas and therefore, play an important role in some learning process. Nevertheless, many high school students have several difficulties to process visual aids in geometry and other areas of mathematics. Visual representations can be a powerful tool to deal with more complex abstractions and are also an important ally in the teaching and learning of mathematics. At the same time, however, understanding a geometric idea by seeing a drawing is not as easy as it may ingenuously seem. It is necessary to learn to see acutely as the skill of processing visual information is not innate.

Technology and Visualization

In the last forty years there has been generated a rebirth of the visualization in mathematics, that is: "Mostly, this revival has been conducted by the technological development. Graphic computation has expanded enormously the range and power of visualization" (Zimmermann & Cunningham, 1991, p.1). Specifically, in the last fifteen years the development of software with didactic purposes has evolved quickly, for instance, at the end of the 80's dynamic geometry programs introduced the "drag effect" which permits to modify the elements in a drawing (performed with a ruler and a compass), keeping unalterable structural relationships. This kind of software can be an important tool to support geometry learning whenever visual approaches develop an important function.

Conceptual Framework

Human action is mediated by instruments which manifest its influence in the subjects that develop "distinctive styles or approaches" (Ruthven, 1990, 1996) associated to the kind of tool used to perform a task. Some research about the use of different instruments to draw circles, Chassapis (1999) states: "Different mediating instruments have a differing impact on thinking and conceptual processes". Accordingly, mental functions are likely to be determined by the mediating instruments used. Following this, Wertch (1993, p.32) states: "Does it make any sense to isolate the mechanism that mediates the action from the mental action of the individual? Actually, is it possible to say that such action is separated from the mediating instruments?". Hölz (1996), referring to the dynamic geometry software, poses the question: " How does this software affect geometry learning?". These questions are the roots of our research.

We assume the perception of the world through our senses and the existence of objects that are not immediately manifested, math concepts are not directly accessible. Thus, an external support is needed to approach them, a medium let them be manifested and manipulated besides a system of representation. To Dufour-Janvier (1987, p. 110) "Representations are inherent to mathematics... we can not pretend we have studied mathematics if we have not learned them… Some representations are closely related to a concept as it is difficult to accept that the concept can be conceived without them". The relevance of the representations can be more evident considering the function of drawings in early learning and teaching of geometry. Dynamic geometry programs also represent geometric objects and introduce a new element: they represent action. Dragging is a represented action. Here resides a distinguishing element of these tools and its potential for learning.

Case Presentation

We have videotaped high school students' dialogues (aged 15-16), working with graphic calculators (GC) TI-92 while interacting with their instructor. In a segment, a pair of students use a marker and a board to do a drawing as the one shown in figure 1. The instructor asks if the displacement of the chord BC, keeping perpendicular to the diameter changes the area of the triangle. The first answer is: " it does not change the area", the argument: "these sides (AB and AC) were shortened and this side (BC) was thickened". Students also mentioned that: "when the perpendicular BC goes bigger, the sides AB and BC become smaller".

Subsequently, students use the graphic calculator (GC) and draw a triangle as the one shown in figure 2. Since point p is found below the center of the circumference o,

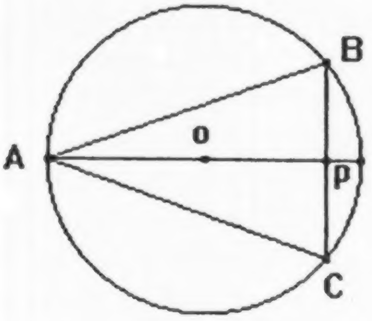


figure 1

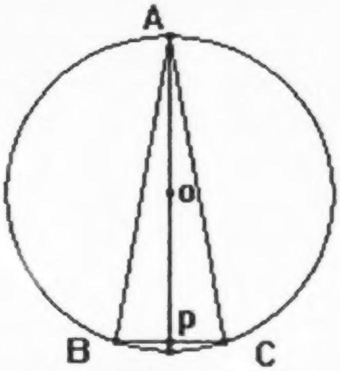


figure 2

students start the exploration dragging point p upwards little by little. They sustain the invariability of the area stating the same compensating argument: "what it increases here, diminishes there". Minutes later, to their surprise, they find the variation of the area. They have observed that by dragging point p upwards beyond point o, the "base" BC and the "height" Ap diminish. Then, the instructor asks which the triangle with the maximum area is and students start a more systematized exploration and conjecture that the triangle of maximum area has a diameter as a "base" and a radius as the "height", "when the point p is in the center", they say.

As they continued the exploration and before the demand of arguments to support their conjecture, they ask if they can measure the area with the calculator, only a visual exploration without measuring the area is allowed in that moment. While one of the students uses the calculator, another observes with attention and says: "the area changes when point p moves upward from the center" but, moving downwards from the center, the area is constant". Later on, they are permitted to measure the area with the tools of the calculator. They rapidly find that the maximum area " seems to belong to a equilateral triangle". They are asked: " what makes you think it is equilateral?" One of them referring to the sides of the triangle says: "At first sight, they seem to be equal". The instructor asks reasons and both of them agree that the angles must measure 60°.

Using the calculator they confirm the size of the angles and state their conjecture: "An equilateral triangle will have a maximum area whenever it is in a circumference. To analyze this video, we have considered three moments: i) When the problem was posed until the perception of the variation of the area. ii) When students' conjecture is that the maximum area triangle is a rectangle ("base equal to diameter"). iii) The discovery of the equilateral triangle as the one of the biggest area.

Since the beginning and for a while, one of the students resists to accept the variation. The other has doubts but once a partner tells him about the compensating justification "what increases here, diminishes there", he is convinced. As in this part they do not use the calculator, they focus on the drawing of the board to wonder about the question. The orientation of the triangle they build up (figure 1) does not help. Although they draw two triangles in the circle, they do not perceive its variation. This representation, the drawing on the board, shows its limitations. In order to grasp the variation, the orientation of the drawing was important (figure 2) as well as the use of the calculator. For quite a long time, one of the students was not able to perceive the variation, when point p is below point o. Subsequently, he accepts that the area diminishes if point p moves upwards from the center but he sustains the invariability of the area when point p moves downwards from the center. His justification: " the height increases but the base diminishes". When they drag point p from the upper part of the construction, they observe that the "base" and the "height" increase at the same time and they easily accept that the area is changing. It is important to mention that since that moment, one of the students perceives the variation of the area to any position of point p over the diameter before the other student. However, both of them perceive, wrongly, that the triangle of bigger area is " the one of longer base" (a diameter) and "height" (a radius) "when p is above o".

For these students, it was necessary the use of the measuring tools of the calculator to conjecture that the triangle of the maximum area was equilateral. For one of them, it was strictly necessary the measurement to accept the variation of the area in any position of point p. In a later talk, one of the students said "of course, the calculator was decisive, without it I was not able to see that the equilateral triangle was the one of maximum area", and besides this, it was easier for him to perceive the variation of the area with the construction of figure 2 because "I saw clearly how the "base" and "height" of the triangle changed. He was asked why the had built up the drawing with the calculator in that way (figure 2) if on the board, they had drawn it with another orientation, his answer was "we did not think about it, but like this, with the base parallel to the horizontal looks easier". While trying to clarify what makes the difference in the perceptions of these two students, we found out, as a possible explanation, a differentiated ability to abstract, one of the students is much better than the other in this field. This is what we found with the questionnaires applied and the follow up of their performance throughout the study.

In another activity without the use of the calculator, students are asked if the angle ACB that belongs to figure 3 changes when point C is moved over the circumference. Some students think that the angle changes and others think that the triangle remains constant. Generally, after using the calculator to displace the point, all the students accept that the angle does not change. In the next step, students start a free exploration with a construction like the one shown in figure 4. Some students found out that the triangle in the semi circumference must be a rectangle, others think that the angle in the center (vertex in o), it is two times the angle with vertex in point A. The words used by the students to state the conjectures unveil differentiated abilities of perception, and particularly, of conception. The students with better resources and math knowledge,

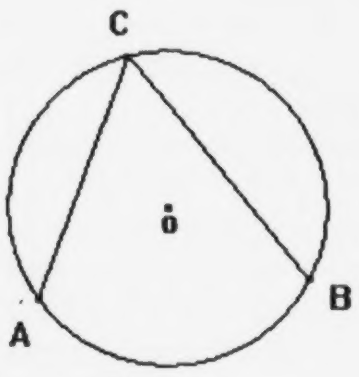


figure 3

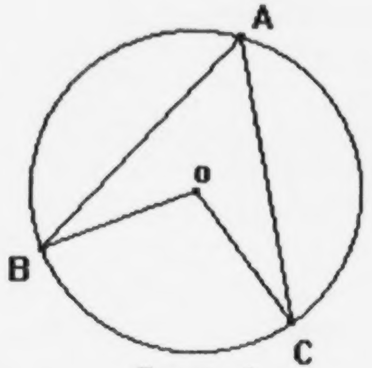


figure 4

the remarkable ones, do better verbal descriptions, the statements of their conjectures are more formal from a mathematical point of view.

An Interpretation

Obtaining data by means of questionnaires, previous and subsequent interviews to this study to students, shows that the compensating arguments "what increases here, diminishes there", are deeply rooted in most high school students (aged 15-17). Then, how did these students' perception work with the use of the calculator? The observation of several students. Working with (GC) make us consider two basic elements: a) the tool, which provides the drag effect and b) the cognitive structure of the students that de-codification that information. What does the drag consist of? It generates the change of position of the objects drawn on screen. A continuous succession creates the sensations of manipulation of the object (the drawing), of movement in a real time. It is the succession of images, perceived as a constant transformation of the object, where there is a chance so that the student perceives the variation and he can release the compensating justifications. On the other hand, the action of the subject is not limited to manipulate the object through the tool and see the screen, he visualizes the variation because he generalizes the succession of images, turns them into a scheme. It means that the specific of a drawing is integrated to the general. Then, visualizing is in a way, a process of generalization.

Population of Study

The research has shown the limitations of most students in basic topics of plane geometry and the difficulties students face to grasp a geometric diagram and describe it in words. By applying questionnaires to participants in this study, we know that students have basic information and knowledge of geometry. For instance, they are not able to state correctly the theorem of Pythagoras, some of them think that it can be applied to all triangles and a few of them can not identify the height of these. They do not know, besides, important theorems such as the central angle or even more basic theorems. The subjects of study have several problems with geometric drawings, for example: a) some of them are not able to describe the essential attributes of triangles and circles; b) they commonly establish their reasoning only on the drawing, without considering the conditions (the statement) that are imposed on the activity where this is used and c) many students' descriptions of geometric representations are vague and irrelevant and most of them have difficulties to generate and work with mental representations of geometric figures. The precarious situation of the participants imposes restrictions and challenges to investigation. The activity just being reported, may seem simple, but it is not for the case of the population of study. To them, it represents real difficulties that help us to show elementary processes of visualization and understanding.

The basic instruction of the calculator was carried out in five two-hour sessions, in which the participants learn to use basic operations and to discover some former properties of the software. In this period of training, the activities are focused on building up constructions and measuring segments, angles and areas.

Initial Results

Most participants improved their perception visualizing some essential relationships in the drawings shown by the calculator. This is very likely to mean that the drag effect is directly responsible for this improvement in the perceptive apprehension of the drawings.

There are some students that just by discovering relevant relationships of the drawings speak them out as statements with some formal structure supporting them on clear references to the tool. The word "drag" is common in the language of the students, and terms such as "enter F2", "mark the angle", etc., reveal the influence of the tool.

Generally, students discover invariant, only some of them are able to speak relevant statements. These students, however, are already outstanding before starting calculator operation. Calculators increases the possibilities of visualization, the student achieves them. We believe that we have found a cognitive threshold (minimum level of knowledge), That is to say that below this, the function of the GC to support understanding is null. We try to achieve the description of this phenomenon that we believe is linked to the ability of abstraction and symbolization of the users of the tool.

What do calculator users see on the screen? A continuity of images that human eye is not able to segment through continuous temporal. The individuals see changes, transformations of a drawing through time. That illusion of continuity is not only related to different images, but also to temporal space relationships that give a new meaning to the drawing. We can say that this kind of technology provides a new way of representation with potential value for geometry learning.

Conclusions

Drawings are essential in geometry teaching, they are needed to support an explanation or to show a kind of relationship. Nevertheless, it is a fact that students and teachers do not see the same in diagrams or drawings. In classrooms, it is very common that for many students visual aids have nothing to do with intuition, for them, what they see in the drawings and diagrams is not enough to see what these drawings represent.

Informative media opens the access to the direct manipulation of the represented objects and offer the opportunity to displace them all over the screen. This effect can be used to visualize invariant and structural properties to the geometric figures. With the tool of dynamic geometry of the calculators, the users have the opportunity to carry out actions which are unreachable with a ruler and a compass. It gives better possibilities to visualize some connections among elements of the figures that are not easy to see using traditional tools. If the structural relationships of a figure are visualized, then the subject is potentially in conditions to formulate coherent statements such as relevant definitions and theorems. That is to say that we have two registrations of representation, the drawing and verbal language, to articulate and the interpretation of the geometric cognition mediated by the calculator. We try to build up a local theory that describes the relationships between those registrations of representation.

The illusion of the continuity of the changes in position of the drawing carried out with the calculator, and the "continuous" transformations generated by the tool, are not only the result of the observation, but also a premise of the visualization, see with understanding.

In this study, we have observed that the GC can be a potential generator of difficulties. The activities with GC must be carefully designed and must consider previous knowledge of the instrument. The calculator, instead of supporting understanding, can provoke difficulties, create wrong ideas and originate obstacles. In this moment we are documenting and classifying the detected problems.

References

Chassapis, D. (1999). The mediation of tools in the development of formal mathematical concepts: The compass and the circle as an example. *Educational Studies in Mathematics, 37*, 275-293.

Dufour-Janvier, B., et al. (1987). Pedagogical considerations concerning the problem of representation. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics*. Erlbaum.

Hölz, R. (1996). How does "dragging" affect the learning of geometry. *International Journal of Computers for Mathematical Learning, 1*, 169-189. Kluwer.

Ruthven, K. (1990). The influence of graphic calculator use on translation from graphic to symbolic form. *Educational Studies in Mathematics, 2*, 431-450.

Ruthven, K. (1996). Calculators in the mathematics curriculum: The scope of personal computational technology. In A. J. Bishop et al., (Eds.), *International handbook of mathematics education*, (pp. 435-468). Kluwer.

Szabó, A. (1960) The transformation of mathematics into deductive science and the beginnings of its foundation on definitions and axioms. *Scripta Mathematica, 27* (1), 27-49.

Wertsch, J. (1993). *Voces de la Mente: Un enfoque sociocultural para el estudio de la acción mediada*. Visor, Madrid.

Zimmermann, W., & Cunningham, S. (Eds.). (1991). *Visualization in teaching and learning mathematics*. In: MAA Notes Series.

EXAMINATION OF A MATHEMATICS TEACHER'S EXPLORATION IN A TECHNOLOGY-SUPPORTED CLASSROOM

Doug McDougall
OISE/University of Toronto
dmcdougall@oise.utoronto.ca

The management of the learning environment must allow for students to construct their own knowledge and to take responsibility for their own learning. Students also need the freedom to discover, through exploration, different ways to build solutions and to spend time working with problems and searching for solutions. As such, it is important that teachers provide students with the opportunity to explore, analyze, and demonstrate their skills.

The use of the computer has been heralded as one teaching tool suitable to mathematics teachers to encourage exploration of mathematics. Its entry into schools, however, has been slow and this is especially true for mathematics classrooms. Many reasons exist for this lack of integration into the mathematics curriculum: teacher anxiety towards computers (Rosen & Weil, 1995; Berebitsky, 1985), challenges of mathematics reform (Smith, 2000: Spillane, 2000; Ross, Hogaboam-Gray, & McDougall, 2000), and teacher control in classrooms (Sandholtz, Ringstaff, & Dwyer, 1997: Cohen, 1989)

This study investigated the use and implementation of geometric construction software in a Grade 6 mathematics classroom. This study examined the reactions of a mathematics teacher as her students explored geometric constructions in a probing for understanding milieu. The teacher was asked to teach the geometric construction unit, normally taught using compass and straightedge tools, using the *Geometer's Sketchpad* computer program.

The teacher experienced an initial loss of control in this environment. As the teacher gained confidence in their own use of the software and recognized that students were experiencing success, the teacher began to regain her sense of control. She was also concerned about the experiences of the students who had been having difficulty with mathematics and how they might be successful in the geometry unit. The implication for mathematics education is that students thrive in dynamic geometric software environments when teachers maintain control over the management of learning, their own personal expectations, and their role as a professional.

Teacher control as a professional is a reality in middle school mathematics teaching. Teacher educators can assist teachers to maintain a level of control over their professional lives by providing them with the tools to be mathematical explorers. Teachers need to be placed in learning environments where they can explore mathematics, interact with their peers though discussion and case studies, and work with dynamic computer environments. These dynamic computer environments provide an environment where teachers and students can interact and share their conjectures and findings with each other. Teacher educators should provide opportunities within their curriculum for teacher exploration in these computer-based tools.

References

Cohen, D. (1989). Educational technology and school organization. In R.S. Nickerson & P. P. Zodhiates (Eds.), *Technology in education: Looking towards 2020* (pp. 231-264). Hillsdale, NJ: Erlbaum.

Rosen, L. D., & Weil, M. M. (1995). Computer availability, computer experience and technophobia among public school teachers. *Computers in Human Behavior, 11* (1), 9-31.

Ross, J. A., Hogaboam-Gray, A., & McDougall, D. (2000). *Elementary school reform and higher order reasoning in mathematics*. Final report of MET Transfer Grant.

Sandholtz, J. H., Ringstaff, C., & Dwyer, D. C. (1997). *Teaching with technology: Creating student-centered classrooms*. New York: Teachers College Press.

Smith, M. S. (2000). Balancing old and new: An experienced middle teacher's learning in the context of mathematics instructional reform. *Elementary School Journal, 100* (4), 351-376.

Spillane, J. P. (2000). A fifth-grade teacher's reconstruction of mathematics and literacy teaching: Exploring interactions among identity, learning, and subject matter. *Elementary School Journal, 100* (4), 307-330.

DEVELOPING PEDAGOGY FOR WIRELESS CALCULATOR NETWORKS[1]

A. Louis Abrahamson
Better Education Inc.
la@bedu.com

Douglas T. Owens
The Ohio State University
owens.93@osu.edu

Michael Meagher
The Ohio State University
meagher.10@osu.edu

Marlena Herman
The Ohio State University
herman.91@osu.edu

With the advent of wireless systems the use of Classroom Communication Systems (CCSs) is about to enter a new era. The TI Navigator, from Texas Instruments, provides teachers with the possibility of a wireless networked classroom. The pedagogical potential of this technology is still in its development stage but preliminary research suggests benefits to active student participation in class and collaborative inquiry in the classroom (Bransford, Brophy, & Williams, 2000; Abrahamson, Davidian & Lippai, 2000). The present study is designed to investigate the extent to which teachers in an in-service program can learn to use CCSs effectively.

Drawing on the work of Bransford, Brown, and Cocking (1999), the study is specifically designed to investigate to what degree is it feasible, given the constraints of a typical one-week intensive in-service teacher enhancement institute, to teach high school mathematics teachers to use a CCS effectively and make their teaching become more learner-centered, knowledge-centered, assessment-centered, and community-centered? Aspects of a learner-centred approach by a teacher include the extent to which the teacher uses questions, tasks, and activities to show existing conceptions that students bring to the classroom, and the extent to which teachers exert an appropriate amount of pressure on students to think through issues, establish positions, and commit to positions. A knowledge-centred approach manifests itself in a focus on conceptual understanding, and the diagnosis and remedy of misconceptions. Assessment-centred instruction concentrates on formative assessment to provide feedback to students and to teachers on student conceptions. Finally, a community-centred approach is reflected in, for example, class discussion, peer interaction, and non-confrontational competition.

Twenty-five high school mathematics teachers are experiencing effective pedagogical techniques for using the TI Navigator system in a one-week intensive institute. Among these techniques are the use of polling as a springboard for class discussion, data collection and dissemination, interactive group simulations, and the provision of immediate feedback to students. The institute will be followed by a series of observational visits to classrooms supported by questionnaires and interviews with both students and teachers. The goal of the data collection is to (a) assess the learner-centeredness, knowledge-centeredness, assessment-centeredness, and community-centeredness of these classrooms, and (b) to provide formative assessment for the teachers on their pedagogical application of the technology.

References

Abrahamson, A. L., Davidian, A., & Lippai, A. (2000, November). *Wireless calculator networks: Where they came from, why they work, and where they're going.* Paper presented at the 13th Annual International Conference on Technology in Collegiate Mathematics, Atlanta, GA.

Bransford, J. D., Brown, A. L., & Cocking, R. R. (Eds.). (1999). *How people learn: Brain, mind, experience, and school.* Washington, DC: National Academy Press.

Bransford, J. D., Brophy, S., & Williams, S. (2000). When computer technologies meet the learning sciences: Issues and opportunities. *Journal of Applied Developmental Psychology, 21* (1), 59-84.

Note

[1] Supported by a grant from the National Science Foundation.

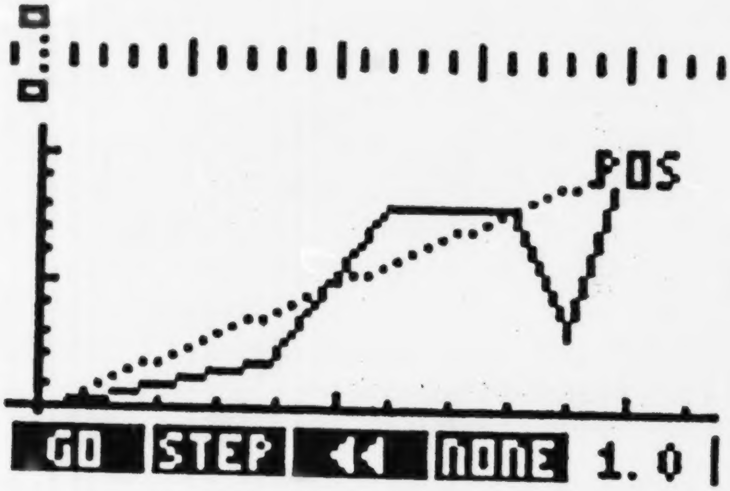
NEW ACTIVITY STRUCTURES EXPLOITING WIRELESSLY CONNECTED GRAPHING CALCULATORS

James J. Kaput
University of Massachusetts-Dartmouth
jkaput@umassd.edu

Stephen Hegedus
University of Massachusetts-Dartmouth
shegedus@umassd.edu

Recent advances in connectivity and the use of hand-held devices are now becoming available to create new forms of mathematics instruction. The authors are engaged in a research project aimed at determining the affordances and constraints of classroom connectivity in mathematics classrooms. Using video from classrooms and demonstrate using linked graphing calculators, we will report on two different activity structures that exploit connectivity in two distinct ways. One uses the ability of quick uploads of student constructions of personally important mathematical objects for "performance" and discussion by the entire classroom. The second uses connectivity to examine parametrically varying mathematical objects, where the students themselves index the parameters. Each activity structure (among several other applications of connectivity being studied) yields intensity and quality of communication that is difficult to attain on a sustained basis in "disconnected" mathematics classrooms.

Example 1: Students create a personally meaningful mathematical object and share it with the class in what amounts to a performance-event. We illustrate with a sample (the position vs. time graphs drive the horizontal motion of the two objects on the top of the calculator screen shown below).



Your job is to write an exciting race-script for your own sack-race with A (Above) that ends in a tie and make a Position graph for B (Below) that makes your race happen. You will then send your MathWorlds document to your teacher who will run it for the whole class to see while you read your exciting script!

A Sample Race B starts off slower than A and falls behind, but then gets a burst of energy and zooms ahead. But then, B falls down, and A catches up and passes B. Now B gets confused, and goes backward! But then B turns around, and in wild finish, catches up with A to tie the race!!

Example 2: Systematic variation. Here students count off in their respective groups of 4, so each student has a number between 1 and 4, inclusive. A travels from 0 to 12 m in 6 seconds.

Part (a) (Staggered Start, Staggered Finish) Your job is to write a linear function Y = mX + b for B, so that you will go at the same speed as A for the same amount of time as A, but starting at your NUMBER. Part (b). (Staggered Start, Simultaneous Finish) Now you must start at 3 times your number and all end in a tie!

Here the teacher uploads and aggregates the graphs of the student functions on the same coordinate axes, yielding a family of parallel lines (and motions) indexed by the respective student numbers. Part (a) illustrates the role of the y-intercept as initial position and slope via the parallelism of the 4 lines. In Part (b) the students must determine the needed slope, and the set of linear functions differ systematically, contextualizing the variation and, especially, the Y = 12 equation, yielding zero velocity. They also experience the idea underlying simultaneous equations—same place, same time. We then ask, what if there was a 5th person in the group?

CALCULUS AND THE WIRED CLASSROOM: EXPLORING THE INTERSTICES OF SOCIAL TECHNOLOGIES

Robert M. Klein
The Ohio State University

This poster presents the results from research on university-level Calculus & *Mathematica* (C&*M*) classrooms that seek to use computer texts to present new material, record homework, and facilitate group work. This study seeks to find answers to the question: How does using the computer as the primary instructional device impact the locus and authorization of knowledge in the Calculus&*Mathematica* classroom?

The framework used in this study offers the opportunity to view the data from behind multiple lenses of interpretation and analysis. Primarily, this study uses elements of critical discourse analysis (CDA) as articulated by Foucault (1991), Mills (1997), Chouliaraki and Fairclough (1999) and Gee (1999) to approach interpretation and representation of the data. Feminist theory perspectives on situated knowledge (Haraway, 1988) are also used.

Critical discourse analysis is a way of looking at data that respects the highly political nature of social contexts. It seeks to interrogate systems of knowledge to look for institutional contradictions, and, by exposing those contradictions, create space for action.

Responsible research ethics demand multiple modes of inquiry. This research uses a crystallization of methods and perspectives from which to view and analyze the data. These methods include document analysis, interviews, observations, and a grounded survey.

Data consists of two sets of taped and transcribed interviews with 3 women from the C&*M* classes, chosen because of their demonstrated abilities to vocalize their feelings in the class (usually away from the teacher). Two classes were observed for one academic quarter, and data was recorded daily in a researcher's log. Further interviews were conducted with engineering advisors and math counselors. Furthermore, the computer text and accompanying online documentation form a large body of data for analysis. Finally, following preliminary analysis of the above data sources, a grounded survey was administered to 131 C&*M* students to confirm and/or deny hypotheses about the classes as a whole.

The results are as follows: 1) Despite a shift from lecturing to computer use as the primary pedagogical device, student experience in the course depends heavily on the instructor. 2) The instructor remains positioned as responsible for authorizing "official knowledge" and in influencing the ways that knowledge is learned. 3) Advisors, students, instructors, and the courseware documentation offer varied descriptions of what the C&*M* course is, yet ultimately the instructor is most responsible for shaping student experience.

Hence, while computing technologies have been used to displace the instructor as "deliverer of calculus," the instructor remains positioned as authorizer of "official knowledge" (Apple, 2000) and maintains a large degree of influence over student perceptions of the *C&M* experience. Current efforts to use technology to teach calculus position the locus of knowledge with the instructor. Creative uses of technology should try to reposition the locus of knowledge in the realm of the social, recognizing that knowledge is socially constructed. This is a challenge for mathematics, often recognized as the science of the "right answer" – a highly structured branch of knowledge.

Efforts to relocate the center of knowledge may involve questioning the nature of the computing technologies involved. What would a computer for three students look like? How do various ways of using computing technology in the classroom reshape the taken-for-granted processes involved in the growth of mathematical understanding? As long as the computer is a "surrogate instructor," programmed with the right answers, the locus of knowledge stays centered away from the social. This is discrepant with currently accepted theories of cognition and also with emerging ways of mathematics instruction.

References

Apple, M. (2000). *Official knowledge: Democratic education in a conservative age.* New York: Routledge.

Chouliaraki, L., & Fairclough, N. (1999). *Discourse in late modernity: Rethinking critical discourse analysis*. Edinburgh: Edinburgh University Press.

Foucault, M. (1991). Politics and the study of discourse. In G. Burchell, C. Gordon, & P. Miller (Eds.), *The Foucault effect: Studies in governmentality.* Chicago: University of Chicago Press.

Gee, J. P. (1999). *An introduction to discourse analysis: Theory and method.* New York: Routledge.

Haraway, D. (1988). Situated knowledge: The science question in feminism and the privilege of partial perspective. *Feminist Studies, 14* (3), 575-599.

Mills, S. (1997). *Discourse.* New York: Routledge.

LEARNING TO FACILITATE AND ANALYZE STUDENTS' PROBLEM SOLVING WITH TECHNOLOGY

Hollylynne Drier Stohl
North Carolina State University
Hollylynne_drier@ncsu.edu

Objective

As teacher educators design and implement technology-based activities, it is important to research the effects of these activities on prospective teachers' mathematical, pedagogical, and technological knowledge. This study reports on prospective teachers as they facilitated 8th grade students using a java applet to solve a ratio problem. The problem and java applet were developed for the ESCOT project and the Math Forum (mathforum.com/escotpow).

Perspective

The role of technology in preparing prospective teachers should be rich with opportunities to learn important mathematical concepts, useful technological skills, and appropriate pedagogy (Garofalo, Drier, Harper, Timmerman, & Shockey, 2000); however, using technology as a tool in their learning is often not enough preparation for prospective teachers (Olive & Leatham, 2000). Prospective teachers also need meaningful experiences working directly with students using technology to solve problems. Instead of relying on this experience happening in a haphazard way during field experiences, I have embedded a sequence of experiences in a course on "Teaching Mathematics with Technology," typically taken 1-3 semesters before student teaching. Since all prospective teachers are engaged in the activities, we were able to discuss the mathematics, role of technology, and pedagogical issues.

Mode of Inquiry and Data Sources

The sequence of activities was planned to give prospective teachers iterative experience-reflection-analysis time in working with technology-based activities with students. By following their development throughout this cycle, I used a coding method to look for similarities across individuals, as well as particular instances that appear to affect a prospective teacher's understanding of mathematics, pedagogy, and use of technology as a representational and problem solving tool. Each prospective teacher first solved the "Fish Farm" problem, discussed the mathematics, and planned facilitating questions they anticipated posing to students. Each prospective teacher worked with two 8th grade students using the java applet to solve the problem. Afterwards, the prospective teachers wrote personal reflections, discussed the experience in small groups, and planned how they would change their interactions with the students. One week later, each prospective teacher worked with two different students on the same problem, reflected on the experience, and compared the two experiences. They also analyzed students' written solutions to the problem, responded to the students in written form, and reflected on what they learned about mathematics, teaching with technology, and students' interactions with the technology.

The data corpus for the study include: videotapes of three computer stations (one prospective teacher and two 8th grade students at each station), both internal computer actions and external social interactions; whole class video tape; video tape of prospective teachers analyzing the students' written solutions; and all written work from prospective teachers and 8th grade students. The three-pronged focus of the analysis included: 1) the prospective teachers' understanding of ratio and problem solving approaches, 2) pedagogical decisions, and 3) use of the java applet tools in helping students solve the problem and understand the ratio concepts embedded in the problem.

Results

This poster session focuses on the development of three case study prospective teachers during this project. The students developed their own understanding of part-part and part-whole relationships, and how to help students transition between two forms of reasoning about a ratio (e.g., 1:2 as 1 part to 2 parts and as a 1/3-2/3 part-to-whole relationship). The prospective teachers recognized and struggled with their ability to ask non-directive questions that engaged the students without "giving it all away." They also tended to improve on their ability to encourage students to justify their reasoning and verbalize their problem solving strategies. The prospective teachers were very reflective about the role of technology in solving this problem. They felt the java applet tools enhanced students' learning. They tended to take advantage of the technology to facilitate their interactions with students and discussed how important pedagogical decisions would be needed to make technology use in a classroom a meaningful experience. The specific results will be shared in an interactive format during the poster session.

Relationship to PME-NA Goals

This study promotes the goal of understanding the psychological aspects of teaching and learning mathematics.

References

Garofalo, J., Drier, H. S., Harper, S. R., Timmerman, M. A. , & Shockey T. (2000). Promoting appropriate uses of technology in mathematics teacher preparation. *Contemporary Issues in Technology and Teacher Education, 1* (1), 66-88. (Available online at http://www.citejournal.org).

Olive, J., & Leatham, K. (2000). *Using technology as a learning tool is not enough.* Paper presented at the International Conference on Technology in Mathematics Education, Auckland. New Zealand

USE OF TECHNOLOGY IN DATA ANALYSIS OF A LONG-TERM LONGITUDINAL STUDY

Elizabeth Uptegrove
Rutgers University
ebuptegrove@aol.com

Purpose

This poster describes how new technology is being used to code and analyze videotape data collected over twelve years of a long-term longitudinal study at the Robert B. Davis Institute for Learning of the Department of Learning and Teaching, Graduate School of Education, Rutgers University, New Brunswick, New Jersey.

Videotape data has been used throughout this study. In the past, this data was viewed through the use of a regular videocassette recorder (VCR) and television set. Transcription, analysis, and coding were generally done by hand and then entered into computer word processing systems. But since early 2000, researchers have been making use of additional computer technology to further automate the process of transcribing, coding, and analyzing this data.

Videotapes are now being converted to digital format for viewing directly on computer monitors. A software program called vPrism allows simultaneous viewing and entry of various kinds of information (transcriptions, summaries, and codes). vPrism also provides for easy cataloging and organization of videotape data. Other software is used to prepare videos for inclusion in reports and projects such as teacher education programs.

Data Sources

A sample application of vPrism is shown, using a portion of an analysis of a videotape of high school students working on a combinatorial problem at an after school research session. Included are video clips, transcripts, analyses, and coding.

Conclusions

The use of technology has improved the accuracy and speed of videotape analysis efforts by automating some of the tedious work and providing automated means for archiving and collating work. But there are pitfalls. For example, software is not completely reliable, and there are compatibility issues (between current systems and also between current and future technology).

References

Kiczek, R. D., & Maher, C. A. (1998). Tracing the origins and extensions of mathematical ideas. *Proceedings of the Twentieth Annual Conference of the North American Group for the Psychology of Mathematics Education* (pp. 159-165), Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Maher, C. A., & Kiczek, R. D. (2000). Long term building of mathematical ideas related to proof making. *Proceedings from the International Congress on Mathematics Education,* Japan.

Maher, C. A., & Speiser, R. (2001). Polynomials, Pascal's triangle, and the building of isomorphisms. *Proceedings of the American Educational Research Association*, Seattle, WA.

Maher, C. A., & Steencken, E. P. (2000). Learning fraction ideas: The classroom as a community. *Proceedings of the American Educational Research Association*, New Orleans, LA.

Powell, A. B., Francisco, J. M., & Maher, C. A. (2001). An evolving analytical model for understanding the development of mathematical thinking using videotape data. *Proceedings of the American Educational Research Association,* Seattle, WA.

Note

This work was supported in part by National Science Foundation grant REC-9814846 (directed by C. A. Maher) to Rutgers, The State University of New Jersey. Any opinion, findings, conclusions, or recommendations expressed in this publication are those of the author and do not necessarily reflect the views of the National Science Foundation.

Author Index

Authors are listed alphabetically followed by the subject code(s) of their paper(s). The subject codes are: Algebraic Thinking (A), Discussion Group (DG), Learning and Cognition (LC), Plenary Paper (PP), Problem Solving (PR), Probability and Statistics (PS), Reasoning and Proof (RP), Socio-Cultural Issues (SC), Teacher Beliefs (TB), Teacher Education (TE), Technology (T) and Working Group (WG).

A. Louis Abrahamson (T)
Silvia Alatorre (LC)
Alice Alston (PR)
Bridget Arvold (TE)
Leslie Aspinwall (PS)
Ann Baker (LC)
Bernadette Baker (A)
Fernando Barrera (PR)
Jeffrey Barrett (TE)
Joanne Rossi Becker (TE)
Carol W. Bellisio (TE)
Jason K. Belnap (TB)
Carol T. Benson (PS)
Sarah B. Berenson (SC) (TE)
Dawn Berk (A)
Elliott Bird (TE)
Maria L. Blanton (A)
Irene Bloom (TE)
Jo Boaler (PP)
Linda A. Bolte (TE)
Katharine Borgen (LC)
Daniel J. Brahier (TE)
M. Lynn Breyfogle (TB)
Lecretia A. Buckley (TB)
Carol Burdell (A)
Cristianne Butto (SC)
Jinfa Cai (PR)
Marilyn P. Carlson (LC)
Guadalupe Carmona (WG)(PR)
José Aires de Castro-Filho (LC)
Laurie O. Cavey (TE)
Jeffrey M. Choppin (TB)
Karen Koellner Clark (TE)
Matthew R. Clark (LC)(TE)
Linda Condron (WG)
Jere Confrey (LC)
José N. Contreras (LC)(TE)
Sandra Crespo (TE)
Emily Dann (TE)
Clarence E. Davis (TE)
Brent Davis (TE)
Gary E. Davis (TE)
Teruni de Silva (LC)
Thomas C. DeFranco (TB)
Galina Dobrynina (A)
Helen Doerr (WG)
Hollis Duncan (PR)
Vari Nelson Durrant (TE)
Amy Burns Ellis (SC)
Susan Empson (TE)
Lyn D. English (DG)(PS)
Diana B. Erchick (WG)
Juan Estrada (PR)
Olimpia Figueras (TB)
William Firestone (TE)
Fulvia Furinghetti (SC)
Karen C. Fuson (LC)(SC)(TB)
Bradley E. Garner (TE)
M. Faaiz Gierdien (SC)
Barbara H. Glass (PR)
Karen Graham (TB)
Theresa J. Grant (TB)
Susan S. Gray (PR)
Linda K. Griffith (TB)
Lourdes Guerrero (PR)
Edith Gummer (PR)
José Guzman (PR)
Lynn C. Hart (TB)

Stephen J. Hegedus (PR)(T)
Karen Heinz (TE)
Clare Hemenway (A)
Beth Herbel-Eisenmann (TB)
Marlena Herman (T)
Abbe H. Herzig (SC)
Catherine Pomar Herzog (TE)
Lisa Hitchcock (TE)
Fernando Hitt (WG)(LC)
Margret Hjalmarson (LC)
Pamela S. Hofbauer (PS)
Amanda Jansen Hoffman (SC)(A)
Peter Howard (LC)
Verónica Hoyos (LC)
Eric Hsu (SC)
Daniel R. Ilaria (RP)
Kathy M. Ivey (TE)
Andrew Izsák (LC)(TE)
Sally Jacobs (LC)
Theresa L. Jardine (SC)
Donna Jenner (LC)
Todd M. Johnson (TE)
Yolanda A. Johnson (PS)
Debbie Junk (TE)
Mindy Kalchman (LC)
Constance Kamii (LC)
Rochelle Goldberg Kaplan (LC)
James J. Kaput (A) (T)
Elham Kazemi (LC)
Regina D. Kiczek (PS)
Thomas Kieren (LC)
Kate Kiln (TB)
Ok-Kyeong Kim (TE)
Margaret Tatem Kinzel (A)(TE)
Robert M. Klein (T)
Hari P. Koirala (TE)
Judith Kysh (SC)
Susan J. Lamon (LC)
Cynthia W. Langrall (PS)
Sean Larsen (LC)
Violeta Lazarovici (A)
Ann Le Sage (TE)
Elena Fabiola Ruiz Ledesma (LC)
Hea-Jin Lee (TE)
Richard Lesh (WG)
Gavrielle Levine (TE)
Gary Lewis (A)
Yeping Li (DG)(PR)
Peter Liljedahl (LC)
Jane-Jane Lo (PR)
Michelle P. Longest (SC)
LouAnn H. Lovin (TE)
P. Michael Lutz (TE)
Carolyn A. Maher (WG)(PS)(RP)
Sitaniselao Stan Manu (LC)
Lyndon C. Martin (LC)(LC)
Tami S. Martin (RP)
Armando M. Martinez-Cruz (TE)
Kay McClain (LC)
Sharon Soucy McCrone (RP)
Douglas McDougall (WG)(TE)(T)
Amy Roth McDuffie (PR)(LC)(SC)
Jean J. McGehee (WG)(TB)
Jean McGivney-Burelle (TB)
Mercedes A. McGowen (TE)
Michael Meagher (T)
Denise S. Mewborn (TE)
James A. Middleton (LC)
Wendy Mitchell (LC)
Lora Monfils (TE)
Edward S. Mooney (PS)
Luis Moreno-Armella (T)
Anne Morris (LC)
Aki Murata (SC)
Tania A. Myrtil (PR)
Cornelia Naude (LC)
Loan H. Nguyen (PR)
Glenda Nixon (LC)
Clara Nosegbe (PR)
Mike Oehrtman (RP)

Jo C. Olson (TE)
Mónica Orendain (SC)
Douglas T. Owens (T)
Saliha Asli Ozgun-Koca (T)
Rosa Páez (LC)
Neil A. Pateman (TE)
Erkki Pehkonen (SC)
Elvia Perrusquía (A)
Bob Perry (LC)
Blake E. Peterson (TE)
Susan E. B. Pirie (LC)
Neil Portnoy (TB)
Thomas Post (WG)
Arthur Powell (DG)
Norma C. Presmeg (LC)
Manya Raman (RP)
Chris Rasmussen (LC)
Antonio Rivera (PR)
Cheryl Roddick (TE)
Teresa Rojano (A)
John A. Ross (TE)
Sharon H. Ross (TE)
Mariana Saiz (TB)
Luis A. Saldanha (PS)
Gabriel Sanchez (SC)
Wendy B. Sanchez (TB)
Marco Antonio Santillán Vázquez (T)
Manuel Santos-Trigo (T)(PR)
Roberta Y. Schorr (TE)
Armando Sepúlveda (PR)
Anna Sfard (PP)
Kenneth L. Shaw (PS)(TE)
Bruce Sherin (LC)
Miriam Sherin (TE)
Martin A. Simon (DG)

Emily Dixon Simpson (TE)
Julie Sliva (TE)
Cynthia Smith (TE)
Jack Smith (A)
Jennifer Christian Smith (LC)
Armando Solares Rojas (A)
Robert Speiser (WG)
Katrina N. Staley (TE)
Darren Stanley (TE)
Jon Star (A)
Dorothea A. Steinke (LC)
Michelle Stephan (LC)
Ana C. Stephens (A)
Hollylynne Drier Stohl (T)
Walter E. Stone Jr. (TE)
Jennifer Thom (LC)
Patrick W. Thompson (PS)
Zulbiye Toluk (LC)
Günter Törner (LC)
Jo Towers (LC)
Maria Trigueros (A) (PR)
Erin Turner (TE)
Elizabeth Uptegrove (T)
Sonia Ursini (SC)
Marta Valdemoros Álvarez (LC)
Jorge Villa (PR)
Sharon B. Walen (TE)
Janet G. Walter (PR)
Tad Watanabe (PR)
Robert G. Wenta (TE)
Dirk Wessels (LC)
Joy Whitenack (TE)
Steven R. Williams (TE)
Judith S. Zawojewski (PR) (WG)
Joseph T. Zilliox (TE)

ISBN 0-88119-001-2

9 780881 190014

1007

U.S. Department of Education
Office of Educational Research and Improvement (OERI)
National Library of Education (NLE)
Educational Resources Information Center (ERIC)

ERIC
Educational Resources Information Center

NOTICE

Reproduction Basis

☒ This document is covered by a signed "Reproduction Release (Blanket)" form (on file within the ERIC system), encompassing all or classes of documents from its source organization and, therefore, does not require a "Specific Document" Release form.

☐ This document is Federally-funded, or carries its own permission to reproduce, or is otherwise in the public domain and, therefore, may be reproduced by ERIC without a signed Reproduction Release form (either "Specific Document" or "Blanket").

EFF-089 (1/2003)